LONG-TERM ORBIT COMPUTATIONS WITH KS UNIFORMLY REGULAR CANONICAL ELEMENTS WITH OBLATENESS

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Abstract. This paper concerns with the study of KS uniformly regular canonical elements with Earth's oblateness. These elements, ten in number, are all constant in the unperturbed motion and even in the perturbed motion, the substitution is straightforward and elementary due to the transformation laws being explicit and closed expression. By utilizing the recursion formulas of Legendre's polynomials, we are able to include any number of Earth's zonal harmonics J_n in the package and also economize the computations. A fixed step-size fourth-order Runge–Kutta–Gill method is employed for numerical integration of the canonical equations.

Utilizing 5 test cases covering a large range of semimajor axis and eccentricity, we have carried out computations to study the effects of Earth's zonal harmonics (up to J_{36}) and integration step-size variation. Bilinear relations and energy equation are used for checking the accuracies of numerical integration. From the application point of view, the package is utilized to study the behaviour of 900 km height near-circular sun-synchronous satellite orbit over a longer duration of 220 days time (nearly 3078 revolutions) and the necessity of including more number of Earth's zonal harmonic terms is noticed. The package is also used to study the effect of higher zonal harmonics on three 900 km height near-circular orbits with inclinations of 60, 63.2, and 65 degrees, by including Earth's zonal harmonics up to J_{24}. The mean eccentricity (e_m) is found to have long-periods of 459.6, 6925.1 and 1077.6 days, respectively. Sharp changes in the variation of ω_m near the minima to e_m are noticed. The values of ω_m are found to be very near to ± 90 degrees at the extrema of e_m. The same orbit is employed to study the effect of variation of inclination from 0 to 180 degrees on long-period (T) of eccentricity with J_2 to J_{24} terms. T is found to increase rapidly as we proceed towards the critical inclinations.

1. Introduction

The problem of orbit computation of artificial satellites has wide applications in mission planning, satellite geodesy and spacecraft navigation, etc. The precision in the orbit computation has become necessary due to the availability of very accurate satellite tracking systems at present. Search is presently in progress for computation techniques which are faster and more accurate. Basically, three mathematical solution techniques, analytical, semianalytical and numerical are used for generating the ephemeris of a satellite. Though numerical integration methods are costly, they provide the most accurate ephemeris of a satellite with respect to any type of perturbing forces.

In the past, orbit computation has been done by integration of the equations in Cartesian coordinates (Cowell's method) or calculating perturbations to a set of Keplerian or equivalent elements. The elements of the orbit need not be the standard Keplerian elements, provided they are slowly varying and completely specify the state of the vehicle. The method of Kustaanheimo and Stiefel (1965), known as KS method, provides such a set of elements by introducing a new independent variable

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called 'fictitious time' and transforming to a four-dimensional coordinate system. In the new system, a set of equations (cf. Stiefel and Scheifele, 1971; p. 91), called KS perturbational equations, is obtained which has the form of a set of perturbed harmonic oscillators. The KS method provides one of the most stable, accurate techniques for orbit prediction presently available. In Sharma (1981), a detaile numerical study was carried out with KS perturbational equations with respect to a force model consisting of Earth's oblateness with Earth's zonal harmonic terms J_2 to J_6. In Sharma (1984) and Sharma and Mani (1985), a detailed numerical study was carried out with these equations by including perturbations due to atmospheric drag (analytical oblate atmospheric model) and oblateness (J_2 to J_6 terms) in the force model and an orbital decay study of the Indian satellite RS-1 was made. The studies clearly established that long-term orbit computations could be done accurately with larger integration step sizes with KS perturbational equations. Another form of KS differential equations (Stiefel and Scheifele, 1971; p 86) was utilized by Sharaf and Awad (1987) to develop an orbit computation package by including any number of Earth's zonal harmonic terms with the help of recursion formulas of Legendre polynomials. However, the package was used for orbit computation during small durations.

Since the canonical approach to a given mechanical system converts the system into a simpler form through transformations, our aim in this paper has been to make a detailed numerical study of a canonical form of the KS theory with respect to a complex force model. The derivation of canonical differential systems describing the perturbed motion, is by no means trivial, since, for instance, the adopted law of the time-transformation must be incorporated in the canonical set. The satisfaction of this requirement implies the knowledge of more refined instruments of general canonical theory as, for instance the enlargement of the phase space and the appropriate restrictions on the initial conditions.

For detailed numerical study, we have chosen the uniformly regular KS canonical elements (Stiefel and Scheifele, 1971; p 251), where all the elements α_j, β_j are constant in the Keplerian motion. We have developed an orbit computation package by including the effect of Earth's oblateness. The recursion formulas of Legendre polynomials are utilized to include up to any number of Earth's zonal harmonic terms J_n. However, the numerical computations are done with terms up to J_{36}.

Four test cases (A, B, C, D) with different perigee and apogee heights to cover large range of semimajor axis and eccentricity are chosen for numerical study. The integration of KS uniform canonical equations of motion is carried out with fixed step size, fourth-order Runge–Kutta–Gill method. Bilinear relations and energy equation are used for checking the accuracies of numerical integration. It is concluded from the study that a larger integration step size (say, 36 steps/rev.) can be utilized for moderate eccentricity cases for accurate orbit computations during long-term intergrations.

From the application point of view, the package is utilized to study the long-term behaviour of 900 km height near-circular sun-synchronous satellite orbit of PSLV mission. The mean orbital elements are generated for 220 days time (nearly 3078 revolutions) with Earth's zonal harmonic terms up to J_{24}. The long-periodic terms in eccentricity and inclination are found to have a period of 129.9 days. The extrema of mean eccentricity and inclination is found to occur very near to ± 90 degrees of mean argument of perigee. Long-term orbital computations are also made for the 900 km height orbit with inclinations nearer to critical inclination (63°26′). The higher zonal harmonics are found to have significant effect on the mean eccentricity. The long-periodic terms in eccentricity are found to have large period. Sharp changes in the variation of ω_m near the minima of e_m are noticed. The same orbit is used to study the effect of orbital inclination variation from 0 to 180 degrees on long-period of eccentricity with J_2 to J_{24} terms. This period increases rapidly as we proceed towards the critical inclinations. For example, the period increases from 459.6 to 6925.1 days as the inclination increases from 60 to 63.2 degrees. The growth will be much higher as we further proceed towards critical inclination, indicating the serious difficulties involved in solving the critical inclination problem with Earth's higher zonal harmonics.

2. KS Canonical Equations of Motion

The sixth-order system of differential equations

$$\ddot{\mathbf{x}} = -\left(\frac{K^2}{r^3}\right)\mathbf{x} - \frac{\partial V}{\partial \mathbf{x}},$$

describing the motion of a particle in the rectangular coordinates (x_1, x_2, x_3) under the perturbed time-independent potential V, can be written in the canonical form as

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial x_k}, \quad (k = 1, 2, 3)$$

with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{K^2}{r} + V,$$

where x_k and p_k are generalized coordinates and momenta, r is the distance of the particle from the central body, t is the time and K^2 is the gravitational constant.

Adding the negative total energy p_0 to the Hamiltonian H, to obtain homogeneous Hamiltonian and then applying the transformation $dt/ds = r$, we obtain the Hamiltonian

$$H_h = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)r + p_0 r + rV - K^2, \tag{1}$$

with the equations of motion

$$\frac{dx_j}{ds} = \frac{\partial H_h}{\partial p_j}, \quad \frac{dp_j}{ds} = -\frac{\partial H_h}{\partial x_j}, \quad (j = 0, 1, 2, 3) \tag{2}$$

where the function x_0 is equal to t.

Employing the canonical KS-transformation given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \Lambda(\bar{\mathbf{x}}) \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{pmatrix}, \; x_0 = \bar{x}_0,$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2(\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 + \bar{x}_4^2)} \Lambda(\bar{\mathbf{x}}) \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \\ \bar{p}_4 \end{pmatrix}, \; p_0 = \bar{p}_0,$$

where

$$\Lambda(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 & -\bar{x}_2 & -\bar{x}_3 & \bar{x}_4 \\ \bar{x}_2 & \bar{x}_1 & -\bar{x}_4 & -\bar{x}_3 \\ \bar{x}_3 & \bar{x}_4 & \bar{x}_1 & \bar{x}_2 \end{pmatrix},$$

to Equations (1) and (2), we obtain the new Hamiltonian as

$$\bar{H} = \frac{1}{8}|\bar{\mathbf{p}}|^2 + p_0|\bar{\mathbf{x}}|^2 + |\bar{\mathbf{x}}|^2 V - \frac{1}{8|\bar{\mathbf{x}}|^2} l^2(\bar{\mathbf{p}}, \bar{\mathbf{x}}) - K^2. \tag{3}$$

The bilinear quantity

$$l(\bar{\mathbf{p}}, \bar{\mathbf{x}}) = \bar{p}_1\bar{x}_4 - \bar{p}_2\bar{x}_3 + \bar{p}_3\bar{x}_2 - \bar{p}_4\bar{x}_1$$

is a first integral of the new canonical equations

$$\frac{\mathrm{d}\bar{x}_k}{\mathrm{d}s} = \frac{\partial \bar{H}}{\partial \bar{p}_k}, \quad \frac{\mathrm{d}\bar{p}_k}{\mathrm{d}s} = -\frac{\partial \bar{H}}{\partial \bar{x}_k}, \quad (k = 0, 1, 2, 3, 4)$$

i.e. $\mathrm{d}l/\mathrm{d}s = 0$.

Hence the Hamiltonian $\bar{H}$ in (3) reduces to

$$\hat{H} = \frac{1}{8}|\bar{\mathbf{p}}|^2 + p_0|\bar{\mathbf{x}}|^2 + |\bar{x}|^2 V - K^2. \tag{4}$$

The basic canonical system with respect to the fictitious time s is obtained by utilizing the canonical transformation

$$\hat{x}_0 = \frac{1}{2}\bar{x}_0, \quad \hat{x}_j = \bar{x}_j, \quad \hat{p}_0 = 2\bar{p}_0, \quad \hat{p}_j = p_j, \quad (j = 1, 2, 3, 4)$$

and the scaling factor $\frac{1}{4}$ to the Hamiltonian (4). The resulting Hamiltonian is

$$H = \frac{1}{2}|\mathbf{w}|^2 + \frac{1}{2}w_0|\mathbf{u}|^2 + \frac{1}{4}|\mathbf{u}|^2 V - \frac{K^2}{4}, \tag{5}$$

where

$$|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad |\mathbf{w}|^2 = \mathbf{w}_1^2 + \mathbf{w}_2^2 + \mathbf{w}_3^2 + \mathbf{w}_4^2,$$

and the new canonical variables are

$$u_k = \hat{x}_k, \quad w_k = \frac{1}{4}\hat{p}_k, \quad (k = 0, 1, 2, 3, 4).$$

The final transformation can be written as

$$x_0 = 2u_0 \ (=t),$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \Lambda(\mathbf{u}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

$$p_0 = 2w_0,$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{2}{|\mathbf{u}|^2} \Lambda(\mathbf{u}) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} .$$

On any solution, the value of p_0 is the negative physical energy and the value of H is zero.

The canonical equations

$$\frac{\mathrm{d}u_k}{\mathrm{d}s} = \frac{\partial H}{\partial w_k}, \quad \frac{\mathrm{d}w_k}{\mathrm{d}s} = -\frac{\partial H}{\partial u_k}, \quad (k = 0, 1, 2, 3, 4).$$

corresponding to the Hamiltonian (5), are the equations of a *perturbed harmonic oscillator*.

The separation of Jacobi's equation corresponding to the unperturbed Hamiltonian of (5) can be achieved through the canonical transformation

$$w_k = \frac{\partial S}{\partial u_k}, \quad \beta_k = \frac{\partial S}{\partial \alpha_k}, \quad (k = 0, 1, 2, 3, 4)$$

having the generating function

$$S = \sum_{k=1}^{4} \int \sqrt{2\alpha_k - \alpha_0 u_k^2}\, \mathrm{d}u_k + \alpha_0 u_0,$$

and the transformed unperturbed Hamiltonian becomes

$$H_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

The perturbed Hamiltonian is

$$H = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \\ + \frac{1}{2\alpha_0} \sum_{k=1}^{4} \alpha_k \sin^2 (\sqrt{\alpha_0}\beta_k) V - \frac{K^2}{4}. \tag{6}$$

The generating function

$$S = \frac{1}{\sqrt{w_0} \sin \sqrt{w_0 s}} \sum_{k=1}^{4} \left[\frac{1}{2} (w_k^2 + \beta_k^2) \cos \sqrt{w_0}\, s - w_k \beta_k \right] - w_0 \beta_0,$$

through the canonical transformation

$$u_j = -\frac{\partial S}{\partial w_j}, \quad \alpha_j = -\frac{\partial S}{\partial \beta_j}, \quad (j = 0, 1, 2, 3, 4)$$

transforms the Hamiltonian (6) to the form

$$\bar{H} = H + \frac{\partial S}{\partial s} = \frac{1}{4} \left[\sum_{k=1}^{4} u_k^2 (\alpha_j, \beta_j) \right] V(\alpha_j, \beta_j) - \frac{K^2}{4}, \tag{7}$$

and the canonical element equations are

$$\frac{\mathrm{d}\beta_j}{\mathrm{d}s} = \frac{\partial \bar{H}}{\partial \alpha_j}, \quad \frac{\mathrm{d}\alpha_j}{\mathrm{d}s} = -\frac{\partial \bar{H}}{\partial \beta_j}, \quad (j = 0, 1, 2, 3, 4). \tag{8}$$

From Equations (8), it follows that the ten elements α_j, β_j are constant in unperturbed motion. This is the *uniformly regular set of elements.*

3. Perturbations and Numerical Results

In the present paper we shall assume that the only forces acting on an artificial satellite are those due to the Earth's gravitational field with axial symmetry in which case, we have

$$V = \frac{K^2}{r} \sum_{n=2}^{\infty} J_n \left(\frac{R}{r} \right)^n P_n (\cos v), \cos v = \frac{x_3}{r}, \tag{9}$$

where R is the equatorial radius, r the distance of the particle from the central body. J_n's are dimensionless constants known as zonal harmonics and P_n are Legendre polynomials of degree n. The values of J_n are taken from Hough (1981).

With respect to V in (9), we have developed an orbit computation package through the uniformly regular KS canonical Equations (8). For the economic computation of V and $\partial V / \partial \mathbf{x}$ with respect to Legendre polynomial of any degree n, we have utilized the recurrence formulas of Legendre's polynomials

$$n P_n(x) = (2n - 1) x P_{n-1}(x) - (n - 1) P_{n-2}(x),$$

having starting values

$$P_0(x) = 1, \quad P_1(x) = x;$$

and

$$P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x),$$

with

$$P'_0(x) = 0.$$

For numerical integration of the Equations (8), we have employed a fixed step size fourth-order Runge–Kutta–Gill method. The computations are done with terms up to J_{36}. Detailed numerical computations are made for 4 test cases A, B, C, and D, whose initial conditions (position, velocity and osculating orbital parameters) are provided in Table I. Case A has a sufficiently large semimajor axis and eccentricity. Case B is highly eccentric with very large semimajor axis. Orbits corresponding to cases C and D are near-Earth orbits with small eccentricities. Orbital inclination in all the cases is 30 degrees.

Orbit computation has been done for the 4 cases A to D up to approximately 25, 10, 25, and 50 revolutions, respectively, with respect to Earth's zonal harmonic terms up to J_2, J_{12}, J_{24}, J_{36} and with integration step sizes of approximately 24, 48, 72, 96, 120 steps/revolution. However, for illustration purpose, we have provided in Table II, the value of the more perturbed parameter, the osculating semimajor axis, under the effect of Earth's oblateness. Table II also provides similar details for the 900 km height near-circular Sun-synchronous orbit (Case E) after 60 revolutions, whose initial orbital parameters are provided in Table a of Section 5. It may be noticed from the table that a larger integration step size between 24 and 48 steps/revolution is sufficient to provide accurate osculating semimajor axis for the cases A, C, D, and E, even after 25, 25, 50 and 60 revolutions, respectively. It has been noticed that the

TABLE I

Initial conditions (position, velocity and osculating orbital parameters)

	CASE			
Variables	A	B	C	D
x_0 (km)	0.0	0.0	0.0	0.0
y_0 (km)	−5888.9727	−5888.9727	−5888.9727	−5888.9727
z_0 (km)	−3400.0	−3400.0	−3400.0	−3400.0
$\dot{x}_0$ (km sec^{-1})	8.3	10.691338	7.8	7.6
$\dot{y}_0$ (km sec^{-1})	0.0	0.0	0.0	0.0
$\dot{z}_0$ (km sec^{-1})	0.0	0.0	0.0	0.0
a (km)	8244.826652	136000.418457	7067.946406	6701.926072
e	0.17524040	0.95000015	0.03791008	0.01463369
i (deg)	30.0	30.0	30.0	30.0
ω (deg)	270.0	270.0	270.0	90.0
Ω (deg)	0.0	0.0	0.0	0.0
M (deg)	0.0	0.0	0.0	180.0
$r_p = a(1-\mathrm{e})$ (km)	6800.0	6800.0	6800.0	6603.8
$r_a = a(1+\mathrm{e})$ (km)	9689.7	265200.8	7335.9	6800.0

TABLE II

Variation of osculating semi-major axis (km) with Earth's zonal harmonics

Case (after s = sec km^{-1})	Time (msd)	Revs. (approx.)	Zonal harmonics up to	Steps/rev.				
				24	48	72	96	120
A	2.153465	25	J_2	8244.8552	8244.8553	8244.8553	8244.8553	8244.8553
	2.153464		J_{12}	8244.8555	8244.8556	8244.8556	8244.8556	8244.8556
	2.153464		J_{24}	8244.8556	8244.8556	8244.8556	8244.8556	8244.8556
	2.153464		J_{36}	8244.8556	8244.8556	8244.8556	8244.8556	8244.8556
B	57.36093	10	J_2	135865.921	136000.975	136002.045	136002.090	136002.100
	57.35911		J_{12}	135839.249	136000.955	136002.419	136002.460	136002.469
	57.35914		J_{24}	135839.760	136000.959	136002.409	136002.452	136002.461
	57.35913		J_{36}	135839.500	136000.983	136002.408	136002.452	136002.464
C	1.7101483	25	J_2	7068.2793	7068.2794	7068.2794	7068.2794	7068.2794
	1.7101486		J_{12}	7068.2807	7068.2808	7068.2808	7068.2808	7068.2808
	1.7101486		J_{24}	7068.2808	7068.2808	7068.2808	7068.2808	7068.2808
	1.7101486		J_{36}	7068.2808	7068.2808	7068.2808	7068.2808	7068.2808
D	3.1582396	50	J_2	6703.3783	6703.3784	6703.3784	6703.3784	6703.3784
	3.1582416		J_{12}	6703.3841	6703.3842	6703.3842	6703.3842	6703.3842
	3.1582416		J_{24}	6703.3842	6703.3843	6703.3843	6703.3843	6703.3843
	3.1582416		J_{36}	6703.3841	6703.3842	6703.3842	6703.3842	6703.3842
E	4.2950012	60	J_2	7283.4876	7283.4889	7283.4889	7283.4889	7283.4889
	4.2949951		J_{12}	7283.4856	7283.4869	7283.4869	7283.4869	7283.4869
	4.2949951		J_{24}	7283.4859	7283.4869	7283.4870	7283.4870	7283.4870
	4.2949951		J_{36}	7283.4849	7283.4869	7283.4870	7283.4870	7283.4870

other orbital parameters are also accurate under similar integrations. However for the high eccentricity case B, as can be noticed from the Table II, a smaller integration step size (96 steps/rev., say) is necessary for accurate computation of semimajor axis after 10 revolutions, including the cases with higher zonal harmonic terms. Though the computational details presented in Sharma and Raj (1986) are not included here, our study points out that the inclusion of higher degree zonal harmonic terms becomes necessary for low-eccentricity cases.

4. Checks During Numerical Integration

The bilinear relations

$$\alpha_4\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 - \alpha_1\beta_4 = 0 \tag{10}$$

and

$$u_4w_1 - u_3w_2 + u_2w_3 - u_1w_4 = 0, \tag{11}$$

satisfied by the canonical variables α_i,β_i and $u_i,w_i(i = 1, 2, 3, 4)$ are used as checks for numerical integration accuracies of the Equations (8) with respect to the force model given by (9). In our computations, the value obtained from the L.H.S. of Equation (10) turns out to be the negative of the value obtained from the L.H.S. of Equation (11). In Table III, we have provided the values obtained from the relation (11) for the four cases A, B, C, D with respect to the zonal harmonic terms J_2 to J_{36}. The table also provides the difference between the initial energy and the energy at the instant of computation from the energy equation

$$h_K = \frac{K^2}{r} - \frac{1}{2}|\dot{\mathbf{x}}|^2 - V. \tag{12}$$

These values also serve as a good test for the accuracies of numerical integration. Though the values from the bilinear relation (11) and energy Equation (12) reported in Table III are only with respect to J_2 to J_{36} terms, we have observed (Sharma and Raj, 1986) that these values change very little with respect to an integration step size when the zonal harmonic terms up to J_2, J_{12} or J_{24} are included in the force model. This clearly indicates that the canonical Equations (8) could be used effectively for numerical integration with respect to complex force models. It may be pointed out that the bilinear relation and energy equation values do not remain constant during a revolution. For illustration purpose, we have depicted in Figure 1 their variations during a revolution for the case A with respect to the force model containing the zonal harmonic terms up to J_{36}. Further it has been noticed from the numerical study of the 4 cases that the parameters $\alpha_1, \ldots, \alpha_4$, $\beta_0, \ldots, \beta_4$ have more uniform variations with relatively less amplitudes than the corresponding orbital parameters a, e, i, Ω, ω and M during a revolution and provide better accuracies during numerical integration.

TABLE III

Bilinear relation and energy equation values with J_2 to J_{36} terms

Steps/rev. (approx.)	Bilinear Relation $x\ 10^{-8}$				Energy Equation $x\ 10^{-10}$			
	CASE							
	A	B	C	D	A	B	C	D
	After revolutions							
	25	10	25	50	25	10	25	50
24	10.12×10^2	19.5×10^4	285.8	−245.7	−95.89	-11.9×10^8	−71.3	−138.3
48	35.03	16.7×10^3	8.49	−7.11	−3.8	-10.9×10^4	−2.43	−4.53
72	8.08	26.27×10^2	−0.17	0.18	−1.32	-45.5×10^2	−0.97	−1.79
96	5.85	671.3	−1.35	1.39	−1.31	-12.8×10^2	−1.07	−2.09
120	6.42	227.2	−2.0	1.54	−1.55	−382.8	−1.31	−2.54

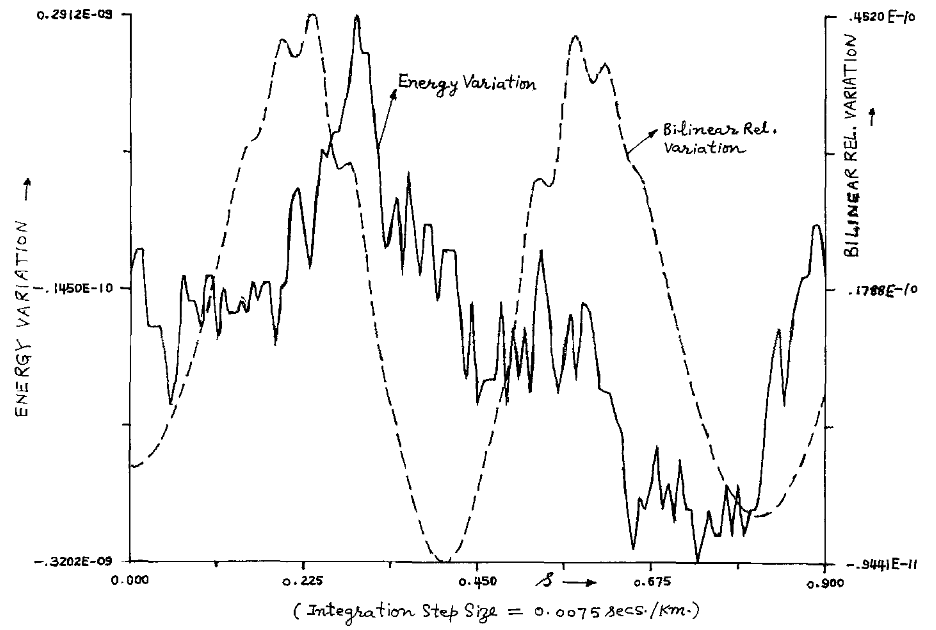

Fig. 1. Variation in energy and bilinear relation with $J_2, \ldots, J_{36}$ terms during one revolution (Case A).

5. Sun-Synchronous Orbit

From the application point of view, we have utilized the developed orbit computation package to generate mean orbital elements of 900 km near-circular Sun-synchronous orbit for 220 days time (nearly 3078 revolutions). Its initial osculating orbital elements chosen for the study along with mean elements are given in Table a:

TABLE a

(Case E)

Parameter	Osculating	Mean
a (km)	7282.7	7277.6969
e	0.00063	0.000717
i (deg)	99.033	99.091146
Ω (deg)	290.033	290.0376
ω (deg)	207.844	180.014
M (deg)	0.0	27.744

The conversion of the osculating orbital elements to mean orbital elements is done through Chebotarev's (1964) first-order short-periodic variations due to J_2. As can be seen from Table II, a metre level accuracy can be obtained in the osculating semi-major axis computation with 36 steps/rev. after 60 revolutions. The mean orbital

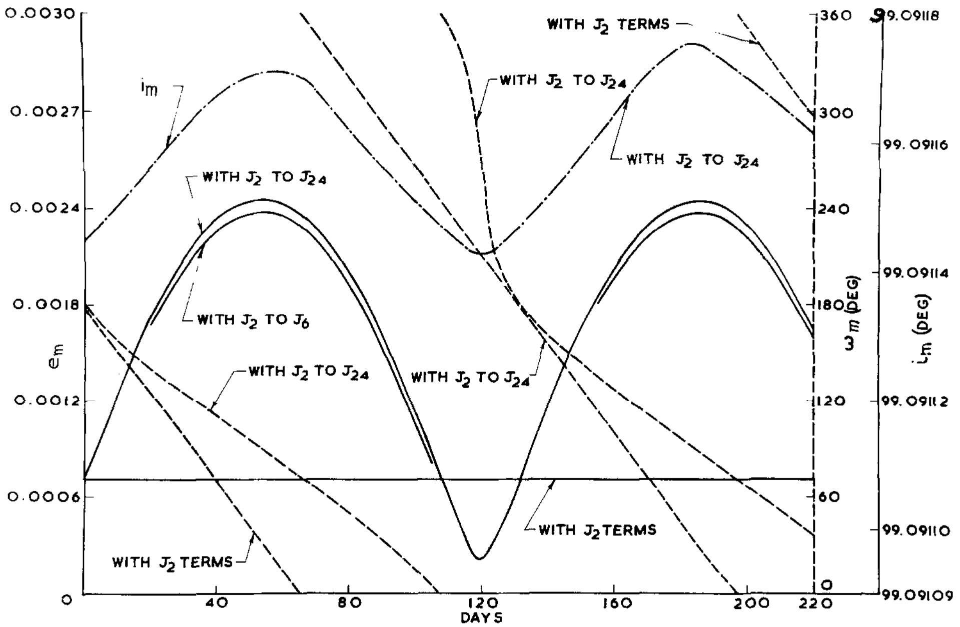

Fig. 2. Variation of mean eccentricity (e_m), argument of perigee (ω_m) and inclination (i_m).

elements for this case are generated with J_2 to J_{24} terms up to 220 days time (nearly 3078 revolutions). It is noted that the mean semimajor axis (a_m) remains nearly constant, while the mean right ascension of ascending node (Ω_m) varies almost linearly during the 220 days time. The variation of mean eccentricity (e_m), argument of perigee (ω_m) and inclination (i_m) are depicted in Figure 2 up to 220 days time. It can be easily noticed from the Figure 2 that the eccentricity and the inclination have long-periodic terms of period 129.9 days and occur almost at the same time. A slight increase in the peak values of i_m is also noted. The extrema of these variations occur when the argument of perigee is near to ± 90 degrees. The argument of perigee varies rapidly near the minimum of e_m and i_m. Variation of e_m and ω_m with terms up to J_2 and J_6 is also shown in the figure to show the effect of higher zonal harmonic terms. It can be easily noticed that the terms J_3 to J_{24} have very significant effect on e_m, ω_m, and i_m. Effect of J_7 to J_{24} is also noticed on e_m. The figure also depicts that J_2 has no long-periodic effect on e_m.

6. Near-Circular Near Critical Inclination Orbits

To show that the higher zonal harmonics have very significant effect near a critical inclination (63°26′) orbit, we have generated the mean orbital elements for the three cases with $i = 60$, 63.2 and 65 degrees. The other initial osculating orbital elements are same as for case E. Variation of e_m and ω_m for $i = 60$ degrees with terms up to J_{24}

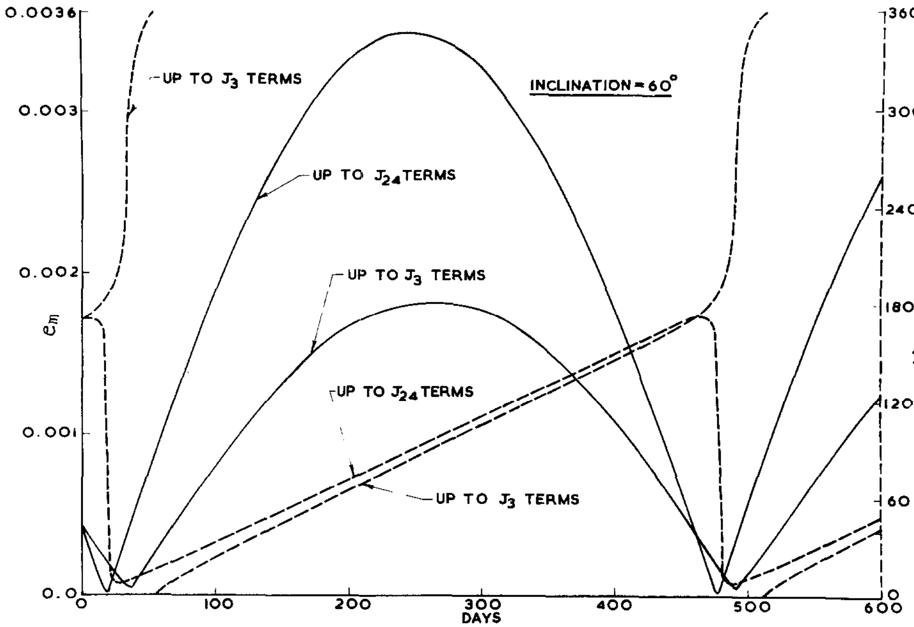

Fig. 3. Variation of mean eccentricity (e_m) and argument of perigee (ω_m).

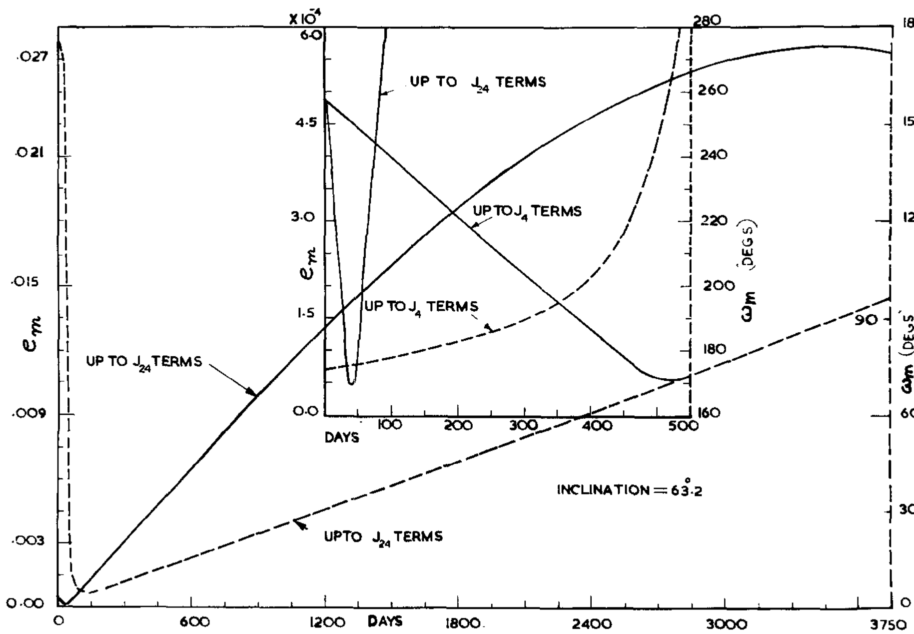

Fig. 4. Variation of mean eccentricity (e_m) and argument of perigee (ω_m).

for 600 days time is shown in Figure 3. Variations in these parameters with J_2, J_3 terms is also shown in the figure. It can be easily noted that the variation of e_m with up to J_{24} terms is much higher than with J_2, J_3 terms and the extrema occur at different times when their respective ω_m are very near to ± 90 degrees. Figure 4 depicts the variation of e_m and ω_m for 3750 days time for $i = 63.2$ degrees case with respect to J_2 to J_{24} force model. In order to show that the higher zonal harmonics have significant effect, we have shown in the figure, the variation of e_m and ω_m with J_2 to J_4 terms for 500 days time. A vast difference is noticed between the timings of minima occurrence of e_m for the two cases and at the extrema of e_m, ω_m are very near to ± 90 degrees. The variation of e_m and ω_m for $i = 65$ degrees case with J_2 to J_{24} terms for 650 days time is plotted in the Figure 5. The figure also contains the variation of these parameters with J_2 to J_4 terms. As can be seen, the variation of e_m and ω_m is quite different in the two cases, showing the significant effect of the higher zonal harmonic terms. Also, ω_m is found to be very near to ± 90 degrees at the extrema of e_m. Though we are using the words 'very near to 90 degrees' due to the numerical nature of our studies, however from the large number of computations we have done with different inclinations, we strongly feel that at the extrema of e_m, ω_m becomes ± 90 degrees. As can be seen from Figures 3 to 5, the long periodic terms in eccentricity have quite large periods (459.6, 6925.1 and 1077.6 days for 60, 63.2, and 65 degrees inclinations, respectively). As we approach the critical inclination (63°26′), this period increases rapidly showing the difficulties involved in the critical inclination problem when Earth's higher zonal harmonics are considered.

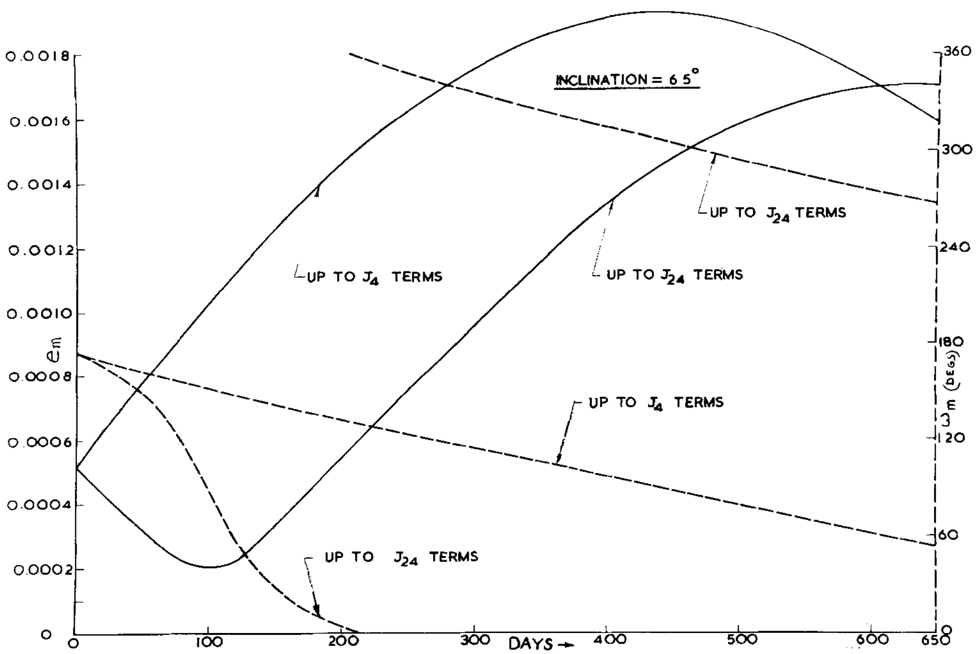

Fig. 5. Variation of mean eccentricity (e_m) and argument of perigee (ω_m).

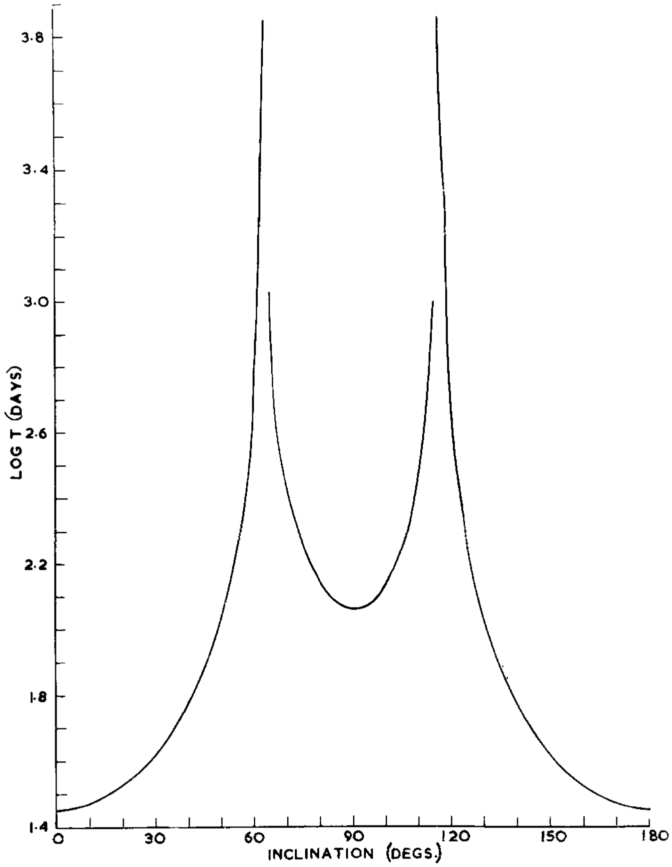

Fig. 6. Variation of long-period (T) of mean eccentricity with inclination with J_2 to J_{24} terms.

Computations are done to study the effect of orbital inclination on long-period (T) of eccentricity of case E with the force model consisting of Earth's zonal harmonics up to J_{24}. Figure 6 provides the variation of log T when the inclination varies from 0 to 180 degrees. T increases rapidly as we proceed towards the critical inclinations (63°26′, 116°34′) from the right of 0 degree and left of 180 degrees. Though we evaluated T at 63.2 degrees inclination, it increases sharply as we proceed towards 63°26′. Again, from 90 degrees, as we proceed towards the critical inclinations, T increases rapidly. It may be noticed that T at 90 degrees inclination is much higher than at 0 or 180 degrees inclination.

7. Computational Time

All the numerical computations in the paper have been done in single precision arithmetic on CDC CYBER 170/730 computed at VSSC. A comparison for orbit computation for 1000 revolutions for the case E was made between single and double

TABLE IV

Computational time (CP secs)

Case	Step size (sec km^{-1})	Zonal harmonic terms up to			
		J_2	J_{12}	J_{24}	J_{36}
A	0.01875	0.858	1.585	2.440	3.330
B	0.076280808005	0.892	1.617	2.490	3.374
C	0.01743	0.842	1.569	2.456	3.301
D	0.016974	0.877	1.605	2.488	3.343
E	0.01767	0.898	1.620	2.472	3.335

Integration step size = 48 steps/rev. approximately.

precision arithmetic computations. It was found that the single precision arithmetic computations are quite acceptable from the practical point of view, even for integrations over long durations. Computational time comparison for single precision arithmetic has been done for the 5 cases A to E through Table IV, which provides the computational time in CP seconds for one revolution with integration step size of approx. 48 steps/revolution with respect to zonal harmonics up to J_2, J_{12}, J_{24} and J_{36}. As can be noticed from the table, the computational time is approximately 0.9, 1.6, 2.5 and 3.4 CP seconds in all the five cases, which shows that we have to spend about four times the computer time when the additional zonal harmonic terms J_3 to J_{36} are included in the force model.

8. Conclusions

KS uniform regular canonical equations with Earth's oblateness perturbations provide a very efficient and accurate integration method for orbit computation even during long durations. Usage of the Legendre polynomials recurrence formulas to compute Earth's potential and its partial derivatives, economizes the computational time. Inclusion of a larger number of zonal harmonic terms in the Earth's potential becomes a necessity for near-circular satellite orbits. Near the critical inclinations, the effect of oblateness is very prominent on some of the orbital parameters of near-circular orbits and the long-periodic terms have very large period.

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