APPLICATION OF HORI TECHNIQUE IN GENERAL PLANETARY THEORY

(Part I: First Step)

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(Received 18 January, 1989)

Abstract. We explain how the first step of Hori–Lie procedure is applied in general planetary theory to eliminate short-period terms. We extend the investigation to the third-order planetary theory. We solved the canonical equations of motion for secular and periodic perturbations by this method, and obtained the first integrals of the system of canonical equations. Also we showed the relation between the determining function in the sense of Hori and the determining function in the sense of Von Zeipel.

1. Introduction

The construction of an artificial satellite theory, lunar theory or planetary theory, by the method of Von Zeipel has the advantage of eliminating all the short-periodic terms and all long periodic terms by two determining functions. The Von Zeipel method has the drawback of being based on the inversion formula of Lagrange for functions of several independent variables, in order to express the initial canonical variables as functions of the variables that result from the elimination of short-periodic terms. Also the Von Zeipel method suffers from an inconvenience of leading to the formulae that are not, in general, invariant in a change of canonical variables. These two inconveniences arise from the fact that the determining function that defines the change of canonical variables contains mixed variables – i.e. the ancient angular variables and the new linear variables. Hori defined a change of canonical variables with the aid of a Lie series which introduces a determining function depending on the new linear and angular canonical variables. Thus he avoided these two difficulties. Hori expresses the ancient canonical variables as functions of the new and reciprocally, without returning to the formula of inversion of Lagrange and he obtained expressions that are invariant in all changes of canonical variables. (Hori, 1966; Deprit, 1969; Yusasa, 1971; Cid and Calvo, 1973; Campbell and Jefferys, 1970; Cid *et al.*, 1975; Rapaport, 1974).

Earth, Moon, and Planets **44**: 275–289, 1989.

2. Change of Canonical Variables Defined by a Lie Series

Let us consider the system of $2n$ *canonical equations*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=\frac{\partial F}{\partial y_i}, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t}=-\frac{\partial F}{\partial x_i} \quad i=1,2,\ldots n \tag{1}$$

with $F(x_1,\ldots,x_n,y_1,\ldots,y_n)$ and apply the change of variables.

$$x_1,\ldots,x_n,y_1,\ldots,y_n\to\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n\ ;$$

defined by the equality

$$f(x_1,\ldots,x_n,y_1,\ldots,y_n)=\sum_{v=0}^{\infty}\frac{1}{v!}D_s^v f(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n) \tag{2}$$

where f is any function of $x_1,\ldots,x_n,y_1,\ldots,y_n$ the operator D_s^v is defined by the following

$$D_s^0 f=f,$$
$$D_s^1 f=(f,S)$$
$$\vdots$$
$$D_s^v f=\underbrace{((f,S),\ldots,S)}_{(v\text{ times})}$$

while

$$S(\xi_1,\ldots\xi_n,\eta_1,\ldots,\eta_n)$$

is by definition the determining function in the sense of Hori and (f,S) the Poisson bracket defined by

$$(f,S)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial\xi_i}\frac{\partial S}{\partial\eta_i}-\frac{\partial S}{\partial\xi_i}\frac{\partial f}{\partial\eta_i}\right). \tag{3}$$

Equation (2) shows that the function f is developable in a Lie series and we suppose that the Lie series is convergent.

Let us define a parameter τ by the auxiliary system of $2n$ canonical equations

$$\frac{\partial\xi_i}{\mathrm{d}\tau}=\frac{\partial S}{\partial\eta_i}, \qquad \frac{\mathrm{d}\eta_i}{\mathrm{d}\tau}=-\frac{\partial S}{\partial\xi_i}, \quad i=1,2,\ldots,n\ ;$$

of which the Hamiltonian is the determining function S, we have, according to (3),

$$D_s^1 f=(f,S)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial\xi_i}\frac{\mathrm{d}\xi_i}{\mathrm{d}\tau}+\frac{\partial f}{\partial\eta_i}\frac{\mathrm{d}\eta_i}{\mathrm{d}\tau}\right)=\frac{\mathrm{d}f}{\mathrm{d}\tau}\ ;$$

from which

$$D_s^2 f = D_s^1(D_s^1 f) = D_s^1 \frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2};$$

and, in general,

$$D_s^v f = \frac{\mathrm{d}^v f}{\mathrm{d}\tau^v}; \quad v = 0, 1, 2, \ldots .$$

If so, we can write Equation (2) as

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{v=0}^{\infty} \frac{1}{v!}\frac{\mathrm{d}^v f}{\mathrm{d}\tau^v}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$$

$$= f(\xi_1(\tau+1), \ldots, \eta_n(\tau+1));$$

from which, supposing that f is a monotone function of its arguments,

$$x_1 = \xi_1(\tau+1), \ldots, y_n = \eta_n(\tau+1). \tag{4}$$

From Equation (4) we see that, since $x_1, \ldots, x_n$; $y_1, \ldots, y_n$ are $2n$ canonical variables, also the same is true for $\xi_1(\tau), \ldots, \eta_n(\tau)$.

Let, moreover,

$$F'(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$$

be the transformed Hamiltonian of $F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in the change of canonical variables. Since F does not depend explicitly on the time t, we have

$$F(x_1, \ldots, x_n, y_1, \ldots, y_n) = F'(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n). \tag{5}$$

Suppose that we have

$$F = \sum_{k=0}^{\infty} F_k, \qquad S = \sum_{k=1}^{\infty} S_k,$$

$$F' = \sum_{k=0}^{\infty} F'_k, \tag{6}$$

$$F_{k'} F'_k; \quad k = 0, 1, 2, \ldots \quad \text{and} \quad S_k; \quad k = 1, 2, \ldots .$$

are of degree k with respect to a small parameter ϵ which in the planetary theory is of the order of planetary masses.

Suppose also that the Hamiltonian $F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ could be developed in a Lie series of the form (2).

The equality (5) could be written according to Equations (2), (6) as

$$\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{v!} D_s^v F_k(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) =$$

$$= \sum_{k=0}^{\infty} F'_k(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n). \tag{7}$$

From (7) we extract the following equalities:

$$F_0 = F'_0 ,$$

$$F_1 + (F_0, S_1) = F'_1 ,$$

$$F_2 + (F_0, S_2) + (F_1, S_1) + \tfrac{1}{2}((F_0, S_1), S_1) = F'_2 ,$$

$$\begin{aligned} &F_3 + (F_0, S_3) + (F_1, S_2) + (F_2, S_1) + \tfrac{1}{2}((F_0, S_1), S_2) + \\ &\quad + \tfrac{1}{2}((F_0, S_2), S_1) + \tfrac{1}{2}((F_1, S_1), S_1) + \\ &\quad + \tfrac{1}{6}(((F_0, S_1), S_1), S_1) = F'_3 . \end{aligned} \tag{8}$$

3. Introduction of Pseudo-time t^* and Calculation of New Hamiltonian F' and of Hori's Determining Functions

Let us define a pseudo-time t^* by the auxiliary system of $2n$ canonical equations of the form

$$\frac{d\xi_i}{dt^*} = \frac{\partial F_0}{\partial \eta_i}, \qquad \frac{d\eta_i}{dt^*} = -\frac{\partial F_0}{\partial \xi_i}, \quad i = 1, 2, \ldots, n ; \tag{9}$$

then we have

$$\begin{aligned} (F_0, S_k) &= \sum_{i=1}^{n} \left(\frac{\partial F_0}{\partial \xi_i} \frac{\partial S_k}{\partial \eta_i} - \frac{\partial S_k}{\partial \xi_i} \frac{\partial F_0}{\partial \eta_i} \right) = \\ &= \sum_{i=1}^{n} \left(-\frac{d\eta_i}{dt^*} \frac{\partial S_k}{\partial \eta_i} - \frac{d\xi_i}{dt^*} \frac{\partial S_k}{\partial \xi_i} \right) = -\frac{dS_k}{dt^*} \end{aligned} \tag{10}$$

with $k = 1, 2, \ldots$.

Let $A(t^*)$ be any function of t^*, such that the expression

$$\frac{1}{T} \int_0^T A(t^*)\, dt^*$$

tends to a finite limit when T tends to $+\infty$. Let us put

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T A(t^*)\, dt^* = A_s(t^*) ;$$

by definition $A_s(t^*)$ is the mean value of $A(t^*)$ when t^* varies from zero to $+\infty$.

Put $A_p(t^*) = A(t^*) - A_s(t^*)$, from which $A(t^*) = A_s(t^*) + A_p(t^*)$. If, in particular, $A(t^*)$ is a periodic function of t^* and of period T_w, we have

$$\frac{1}{T_w} \int_0^{T_w} A(t^*)\, dt^* = A_s(t^*) .$$

Let $\xi_1(t^*), \ldots, \eta_n(t^*)$ is a solution of (9). The Hamiltonians $F_1, F_2, \ldots$ are functions of t^* by the intermediate $\xi_1, \ldots, \eta_n$, and suppose that the mean values of $F_1(t^*)$, $F_2(t^*), \ldots$ exist.

The second equality of (8) could be written, according to (10) and above definitions, as

$$F_{1s} + F_{1p} - \frac{dS_1}{dt^*} = F'_1 . \tag{11}$$

Let us put

$$F_{1p} - \frac{dS_1}{dt^*} = 0,$$

from which

$$S_1 = \int F_{1p} \, dt^* . \tag{12}$$

According to (12), Equation (11) could be rewritten as

$$F_{1s} = F'_1 . \tag{13}$$

The equalities (12) and (13) give, respectively, the values of the determining function and the new Hamiltonian in a first-order theory. The third equality of (8) written according to (10) and according to the second equality of (8).

$$F_2 - \frac{dS_2}{dt^*} + (F_1, S_1) + \tfrac{1}{2}(F'_1 - F_1, S_1) = F'_2 .$$

This means that

$$F_2 - \frac{dS_2}{dt^*} + \tfrac{1}{2}(F'_1 + F_1, S_1) = F'_2 ,$$

from which according to (13)

$$F_{2s} + F_{2p} - \frac{dS_2}{dt^*} + \tfrac{1}{2}(2F_{1s} + F_{1p}, S_1)_s + \tfrac{1}{2}(2F_{1s} + F_{1p}, S_1)_p = F'_2 . \tag{14}$$

Put

$$F_{2p} - \frac{dS_2}{dt^*} + \tfrac{1}{2}(2F_{1s} + F_{1p}, S_1)_p = 0 ;$$

from which

$$S_2 = \int (F_{2p} + \tfrac{1}{2}(2F_{1s} + F_{1p}, S_1)_p) \, dt^* . \tag{15}$$

According to (15) we can write Equation (14) as

$$F_{2s}+\tfrac{1}{2}(2F_{1s}+F_{1p},S_1)_s=F'_2\,. \tag{16}$$

Equation (15) in which S_1 is replaced by its value expressed by (12) gives the determining function S_2; while Equation (16) gives the new Hamiltonian F'_2.

If, in particular, F is reduced to F_0+F_1, Equation (15) could be written as

$$S_2=\tfrac{1}{2}\int(2F_{1s}+F_{1p},S_1)_p\,\mathrm{d}t^*\,;$$

and (16), as

$$F'_2=\tfrac{1}{2}(2F_{1s}+F_{1p},S_1)_s\,.$$

The fourth of Equations (8) may be written according to (10) and the second and third equation of (8) as

$$\begin{aligned}F_3-\frac{\mathrm{d}S_3}{\mathrm{d}t^*}&+(F_1,S_2)+(F_2,S_1)+\tfrac{1}{2}(F'_1-F_1,S_2)+\tfrac{1}{2}(F'_2-F_2,S_1)-\\&-\tfrac{1}{2}((F_1,S_1),S_1)-\tfrac{1}{4}((F'_1-F_1,S_1),S_1)+\tfrac{1}{2}((F_1,S_1),S_1)\\&+\tfrac{1}{6}((F'_1-F_1,S_1),S_1)=F'_3\,.\end{aligned}$$

Let

$$\begin{aligned}F_3-\frac{\mathrm{d}S_3}{\mathrm{d}t^*}&+\tfrac{1}{2}(F'_1+F_1,S_2)+\tfrac{1}{2}(F'_2+F_2,S_1)-\\&-\tfrac{1}{12}((F'_1-F_1,S_1),S_1)=F'_3\,,\end{aligned}$$

form which and according to (13) and (16)

$$\begin{aligned}F_{3s}+F_{3p}-\frac{\mathrm{d}S_3}{\mathrm{d}t^*}&+\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_s+\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_p+\\&+\tfrac{1}{2}(2F_{2s}+F_{2p},S_1)_s+\tfrac{1}{2}(2F_{2s}+F_{2p},S_1)_p+\\&+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_s+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_p+\\&+\tfrac{1}{12}((F_{1p},S_1),S_1)_s+\tfrac{1}{12}((F_{1p},S_1),S_1)_p=F'_3\,.\end{aligned} \tag{17}$$

Put

$$\begin{aligned}F_{3p}-\frac{\mathrm{d}S_3}{\mathrm{d}t^*}&+\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_p+\tfrac{1}{2}(2F_{2s}+F_{1p},S_1)_p+\\&+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_p+\tfrac{1}{12}((F_{1p},S_1),S_1)_p=0\,,\end{aligned}$$

from which

$$\begin{aligned}S_3=\int[F_{3p}&+\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_p+\tfrac{1}{2}(2F_{2s}+F_{2p},S_1)_p+\\&+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_p+\tfrac{1}{12}((F_{1p},S_1),S_1)_p]\,\mathrm{d}t^*\,.\end{aligned} \tag{18}$$

According to (18), Equation (17) may be written as

$$F_{3s}+\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_s+\tfrac{1}{2}(2F_{2s}+F_{2p},S_1)_s+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_s+ \\ +\tfrac{1}{12}((F_{1p},S_1),S_1)_s=F'_3\,; \tag{19}$$

while Equation (18) in which S_1 is replaced by its value (12) and S_2 by its value (15) gives the determining function S_3. The equality (19) gives the new Hamiltonian F'_3. If, in particular, F is reduced to F_0+F_1, Equation (18) may be rewritten

$$S_3=\int[\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_p+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_p+ \\ +\tfrac{1}{12}((F_{1p},S_1),S_1)_p]\,\mathrm{d}t^*\,,$$

and (19) could be written as

$$F'_3=\tfrac{1}{2}(2F_{1s}+F_{1p},S_2)_s+\tfrac{1}{4}((2F_{1s}+F_{1p},S_1)_s,S_1)_s+ \\ +\tfrac{1}{12}((F_{1p},S_1),S_1)_s\,.$$

Thus we determine step-by-step for each value $k=1,2,\ldots$ the determining function S_k and the new Hamiltonian F'_k. The formulae become longer and longer but the calculation is tolerable for small powers of the eccentricities and inclinations. The difficulties in the planetary theory arise from the development of the perturbing function.

4. Expression for the Initial Canonical Variables $x_1,\ldots,x_n,y_1,\ldots,y_n$ as Functions of the New $\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n$ and Reciprocal Variables ξ_n and η_n

We obtain that by writing successively

$$f(x_1,\ldots,x_n,y_1,\ldots,y_n)\equiv x_i$$

and

$$f(x_1,\ldots,x_n,y_1,\ldots,y_n)\equiv y_i\,.$$

Hence, according to (2) we have

$$x_i=\sum_{V=0}^{\infty}\frac{1}{V!}D_s^V\xi_i=\xi_i+\sum_{V=1}^{\infty}\frac{1}{V!}D_s^{V-1}D_s^1\xi_i\,, \tag{20}$$

$$y_i=\sum_{V=0}^{\infty}\frac{1}{V!}D_s^V\eta_i=\eta_i+\sum_{V=1}^{\infty}\frac{1}{V!}D_s^{V-1}D_s^1\eta_i\,, \tag{21}$$

but

$$D_s^1\xi_i=(\xi_i,S)=\frac{\partial S}{\partial\eta_i}\,,$$

$$D_s^1\eta_i=(\eta_i,S)=-\frac{\partial S}{\partial\xi_i}\,. \tag{22}$$

According to (22) Equations (20) and (21) may be rewritten as

$$x_i = \xi_i + \frac{\partial S}{\partial \eta_i} + \frac{1}{2}\left(\frac{\partial S}{\partial \eta_i}, S\right) + \frac{1}{6}\left(\left(\frac{\partial S}{\partial \eta_i}, S\right), S\right) + \cdots$$

$$y_i = \eta_i - \frac{\partial S}{\partial \xi_i} - \frac{1}{2}\left(\frac{\partial S}{\partial \xi_i}, S\right) - \frac{1}{6}\left(\left(\frac{\partial S}{\partial \xi_i}, S\right), S\right) - \cdots$$

from which if we neglect the powers of ϵ higher than the third,

$$x_i = \xi_i + \frac{\partial S_1}{\partial \eta_i} + \frac{\partial S_2}{\partial \eta_i} + \frac{\partial S_3}{\partial \eta_i} + \frac{1}{2}\left(\frac{\partial S_1}{\partial \eta_i}, S_1\right) + \frac{1}{2}\left(\frac{\partial S_1}{\partial \eta_i}, S_2\right) + $$
$$+ \frac{1}{2}\left(\frac{\partial S_2}{\partial \eta_i}, S_1\right) + \frac{1}{6}\left(\left(\frac{\partial S_1}{\partial \eta_i}, S_1\right), S_1\right) + O(\epsilon^4), \tag{23}$$

$$y_i = \eta_i - \frac{\partial S_1}{\partial \xi_i} - \frac{\partial S_2}{\partial \xi_i} - \frac{\partial S_3}{\partial \xi_i} - \frac{1}{2}\left(\frac{\partial S_1}{\partial \xi_i}, S_1\right) - \frac{1}{2}\left(\frac{\partial S_1}{\partial \xi_i}, S_2\right) - $$
$$- \frac{1}{2}\left(\frac{\partial S_2}{\partial \xi_i}, S_1\right) - \frac{1}{6}\left(\left(\frac{\partial S_1}{\partial \xi_i}, S_1\right), S_1\right) + O(\epsilon^4), \tag{24}$$

with $i = 1, 2, \ldots, n$.

In (23) and (24), S_1, S_2, S_3 are replaced by their values given by (12), (15), (18).

Let us consider again the equality (2), it could be written as

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \exp D_s f(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n), \tag{25}$$

while the operator $\exp D_s$ is defined in terms of D_s by

$$\exp D_s = \sum_{V=0}^{\infty} \frac{1}{V!} D_s^V .$$

From (25) we deduce that

$$f(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) = (\exp D_s)^{-1} f(x_1, \ldots, x_n, y_1, \ldots, y_n) .$$

But

$$(\exp D_s)^{-1} = \exp D_{-s}$$

we have then

$$f(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) = \exp D_{-s} f(x_1, \ldots, x_n, y_1, \ldots, y_n)$$
$$= \sum_{V=0}^{\infty} \frac{1}{V!} D_{-s}^V f(x_1, \ldots, x_n, y_1, \ldots, y_n),$$

from which, in particular,

$$\xi_i = \sum_{V=0}^{\infty} \frac{1}{V!} D_{-s}^V x_i, \qquad \eta_i = \sum_{V=0}^{\infty} \frac{1}{V!} D_{-s}^V y_i ,$$

i.e.,

$$\xi_i = x_i - \frac{\partial S}{\partial y_i} + \frac{1}{2}\left(\frac{\partial S}{\partial y_i}, S\right) - \frac{1}{6}\left(\left(\frac{\partial S}{\partial y_i}, S\right), S\right) + \cdots,$$

$$\eta_i = y_i + \frac{\partial S}{\partial x_i} - \frac{1}{2}\left(\frac{\partial S}{\partial x_i}, S\right) + \frac{1}{6}\left(\left(\frac{\partial S}{\partial x_i}, S\right), S\right) - \cdots.$$

From which – by neglecting powers of ϵ higher than the third – it follows that

$$\xi_i = x_i - \frac{\partial S_1}{\partial y_i} - \frac{\partial S_2}{\partial y_i} - \frac{\partial S_3}{\partial y_i} + \frac{1}{2}\left(\frac{\partial S_1}{\partial y_i}, S_1\right) + \frac{1}{2}\left(\frac{\partial S_1}{\partial y_i}, S_2\right) + $$
$$+ \frac{1}{2}\left(\frac{\partial S_2}{\partial y_i}, S_1\right) - \frac{1}{6}\left(\left(\frac{\partial S_1}{\partial y_i}, S_1\right), S_1\right) + O(\epsilon^4), \tag{26}$$

$$\eta_i = y_i + \frac{\partial S_1}{\partial x_i} + \frac{\partial S_2}{\partial x_i} + \frac{\partial S_3}{\partial x_i} - \frac{1}{2}\left(\frac{\partial S_1}{\partial x_i}, S_1\right) - \frac{1}{2}\left(\frac{\partial S_1}{\partial x_i}, S_2\right) - $$
$$- \frac{1}{2}\left(\frac{\partial S_2}{\partial x_i}, S_1\right) + \frac{1}{6}\left(\left(\frac{\partial S_1}{\partial x_i}, S_1\right), S_1\right) + O(\epsilon^4). \tag{27}$$

From Equations (26), (27) we find that the new canonical variables ξ_i, η_i are inversely expressed as functions of the old canonical variables x_i, y_i on changing in (23), (24) S by $-S$ and on permuting x_i and ξ_i, y_i, and η_i.

5. First Integrals of the System of $2n$ Canonical Equations of the Variables $\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n$

Equation (10) holds good for all functions $U(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$. It is particularly true for the Hamiltonian F' of the system of $2n$ canonical equations.

$$\frac{d\xi_i}{dt} = \frac{\partial F'}{\partial \eta_i}, \qquad \frac{d\eta_i}{dt} = -\frac{\partial F'}{\partial \xi_i}. \tag{28}$$

If we transform Equation (1) in the change of canonical variables $x_1, \ldots, x_n, y_1, \ldots, y_n \to \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ as defined by Equation (2), we have

$$(F_0, F') = -\frac{dF'}{dt^*}. \tag{29}$$

Therefore, F' does not depend explicitly on t, it is a first integral of (28) and we have

$$F'(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) = \text{Constant};$$

from which and according to Equations (28), (29) and the first equality of (8)

$$\frac{\mathrm{d}F'}{\mathrm{d}t^*} = 0 = -(F_0, F') =$$

$$= -\sum_{i=1}^{n} \left(\frac{\partial F_0}{\partial \xi_i} \frac{\partial F'}{\partial \eta_i} - \frac{\partial F'}{\partial \xi_i} \frac{\partial F_0}{\partial \eta_i} \right) =$$

$$= -\sum_{i=1}^{n} \left(\frac{\partial F_0}{\partial \xi_i} \frac{\mathrm{d}\xi_i}{\mathrm{d}t} + \frac{\partial F_0}{\partial \eta_i} \frac{\mathrm{d}\eta_i}{\mathrm{d}t} \right) =$$

$$= -\frac{\mathrm{d}F_0}{\mathrm{d}t} = -\frac{\mathrm{d}F'_0}{\mathrm{d}t} .$$

Consequently,

$$F'_0 = \text{Constant.}$$

Accordingly, Equation (28) yields the first two integrals $F' = \text{constant}$ and $F_0 = \text{constant}$. Its resolution leads to a system of order $2n-2$. On the other hand we can write

$$\frac{\mathrm{d}\xi_2}{\mathrm{d}\xi_1} = \frac{\partial F_0}{\partial \eta_2} \Big/ \frac{\partial F_0}{\partial \eta_1}, \ldots, \frac{\mathrm{d}\eta_n}{\mathrm{d}\xi_1} = -\frac{\partial F_0}{\partial \xi_n} \Big/ \frac{\partial F_0}{\partial \eta_1} . \tag{30}$$

Equations (30) is a system of $2n-1$ differential equations in, $\xi_2, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ with respect to the independent variable ξ_1. Let $\xi_2 = \xi_2(\xi_1, C_2, \ldots, C_{2n}), \ldots \eta_n(\xi_1, C_2, \ldots, C_{2n})$ is its general solution. From Equation (9) and the general solution of (30), we can extract

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t^*} = \frac{\partial}{\partial \eta_1} F_0(\xi_1, \xi_2(\xi_1, C_2, \ldots, C_{2n}), \ldots, \eta_n(\xi_1, C_2, \ldots, C_{2n})) ,$$

from which

$$t^* + C =$$

$$= \int \mathrm{d}\xi_1 \Big/ \frac{\partial}{\partial \eta_1} F_0(\xi_1, \xi_2(\xi_1, C_2, \ldots, C_{2n}), \ldots, \eta_n(\xi_1, C_2, \ldots, C_{2n})) . \tag{31}$$

Furthermore, from (31), we can write

$$\xi_1 = \xi_1(t^* + C, C_2, \ldots, C_{2n}) , \tag{32}$$

while from (32) and the general solution of (30), we deduce that

$$\xi_j = \xi_j(t^* + C, C_2, \ldots, C_{2n}) ,$$

$$\eta_j = \eta_j(t^* + C, C_2, \ldots, C_{2n}), \quad j = 1, 2, \ldots, n . \tag{33}$$

Equation (33) is the general solution of the $2n$ canonical equations (9) of the

Hamiltonian F_0, from which it follows that

$$t^* + C = \phi(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)\,, \tag{34}$$

$$C_j = \psi_j(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n), \quad j = 2, 3, \ldots, 2n\,. \tag{35}$$

The Equations (34), (35) are fundamental because with their aid we can operate the integrations with respect to t^* and the searching for mean values of the functions of t^* necessary for the calculations of the determining functions $S_1, S_2, S_3, \ldots$, and the new Hamiltonians $F'_1, F'_2, \ldots$. They are equivalent to the general integral (33) of the auxiliary system of the $2n$ canonical Equation (9) of the Hamiltonian F_0, that means in the calculations of $S_1, S_2, \ldots; F'_1, F'_2, \ldots$, it is necessary to know *a priori* the general integral of (9). But in most of the problems of celestial mechanics F_0 has a very simple expression. That is the particular case of planetary theory in which it does not depend on the angular variables $\eta_1, \ldots, \eta_n$ and depend only on the linear variables $\xi_1, \ldots, \xi_p$; $p < n$ conjugate to the angular variables $\eta_1, \ldots, \eta_p$ of the short period terms.

The solution of Equation (9) leads to $n + (n - p) = 2n - p$ quadratures expressing that $\xi_1, \ldots, \xi_n, \eta_{p+1}, \ldots, \eta_n$ are constants and to p quadratures expressing that $\eta_1, \ldots, \eta_p$ are the linear functions of time t (Meffroy, 1970–1982).

6. Relation Between the Determining Function in the Sense of Von Zeipel and the Determining Function in the Sense of Hori

Let $\tilde{S}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n)$ be the determining function in the sense of Von Zeipel applied to a system of $2n$ canonical equations

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, \ldots, n\,.$$

We have

$$\frac{\partial \tilde{S}}{\partial \xi_i} = \eta_i \qquad \frac{\partial \tilde{S}}{\partial y_i} = x_i \quad i = 1, 2, \ldots, n \tag{36}$$

and

$$\tilde{S} = \sum_{k=0}^{\infty} \tilde{S}_k\,,$$

$\tilde{S}_k$ being of order k with respect to a small parameter of the order of masses, and S_0 being equal to

$$\tilde{S}_0 = \xi_1 y_1 + \cdots + \xi_n y_n\,.$$

From Equation (36), neglecting the powers of ϵ – higher than the second, we find that

$$\eta_i = y_i + \frac{\partial \tilde{S}_1}{\partial \xi_i} + \frac{\partial \tilde{S}_2}{\partial \xi_i} + O(\epsilon^3), \quad i = 1, 2, \ldots, n$$

$$x_i = \xi_i + \frac{\partial \tilde{S}_1}{\partial y_i} + \frac{\partial \tilde{S}_2}{\partial y_i} + O(\epsilon^3), \tag{37}$$

where S_1 and S_2 are the determining functions in the sense of Hori of degree 1, 2, respectively, with respect to ϵ, we have as we have shown above.

$$x_i = \xi_i + \frac{\partial S_1}{\partial \eta_i} + \frac{\partial S_2}{\partial \eta_i} + \frac{1}{2}\left(\frac{\partial S_1}{\partial \eta_i}, S_1\right) + O(\epsilon^3)$$

$$y_i = \eta_i - \frac{\partial S_1}{\partial \xi_i} - \frac{\partial S_2}{\partial \xi_i} - \frac{1}{2}\left(\frac{\partial S_1}{\partial \xi_i}, S_1\right) + O(\epsilon^3). \tag{38}$$

From (37) and (38) we extract

$$\frac{\partial \tilde{S}_1}{\partial y_i} + \frac{\partial \tilde{S}_2}{\partial y_i} = \frac{\partial S_1}{\partial \eta_i} + \frac{\partial S_2}{\partial \eta_i} + \frac{1}{2}\left(\frac{\partial S_1}{\partial \eta_i}, S_1\right)$$

$$\frac{\partial \tilde{S}_1}{\partial \xi_i} + \frac{\partial \tilde{S}_2}{\partial \xi_i} = \frac{\partial S_1}{\partial \xi_i} + \frac{\partial S_2}{\partial \xi_i} + \frac{1}{2}\left(\frac{\partial S_1}{\partial \xi_i}, S_1\right). \tag{39}$$

From the first n equalities of (37) we find that

$$y_i - \eta_i = O(\epsilon).$$

$$\frac{\partial \tilde{S}_j}{\partial y_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) =$$

$$= \frac{\partial \tilde{S}_j}{\partial y_i}\left(\xi_1, \ldots, \xi_n, \eta_1 - \frac{\partial \tilde{S}_1}{\partial \xi_1} - \cdots, \ldots, \eta_n - \frac{\partial \tilde{S}_1}{\partial \xi_n} - \cdots\right) =$$

$$= \frac{\partial \tilde{S}_j}{\partial y_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) - \frac{\partial^2 \tilde{S}_j}{\partial y_i \, \partial \eta_1} \frac{\partial \tilde{S}_1}{\partial \xi_1} - \cdots$$

$$\cdots - \frac{\partial^2 \tilde{S}_j}{\partial y_i \, \partial \eta_n} \frac{\partial \tilde{S}_1}{\partial \xi_n} + \cdots \tag{40}$$

$$\frac{\partial \tilde{S}_j}{\partial \xi_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) = \frac{\partial \tilde{S}_j}{\partial \xi_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) -$$

$$- \frac{\partial^2 \tilde{S}_j}{\partial \xi_i \, \partial \eta_1} \frac{\partial \tilde{S}_1}{\partial \xi_1} - \cdots - \frac{\partial^2 \tilde{S}_j}{\partial \xi_i \, \partial \eta_n} \frac{\partial \tilde{S}_1}{\partial \xi_n} + \cdots \tag{41}$$

with $j = 1, 2$.

$\tilde{S}$ is a continuous function of its arguments particularly of $y_1, \ldots, y_n$. Similarly for $\tilde{S}_1$, $\tilde{S}_2$ and their partial derivatives.

$$\frac{\partial \tilde{S}_1}{\partial y_i}, \frac{\partial \tilde{S}_2}{\partial y_i}, \quad i = 1, 2, \ldots, n.$$

From (40), (41) we can find for a first order theory

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_1}{\partial y_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq \frac{\partial \tilde{S}_1}{\partial \eta_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n),$$

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_1}{\partial \xi_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq \frac{\partial \tilde{S}_1}{\partial \xi_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n); \tag{42}$$

and for a theory of the second order, with $i = 1, 2, \ldots, n$,

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_1}{\partial y_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq$$

$$\simeq \frac{\partial \tilde{S}_1}{\partial \eta_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) - \frac{\partial^2 \tilde{S}_1}{\partial \eta_i \, \partial \eta_1} \frac{\partial \tilde{S}_1}{\partial \xi_1} - \cdots - \frac{\partial^2 \tilde{S}_1}{\partial \eta_i \, \partial \eta_n} \frac{\partial \tilde{S}_1}{\partial \xi_n},$$

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_1}{\partial \xi_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq$$

$$\simeq \frac{\partial \tilde{S}_1}{\partial \xi_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) - \frac{\partial^2 \tilde{S}_1}{\partial \xi_i \, \partial \eta_1} \frac{\partial \tilde{S}_1}{\partial \xi_1} - \cdots - \frac{\partial^2 \tilde{S}_1}{\partial \xi_i \, \partial \eta_n} \frac{\partial \tilde{S}_1}{\partial \xi_n},$$

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_2}{\partial y_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq \frac{\partial \tilde{S}_2}{\partial \eta_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$$

$$\lim_{y_i \to \eta_i} \frac{\partial \tilde{S}_2}{\partial \xi_i}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n) \simeq \frac{\partial \tilde{S}_2}{\partial \xi_i}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$$

$$i = 1, 2, \ldots, n. \tag{43}$$

According to (42), (43) we can write equalities (39) as

$$\frac{\partial \tilde{S}_1}{\partial \eta_i} + \frac{\partial \tilde{S}_2}{\partial \eta_i} - \sum_{j=1}^{n} \left(\frac{\partial^2 \tilde{S}_1}{\partial \eta_i \partial \eta_j} \frac{\partial \tilde{S}_1}{\partial \xi_j} \right) = \frac{\partial S_1}{\partial \eta_i} + \frac{\partial S_2}{\partial \eta_i} +$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial^2 S_1}{\partial \eta_i \, \partial \xi_j} \frac{\partial S_1}{\partial \eta_j} - \frac{\partial S_1}{\partial \xi_j} \frac{\partial^2 S_1}{\partial \eta_i \, \partial \eta_j} \right),$$

$$\frac{\partial \tilde{S}_1}{\partial \xi_i} + \frac{\partial \tilde{S}_2}{\partial \xi_i} - \sum_{j=1}^{n} \left(\frac{\partial^2 \tilde{S}_1}{\partial \xi_i \, \partial \eta_j} \frac{\partial \tilde{S}_1}{\partial \xi_j} \right) = \frac{\partial S_1}{\partial \xi_i} + \frac{\partial S_2}{\partial \xi_i} +$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial^2 S_1}{\partial \xi_i \, \partial \xi_j} \frac{\partial S_1}{\partial \eta_j} - \frac{\partial S_1}{\partial \xi_j} \frac{\partial^2 S_1}{\partial \xi_i \, \partial \eta_j} \right). \tag{44}$$

From (44) we deduce that

$$\frac{\partial \tilde{S}_1}{\partial \eta_i} = \frac{\partial S_1}{\partial \eta_i}, \qquad \frac{\partial \tilde{S}_1}{\partial \xi_i} = \frac{\partial S_1}{\partial \xi_i} \quad i = 1, 2, \ldots, n$$

from which

$$\tilde{S}_1 = S_1 . \tag{45}$$

From (44), (45) we extract

$$\frac{\partial \tilde{S}_2}{\partial \eta_i} = \frac{\partial S_2}{\partial \eta_i} + \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial^2 S_1}{\partial \eta_i \, \partial \xi_j} \frac{\partial S_1}{\partial \eta_j} + \frac{\partial S_1}{\partial \xi_j} \frac{\partial^2 S_1}{\partial \eta_i \, \partial \eta_j} \right),$$

$$\frac{\partial \tilde{S}_2}{\partial \xi_i} = \frac{\partial S_2}{\partial \xi_i} + \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial^2 S_1}{\partial \xi_i \, \partial \xi_j} \frac{\partial S_1}{\partial \eta_j} + \frac{\partial S_1}{\partial \xi_j} \frac{\partial^2 S_1}{\partial \xi_i \, \partial \eta_j} \right);$$

i.e.,

$$\frac{\partial \tilde{S}_2}{\partial \eta_i} = \frac{\partial S_2}{\partial \eta_i} + \frac{1}{2} \frac{\partial}{\partial \eta_i} \sum_{j=1}^{n} \left(\frac{\partial S_1}{\partial \xi_j} \frac{\partial S_1}{\partial \eta_j} \right), \tag{46}$$

$$\frac{\partial \tilde{S}_2}{\partial \xi_i} = \frac{\partial S_2}{\partial \xi_i} + \frac{1}{2} \frac{\partial}{\partial \xi_i} \sum_{j=1}^{n} \left(\frac{\partial S_1}{\partial \xi_j} \frac{\partial S_1}{\partial \eta_j} \right). \tag{47}$$

From (46), (47) we obtain

$$\tilde{S}_2 = S_2 + \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial S_1}{\partial \xi_j} \frac{\partial S_1}{\partial \eta_j} \right) \tag{48}$$

and from (45), (48) we extract

$$S_1 = \tilde{S}_1, \qquad S_2 = \tilde{S}_2 - \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial \tilde{S}_1}{\partial \xi_i} \frac{\partial \tilde{S}_1}{\partial \eta_j} \right). \tag{49}$$

The equalities (45), (48) give the expressions for the determining functions $\tilde{S}_1$, $\tilde{S}_2$ in the sense of Von Zeipel, as function of determining functions in the sense of Hori S_1 and S_2, and reciprocally the equalities (49) give the expressions for the determining functions in the sense of Hori S_1 and S_2 as function of the determining functions in the sense of Von Zeipel $\tilde{S}_1$ and $\tilde{S}_2$. We particularly notice that we obtain the expression of S_2 from the expression of $\tilde{S}_2$ by permuting the S and the $\tilde{S}$ in (48) and by replacing the coefficient $+\frac{1}{2}$ by the coefficient $-\frac{1}{2}$. We can calculate $\tilde{S}_3, \tilde{S}_4, \ldots$ as functions of $S_1, S_2, S_3, S_4, \ldots$ and we can find the corresponding expressions of $S_3, S_4, \ldots$ as function of $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4, \ldots$. The formulae become increasingly more complicated, but their explicit form is not difficult to establish. In particular,

$$S_3 = \tilde{S}_3 - \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial \tilde{S}_1}{\partial \xi_j} \frac{\partial \tilde{S}_2}{\partial \eta_j} + \frac{\partial \tilde{S}_2}{\partial \xi_j} \frac{\partial \tilde{S}_1}{\partial \eta_j} \right) + \frac{1}{12} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \tilde{S}_1}{\partial \xi_j} \frac{\partial \tilde{S}_1}{\partial \xi_k} \frac{\partial^2 \tilde{S}_1}{\partial \eta_j \, \partial \eta_k} + \right.$$
$$\left. + 4 \frac{\partial \tilde{S}_1}{\partial \xi_j} \frac{\partial \tilde{S}_1}{\partial \eta_k} \frac{\partial^2 \tilde{S}_1}{\partial \eta_j \, \partial \xi_k} + \frac{\partial \tilde{S}_1}{\partial \eta_j} \frac{\partial \tilde{S}_1}{\partial \eta_k} \frac{\partial^2 \tilde{S}_1}{\partial \xi_j \, \partial \xi_k} \right) \tag{50}$$

(Mersman, 1970, 1971); see Meffroy (1973, 1975, 1978), Von Zeipel (1916–17), or Meffroy (1970–82).

Aspects of Future Research Work

In subsequent papers we shall indicate how the second step of Hori's method is applied to eliminate the long-period terms in general planetary theory. We shall refer to Mersman's improvement of Hori's technique adapted in celestial mechanics theory of perturbation. We shall complete the construction of the third order Uranus–Neptune theory, and we shall establish the Jupiter–Saturn third order theory, then we shall build the four major planets third order theory J–S–U–N. In all the above theories we shall take into consideration the main as well as the indirect part of the planetary perturbing function.

Acknowledgments

This work has been carried out under the direction of the late Professor J. Meffroy, Institute of Mathematics, Montepellier University, France, through several communications with him in the years 1970–1982.

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