CORRECTIONS: DISCUSSION: "J Integral Analysis of Stable Crack Growth," by I.-H. Lin, J. P. Hirth, and A. R. Rosenfield*

S. J. Burns

Materials Science Program, Department of Mechanical and Aerospace Sciences
University of Rochester, Rochester, New York 14627 USA
tel: (716)275-4082

A method for deducing the J crack driving force value versus crack extension curve, the J_R curve, during stable crack growth has recently been reformulated by Lin, *et al.* This new proposed relation for J is intended to apply to rectangular bend beams with a through thickness crack subject to bending only. It is shown here that in general, the proposed, reformulated J relationship applies only to a very restricted class of test specimens - constant J specimens. Linear elastic, deeply notched and unstable, non-linear elastic bend beams are not constant J test specimens. The proposed reformulated J relationships which involve extensions will not apply to bend beams since in these test pieces both J and the bending moment may not, in general, be held constant simultaneously while the crack extends.

The incremental isothermal, Helmholtz free energy for a fracture specimen mechanically loaded with the bending moment M, a conjugate displacement θ_c , and an uncracked ligament length b, in a sample of thickness B, is

$$dF = Md\theta_c + JBdb \quad (1)$$

$Md\theta_c$ is the incremental work done on the bend beam and $JBdb$ is the work done in advancing the crack. The sum of these two incremental work terms is the incremental change in the isothermal Helmholtz energy. The isothermal complementary energy function of (1) is $G = M\theta_c - F$.

Thus,

$$dG = \theta_c dM - JBdb \quad (2)$$

From (1) it follows that

$$\left. \frac{\partial F}{\partial \theta_c} \right|_b \equiv M, \quad \left. \frac{1}{B} \frac{\partial F}{\partial b} \right|_{\theta_c} \equiv J$$

and

$$\left. \frac{1}{B} \frac{\partial M}{\partial b} \right|_{\theta_c} = \left. \frac{\partial J}{\partial \theta_c} \right|_b \quad (3)$$

**International Journal of Fracture* 16 (1980) R11-R14. Discussion originally published in *International Journal of Fracture* 16 (1980) R153-R156.

From (2) it follows that

$$\left. \frac{\partial G}{\partial M} \right|_b \equiv \theta_c, \quad \left. \frac{1}{B} \frac{\partial G}{\partial b} \right|_M = -J$$

and

$$\left. \frac{1}{B} \frac{\partial \theta_c}{\partial b} \right|_M = - \left. \frac{\partial J}{\partial M} \right|_b \quad (4)$$

Now, it is noted that integration of (4) yields

$$J = \frac{1}{B} \int_0^M - \left. \frac{\partial \theta_c}{\partial b} \right|_M dM \quad (5)$$

provided b is held constant during integration. Eq. (5) is the Lin, *et al.* eq. (1) without the restriction of constant B during integration. Eq. (5) was initially proposed by Rice, Paris, and Merkel [1]. There, (5) was used only with b constant, but it was not explicitly stated that b must be held constant during integration. On the other hand, if it is not explicitly stated how b changes during integration then (5) is not unique since $\partial \theta_c / \partial b|_M$ contains both b and M dependences. Lin, *et al.* derive from their eq. (1) an incremental form for dJ . (In their case, b has been permitted to change during integration of (5).)

$$dJ = \left. \frac{1}{B} \frac{\partial M}{\partial b} \right|_{\theta_c} d\theta_c - \left. \frac{\partial M}{\partial b} \right|_{\theta_c} \left. \frac{\partial \theta_c}{\partial b} \right|_M db \quad (6)$$

I will now show that (6) is valid only with the restriction that $\partial J / \partial b|_M = 0$. Chose θ_c and b as the independent variables in this fracture problem. It is shown in [2] that, in general non-linear elastic fracture problems, two independent state variables are necessary to describe the equation of state, $J = J(\theta_c, b)$:

$$dJ = \left. \frac{\partial J}{\partial \theta_c} \right|_b d\theta_c + \left. \frac{\partial J}{\partial b} \right|_{\theta_c} db \quad (7)$$

Comparing (6) and (7) shows that if (6) is always to be true then

$$\left. \frac{1}{B} \frac{\partial M}{\partial b} \right|_{\theta_c} = \left. \frac{\partial J}{\partial \theta_c} \right|_b \quad (8)$$

and

$$- \left. \frac{\partial M}{\partial b} \right|_{\theta_c} \left. \frac{\partial \theta_c}{\partial b} \right|_M = \left. \frac{\partial J}{\partial b} \right|_{\theta_c} \quad (9)$$

Eq. (8) is always valid since it is the cross derivative of an energy function; see (3). Eq. (9), however, is not always valid but imposes a restriction on the specimen type. The derivatives in (9) calculated with the aid of [2] are

$$\left[\frac{1}{B} \frac{\partial M}{\partial b} \right]_{\theta_c}^2 = \left[\frac{1}{B} \frac{\partial J}{\partial b} \right]_{\theta_c} \left[\frac{\partial M}{\partial \theta_c} \right]_b \quad (10)$$

Again using [2] to calculate a partial derivative, it may be shown that

$$\left[\frac{1}{B} \frac{\partial J}{\partial b} \right]_M = \left[\left[\frac{1}{B} \frac{\partial M}{\partial b} \right]_{\theta_c}^2 / \left[\frac{\partial M}{\partial \theta_c} \right]_b \right] - \left[\frac{1}{B} \frac{\partial J}{\partial b} \right]_{\theta_c} \quad (11)$$

Thus, by comparing (10) with (11), it is clear that (9) is valid only when

$$\left[\frac{\partial J}{\partial b} \right]_M = 0 \quad (12)$$

A test specimen that is restricted to have both J and M constant while b changes is a non-linear constant J specimen (at constant generalized load).

In linear elastic fracture mechanics, this would be regarded as a constant K specimen. The linear elastic bend bar is not a constant K specimen because its compliance is not proportional to b [3]; the non-linear, unstable, elastic bend bar is not a constant J specimen, at constant M , because it is (nearly) a constant J specimen, at constant θ_c [4]; the deeply notched bend bar [1] is not a constant J specimen at constant M because it does not satisfy (11) with the left side equal to zero. The following derivatives for the deeply notched specimens restricted to $\theta_c = f(M/b^2)$ are

$$\alpha = \left[\frac{1}{B} \frac{\partial J}{\partial b} \right]_{\theta_c} = J/Bd \quad (13)$$

$$\beta = \left[\frac{1}{B} \frac{\partial M}{\partial b} \right]_{\theta_c} = 2M/Bb \quad (14)$$

and

$$\gamma = \left[\frac{\partial M}{\partial \theta_c} \right]_b = \frac{b^2}{f'(M/b^2)} \quad (15)$$

Eq. (10) is valid only for the deeply notched bend bar if $\beta^2 = \alpha\gamma$, or if

$$f' = Jb^3 B/4M \quad (16)$$

Eq. (16) is a highly restricted condition on f' and therefore on θ , and it certainly is not valid for all deeply notched bend bars.

Acknowledgement: This work was supported by U. S. Department of Energy under Contract DE-AS02-76ER02422.

REFERENCES

- [1] J. R. Rice, P. C. Paris, and J. G. Merkle, in *Progress in Flaw Growth and Fracture Toughness Testing*, STP 536, American Society for Testing and Materials, Philadelphia (1973) 231-245.
- [2] S. J. Burns, J. -C. Pollet, and C. Lun Chow, *International Journal of Fracture* 14 (1978) 311-326.
- [3] S. J. Burns, J. -C. Pollet, and C. Lun Chow, *International Journal of Fracture* 15 (1979) R63-R64.
- [4] P. W. Early and S. J. Burns, *International Journal of Fracture* 16 (1980) (to appear).

7 April 1980 (corrections received 20 October 1980)