Chapter 3
The Impact of Denomination Choice on Commercial Trading: A Policy Evaluation of a New Iraqi Monetary System

So he saluted him and Ma'an said to him "What bringeth thee, O brother of the Arabs?" Answered the Badawi, "I hoped in the Emir, and have brought him curly cucumbers out of season." Asked Ma'an, "And how much dost thou expect of us?" "A thousand dinars," answered the Badawi. "This is far too much," quoth Ma'an. Quoth he, "Five hundred." "Too much!" "Then three hundred." "Too much!" "Two hundred." "Too much!" "One hundred." "Too much!" "Fifty." "Too much!" At last the Badawi came down to thirty dinars; but Ma'an still replied, "Too much!" So the Badawi cried, "By Allah, the man who met me in the desert brought me bad luck! But I will not go lower than thirty dinars." Quoted from the 271th Night, "Ma'an Son of Zaidah and the Badawi", from *the Book of The Thousand Nights and a Night*, Vol. 4 by Richard F. Burton, Shammar Edition, 1885.

3.1 Introduction

Ever since 1991, the Gulf War, starting the invasion of Iraq to Kuwait, economic sanctions were imposed on Iraq for a long time and oil exports, which was the major source of foreign currency, were also restricted. As a result, the Iraqi government lacked fiscal revenues, thus faced large fiscal deficits. Inevitably, the government financed its expenditures by printing money as exchanges of government bonds. The nation experienced a rapid inflation, as we can easily imagine. The Saddam Hussein regime collapsed after the Iraq war in March 2003. Under the interim Iraqi Administration, new Iraqi dinar notes were issued and replaced old notes between October 15, 2003, and January 15, 2004, in three months.

This chapter considers the role of currency under a rapidly changing society in Iraq. In particular, we will focus on the new currency regime after January 15, 2004.

After the Gulf War in 1991, no reliable macroeconomic statistics were available except for a part of the UN statistics. We can use the information only on currency denominations and fragmentary price data. Iraqi currency dropped to ten-thousandth its value over 10 years' economic chaos following the Gulf War, economic sanctions, and the Iraq War. In order to evaluate the Iraqi currency system in the future, we shall use an appropriate exchange rate under normal circumstances, not an extremely undervalued exchange rate under the chaotic economic conditions. This chapter was

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originally written in August 2004. We did not have sufficient information on macroeconomic statistics to understand the normal economic activity in Iraq. This chapter handles a very limited part of the problems the Iraqi economy faced at that time.

The chaos in the Iraqi economy can be understood as follows. In general, economic institutions normally evolve from the primitive stage to the matured stage. In Iraq, the modern economic institutions under the British Occupation during the First World War were introduced and functioned rather well. After independence in 1932, the Socialist Revolution in July 14, 1958, and Saddam Hussein's regime in July 16, 1979, economic institutions gradually deteriorated. Iraq's fundamental economic problems rest on malfunctioning of economic institutions in general and on the inability of the administration to understand the problems and inability of fixing them in particular.[1] That is to say, when facing the process of collapsing economic and social order, how can we amend and improve a new economic and social order?

Restricted to the currency system, if there is a regime change in a fundamental sense, a new currency system should be introduced to replace the old system. The government imposes deposit freezing of the old currency, encourages renegotiation of the old economic contracts in general, debt contracts in particular, and debtors repaying debt in a full or in a discounted amount.

The Iraqi currency reform was conducted in three months under the coalition provisional authority. Although the old currency was replaced by the new currency, old economic contracts were untouched. The exchange rate between the old and the new currency was fixed at 1:1. Consequently, economic impacts of the new currency system remained limited.

An exception was to unify the two currency systems prevailing in the Kurdistan Region, i.e., the old Iraqi currency and the Swiss dinar currency[2] with the exchange rate at 1:150. This currency unification with the Kurdistan region was noteworthy. Of course, as we all know, the situation was not that simple. In recent years, the 2017 Kurdistan Region independence referendum took place on September 25, with 92.73% voting in favor of independence. This triggered a military operation in which the Iraqi government retook control of Kirkuk and surrounding areas and forced the Kurdistan Regional Government to annul the referendum.

Going back to 2004, many money changers around Al-Kadhimiya Mosque actively traded in various currencies with a fixed handling fee of 2%. Money changers decided the exchange rates among currencies, willingly including the US dollar, Iranian rial (toman), Swiss dinar, and Saddam dinar, which were not necessarily at the rates in the Foreign Exchange Market. Because of the introduction of the new currency and anti-US sentiment, dollarization did not happen. Money changing was an attractive business with foreigners, given the normal economic activity did not occur.

[1] Similar problems were found in many Asian and African countries that obtained independence after the Second World War.

[2] This was the currency introduced before the Saddam Hussein regime, for which the original plates for banknotes were engraved in Switzerland and thus was called the Swiss dinar.

The new currency system included six denominations (50, 250, 1,000, 5,000, 10,000, and 25,000 dinars). As the minimum denomination of the old Saddam dinar was 250 dinar, the minimum denomination dropped to 50 dinars in the new currency system and difficulties with small payments were eased. However, it remained a big problem without smaller change below 50 dinars. In this chapter, we consider the small change problem in which small denominations are not available in market transactions. This problem was well known in medieval Europe, as discussed by Sargent and Velde (2002).[3] The same problem has happened in Iraq in recent years. In medieval Europe, it was said that the rulers did not have any incentive to provide small coins (denominations). In Iraq nowadays, it may be true that the costs of providing small coins (denominations) exceed the face value of the coins. That is, these small coins would not yield any seigniorage. As of April 24, 2004, the exchange rate was 1 dollar = 1,150 dinars, 100 yen = 1,077.43 dinars (approximately 50 dinar = 5y en). It may not be a big problem to ignore denominations below 50 dinars.

If this circumstance continues, this argument can be valid. But in 1991, just before the Gulf War, the exchange rate was 1 dinar = 3.22 dollars = 436 yen. 50 dinars was equivalent to 21,800 yen. It was a sufficiently large amount and it may not be the case that we can ignore any amount below 50 dinars. In this case, we may need to consider supplementary coins (5, 10, 25, and 100 fils as 1 dinar = 1,000 fils).[4] If Iraq could concentrate its effort on recovering the economic and social system as it was before the Gulf War, its economic activity would be at least as high as that before the Gulf War period. It is necessary to prepare small denominations (coins and notes) below 50 dinars.

Furthermore, in principle, without the basic unit of accounting, 1 dinar, payable amounts are largely restricted. That creates non-negligible distortions in small payments (transactions). Imagine, with the current purchasing power, you can drink a cup of tea with 50 dinars; a lot of other goods and services can be purchased at lower than 50 dinars. It is difficult to make payment in multiples of 50 dinars if a priori quantity adjustment is not feasible with daily-use goods like gas, water, and electricity. In order to make payment exactly and flexibly, the basic unit of account, 1 dinar, is needed. To put it differently, introduction of the basic unit, 1 dinar, would make small payments easier by far, given the inconvenience and distortions without small denominations (change). We will come back to this problem in Sect. 3.3.

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[3] In Japan, the central government (Bakufu) did not mint small coins; the merchants imported Chinese coins and used them regardless of the exchange rate.

[4] Currently supplementary coins (fils) in Iraq were not used because of high inflation.

3.2 History of Iraqi Currency System and the Facts

3.2.1 History of Iraqi Currency Denominations

In this section, we look at the history of Iraqi currency system. In 1831, the Ottoman Empire managed to overthrow the Mamluk regime and again imposed their direct control over Iraq. During the Ottoman rule, the Ottoman pound was the legal tender. Nonetheless, actually the Indian rupee, which was linked to the British pound sterling, was most widely circulated in the market. After colonization of the Ottoman Empire, the British government adopted the Indian rupee as the official currency in Iraq and consequently abolished the Ottoman pound. This was, in modern terms, equivalent to the dollarization of Iraq.

In 1932, the monarch Faisal I of Iraq (1885–1933) achieved independence from the British Mandate and the Iraqi dinar went into circulation on April 2, 1932, replacing the Indian rupee. The Central Bank of Iraq was established on November 16, 1947. After the Iraqi Revolution on July 14, 1958, the design of the bank notes was changed, but the currency denomination structure remained the same.

At that time, under the strong British influence, coins were minted at the Royal Mint, bank notes were printed at De La Rue. Both were British companies. Coins were introduced from 1932 to 1990 and in denominations of 1, 5, 10, 25, 50, 100, 250, 500 fils and 1 dinar(= 1,000 fils). Banknotes were issued in denominations of ¼, ½, 1, 5, 10, 50, 100,and 250 dinars. In 2003, new banknotes were issued consisting of six denominations; 50, 250, 1,000, 5,000, 10,000, and 25,000dinars. These denominations of coins and banknotes in 1982 included the coins for a commemorative series celebrating Babylonian achievements and small face-value banknotes (i.e., less than 10 dinars) were taken out of circulation. De La Rue issued a 25 dinar note in 1986, bearing an idealized engraving of Saddam Hussein, then president of Iraq.

Following the 1991 Gulf war, Iraq was divided into the region controlled by Saddam Hussein in the south and the Kurdish region in the north. The UN Security Council adopted Resolution 661, which imposed economic sanctions on Iraq. As a result, the British Royal Mint and De La Rue withheld their business with Iraq. Iraqi currency was printed both locally and in China, using poor-grade wood pulp paper and inferior offset printing. Its denominations were 5, 10, 25, 50 and 100 dinars without any counterfeit technology. On May 5, 1992, the Iraqi government announced that because foreign countries had provided counterfeits of Iraqi currency and tried to destroy the Iraqi economy, the popular 50 and 100 dinar notes would no longer be used.

In May 1993, Iraq closed its border with Jordan and abolished the De La Rue 25 dinar note. This note was very popular, so that it acquired a premium value. Jordanian merchants hoarded the De La Rue 25 dinar notes on a large scale and lost 1.5 billion dinars.[5] Replacement of the old currency by the new currency took place in only

[5] The purposes of this border closing were to make Iraqi currency in foreigners' hands useless, to reduce the amount of currency circulation, and thus to maintain the value of Iraqi currency.

three weeks, beginning May 5, 1993. While currency in circulation was 22 billion dinars in 1991, in 1995 climbed to 584 billion dinars. As a result, the annual average inflation rate shot up 250%. Small face-value notes and coins became valueless, and only the 25 dinar note was used in cash transactions.

Iraqi Kurdistan in the north hoarded 5 billion old Swiss 25 dinar notes[6] out of the total 7 billion in Iraq. It was circulated without replacement by the Saddam dinar. The Swiss dinar had not been issued since 1989. The Central Bank in charge of issuing the Swiss dinar no longer existed. It was fiat money without any guarantee from the government or the Central Bank. Nevertheless, the remaining notes kept circulating in the market. This was a rare event in monetary history. Furthermore, the Swiss dinar had a higher quality than the Saddam dinar, used anti-counterfeit techniques, and thus had a lower counterfeit risk premium.[7]

Look at the exchange rate between the Swiss dinar and the Saddam dinar: as of July 1997, it was 60 Saddam dinars = 1 Swiss dinar. Over time, the rate dropped to 200–400 Saddam dinars = 1 Swiss dinar after July 2002. When the Saddam Hussein regime collapsed in March 2003, it was 300 Saddam dinars = 1 Swiss dinar. This phenomenon can be explained by the fact that the exchange rate reflected the balance between the fixed-amount Swiss dinar and the ever-increasing (money supply) Saddam dinar.

How can we explain that the exchange rate between the Swiss dinar, which was not-supported by the Iraqi government, and the US dollar was 1 dollar = 18 Swiss dinard in May 2002 and rose to 1dollar = 6Swiss dinars after the Iraq War ended in May 2003? As King (2004, pp. 8–9) argues, that appreciation reflected expectations about (1) the durability of the political and military separation of the Kurdish region from Saddam-controlled Iraq and (2) the likelihood that a new institution would be established governing monetary policy in Iraq as a whole that would retrospectively back the value of the Swiss dinar. However, the government that issued the Swiss dinar no longer existed and the Central Bank of Iraq did not guarantee the exchange between the Swiss dinar and the Saddam dinar. The value of the Swiss dinar had everything to do with policies and nothing to do with the economic policies of the government issuing the Swiss dinar.

The Iraq War begun in March 2003 and the Saddam Hussein regime collapsed in May 2003. In July 7, 2003, Paul Bremer, Interim Civil Administrator (the head of the Coalition Provisional Authority), announced that a set of new Iraqi dinars (50, 250, 1,000, 5,000, 10,000, 25,000) would be printed and exchanged for the two existing currencies (i.e., 250 10.000 dinars) at a parity rate (i.e., 1 to 1) and 1 Swiss dinar = 150 new dinars The exchange was to take place over the period from October 15, 2003, to January15, 2004. The design of the new dinars was similar to the notes

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[6] This banknote was printed in British De La Rue, but the original picture plate was engraved in Switzerland. It is called the Swiss dinar.

[7] On the contrary, the Saddam dinar under the economic sanctions, was made with very primitive printing techniques, and counterfeit banknotes, in particular 1,0000 dinar notes, often appeared. This note was rarely accepted in practice due to fears of looting and counterfeiting. This forced the Iraqi people to use 250 dinar banknotes for daily shopping.

issued by the Central Bank of Iraq in the 1970s and 1980s, given a very short time of preparation and printing by De La Rue.

As discussed before, the minimum denomination being 50 dinar (about 5 yen) had to do with the manufacturing cost. It is said that 1 banknote costs 10–20yen to produce in Japan. If the cost is similar to this, 100 dinar (about 10 yen) would be the profit line. A lower denomination than 100 dinar might generate losses (negative profit) so that the smaller denominations might not be feasible.

The exchange rate between the Swiss dinar and the Saddam dinar was 1:150 in September 2002. It shot up to 1:300 at the beginning of the Iraq War on March 20, 2003, and converged to 1:150 on the day the new currency system was announced, July 7, 2003. King (2004, p. 10) elaborated that the exchange rate hovered above 150 after parity was announced on July 7, considering (1) the counterfeit risk premium on the Saddam dinar, and (2) before the capture of Saddam Hussein in December 2003, some uncertainty about the prospects of the new regime and the new currency system was inevitable.

King's final remark on this episode is worth quoting: "The value of money depends on beliefs about the probability of survival of the institutions that define the state itself." (King, 2004, p. 10).

3.2.2 Price Level and Exchange Rate in Iraq

As discussed above, comprehensive economic statistics after the Gulf War in Iraq did not exist. Only partial and insufficient data were available. UN National Account statistics were available only in nominal value. The nominal GDP growth in 1990 was 10.79%, in 1991 was –14.42%, in 1992 was 148.04%, in 1993 was 151.68%, in 1994 was 395.02%, in 1995 was 214.99%, in 1996 was 22.58%, in 1997 was 37.43%, in 1998 was 39.76%, in 1999 was 49.01%, and in 2000 was 8.65%.

During this period, the Iraqi economy experienced hyper-inflation. It was said that the real GDP dropped by 57% between 1989 and 1992 and that it reached the level of the 1970s (of course, these are not exact statistics). Suppose the real GDP growth rate was 0%: the nominal GDP growth rate actually reflected the inflation rate. From 1992 to 1995, there were a strong signs of hyper-inflation.

We can observe this phenomenon from information on the household income. According to one source, a high-ranking bureaucrat's monthly salary was 775 dinars in 1993, reaching 5,000 dinars in 1996 and further increasing to 50,000–100,000dinars later. Supplementary information shows that the average monthly expenditure on food for a six-member household was 5,400 dinars in 1993. It is easy to imagine that the standard of living would be very low even if two or three members of the household worked.[8]

[8] After the Iran–Iraq war, rationing of basic goods was stopped. After the Gulf War, it was resumed. The rationed materials included wheat (9 kg), rice (3 kg), vegetable oil, sugar, salt, tea, and detergent.

Inflation was reflected in the exchange rate. From this perspective, the exchange rate was 1 dinar = 3.39 dollars in 1979, when Saddam Hussein became the president and chairman of the Revolutionary Command Council in Iraq. 1 dinar = 3.2164 dollars in 1985, 3.2249 dollars in 1990, Until then the exchange rate had been very stable. But in 2003, it dropped to the level of 1 dollar = 4,000dinars (1 dinar = 0.00025dollar = 0.025cents), 1/13560 of its earlier value. Immediately after the Gulf War in 1991, it dropped to 1/600 of its value. However, the exchange rate recovered to 1 dollar = 1,150dinars on April 24, 2004. The value of the Iraqi dinar increased four times from its level in 2003.

The fluctuation of the exchange rate cannot be explained by the inflation rate of a country, because it is essentially a relative measure. Suppose the inflation rate of the US in the 1990s remained very stable, *ceteris paribus*; Iraq experienced more than a 100% inflation rate *per annum*. From another source, it is said that before the Gulf War, the annual average inflation rate was 45%; after the war, it shot up above 500%. It may not be badly wrong to assume a 100% inflation rate *per annum* during the 1990s. For those who stayed in Iraq during and after the Saddam Hussein regime, all monetary transactions took place only by means of the 250 dinar note (the 1,000dinar note was rarely used). Smaller- denomination coins were not used at all. We have only partial information on individual prices of goods and services and we do not know how these prices represent the whole economy. For example, 1 kg mutton was 3,000 dinars before the Iraq War (2003) and 8,000dinars after the war. Three eggs were 250 dinars before the Iraq war and 450 dinars after the war. 1 piece of pita bread was 25 dinars before the war and 50 dinars afterward. After the Iraq war, the price levels became double. The prices were adjusted in multiples of 250 before the war and in the multiples of 50 after the war because of the availability of minimum denominations. If a unit price was below the minimum denomination, units were bundled quantitatively to make a price, multiples of the minimum denomination.

3.3 Use-Value of the New Currency System in Iraq and Its Problems

A typical pattern of shopping before the new currency system (i.e., two currency denomination system) went as follows. If you were not Iraqi, you exchanged your 100 dollars, for example, for 250 dinar banknotes (about 25 yen), then you received 1,000–2,000 in 250 dinar notes (usually 10–20 blocks in a bundle of 100 notes) in a plastic bag. You shopped with those. Until January 2004, only 250 dinar notes were circulated in the market. Payments were made in multiples of 250 dinar notes. In case of a large payment, you could mix dollar notes with dinar notes. But in case of bundles of dinar notes, merchants checked only the numbers of the first and last note to assume a bundle contain a hundred 250 dinar notes.

This pattern of shopping inevitably became very rough because there were no high-denomination notes or low-denomination coins. In the following, let us discuss use-value of the new currency system in Iraq and its problem from theoretical viewpoints.

First of all, the economic price (producer price) of a good is determined by the marginal cost of its production, which is independent of the currency denomination structure. On the other hand, the monetary price (consumer price) is set at the payment stage, taking into account available currency denominations, tax, and other factors. Without VAT (consumption tax) and with a perfect competition in the market, the economic efficiency can be achieved by equilibrating the economic price and the monetary price. However, this condition is not satisfactory because the infeasibility of the payment amount due to limited denominations. We need to understand that a multiple of 250 is a very small portion of the natural number as a whole. It is clear how difficult it is to set a price under which the producer price and the consumer price may not coincide. Given currency denominations, we need to decide how to make a payment with these notes. As we experience daily, it is convenient for consumers to have many ways to settle a payment. With currency denominations in Iraq, there are far fewer ways to make payments than those with Japanese denominations (see Appendix 3.1). How do merchants set prices, given denominations? Merchants play a role as an intermediary between producers and consumers. If the currency denomination structure prevents flexible (marginal) pricing, the second-best solution would make quantitative adjustments to meet a payable amount with its denominations. Of course, quantity could easily exceed what a consumer wants. This type of pricing can be found in supermarkets and grocery stores.[9] In the Middle East, from my limited knowledge, the standard pricing in bazaar is to set prices for a unit amount (say, kilograms of meat, dozens of eggs, liters of milk). In this case, it could set prices for a payable amount by adjusting the unit (say, per 2 kg, per 2 dozen, per 2 L). However, merchants usually bargain the price up and down according the customers. This becomes possible under a bilateral trading (the bazaar system). In economic theory, this could be handled in the framework of non-linear pricing under which customers are differentiated according to consumption amounts or to long-term trade relationships.

Secondly, let us think about how Iraqi merchants reacted to the small-change problem, that is, a shortage of small denominations. In general, there could be three reactions: (1) disregard small change, (2) use alternatives to substitute for change. For example, use candy or foreign coins as substitutes for small change, and (3) record transactions in a trading account and settle them after a certain period for repeated customers. For (1), we could consider it as the merchant's pricing behavior. For (2), it is possible and we know this method was used in the past. We will not consider it here. For (3), it can be considered as a type of credit formation.

Thirdly, it may be awkward to consider small-dinar denominations when high inflation devalues the Iraqi dinar substantially. However, when inflation calms down

[9] Basu (1997) and Shy (2000) studied reasons that supermarkets tend to make prices like $99 and $299. This is not only a quantitative adjustment but also a psychological manipulation due to consumers' behavioral reactions.

and the exchange rate returns to the long-run parity level, the value of the Iraqi dinar would go back to the pre-Gulf War standard. In such a case, the Iraqi dinar would have a substantial purchasing power. For example, if the value of the dinar goes up to 100 times the current level, 1 dinar could easily reach 10 yen. For the same reason, without considering the long-run steady-state level, judging solely from the status quo, it is a highly questionable practice to insist that the current 50 dinar should be replaced by a new 1 dinar via rescaling the denominations.[10] It is understandable that the issue cost below the 50 dinar denomination may exceed the face value, so that the issuer (i.e., the government or the central bank) could not earn any seigniorage (may generate a loss to the government). Under this circumstance, if small denominations are issued in the form of metallic coins, these coins would be melted and sold for alternative uses, and we could expect that no metallic coins would be circulated in the market. To avoid this risk, small denominations should be issued in the form of (paper) banknotes. The banknotes will not be used for alternative purposes, and there would be no incentive to produce counterfeits, considering the substantial issue cost.[11]

3.3.1 The Ratio of Payable–Non-payable Amounts and the Ways of Payment in Payable Amounts

The first problem here is to consider a share of a multiple of S (the smallest denomination) over any natural number (N) that the payment amount may take. The total number by which the smallest denomination can generate is N/S. The share of the total number (N/S) over the whole natural number (N) can be expressed as

$$\frac{\frac{N}{S}}{N} = \frac{1}{S} \tag{3.1}$$

[10] In Japan, rescaling denominations is not fully understood. That is to say, we need to distinguish the case in which a country that experienced hyperinflation decided to replace an old 10,000 unit note by a new 1 unit note because the value of the old 10,000 unit note dropped so that it does not make any sense to consider the value below old 10,000 unit note. The case in which a country like Japan decides to replace the current 100 yen by a new 1 yen under the name of rescaling denominations, consumers in Japan can purchase many goods below the current 100 yen level so that the Japanese government must issue supplementary denominations below the new 1 yen. In the latter case, the replacement of the old currency by the new one, in principle, implies a rescaling of currency denominations and it does not change any methods of payments and quantities of currency in circulation. Note that the value of the minimum currency unit 1 cent in the US or the Euro zone is at parity, more or less, with that of the 1yen in a sense: the minimum purchasing power is set as equal among the major international currencies.

[11] Sargent and Velde (2002) pointed out that if the value of metallic coins is higher than the face value, there would be a possibility of their being melted, and if the value of coin is low, then people would refuse to accept it. The metallic value and the face value of coins must be balanced, if the issuer want metallic coins to be circulated normally. Historical episodes indicate that was difficult in reality.

Regardless of the upper limit of the natural number (N), the share is 1/S. Intuitively, if S = 250, you encounter a payable amount with 250 dinar denominations once in every 250 continuous natural numbers. In a numerical expression, 1/250 = 0.004 (0.4%). If the smallest denomination is 50, then 1/50 = 0.02 (2%). If that is 1, then 1/1 = 1 (100%), i.e., with the basic unit denomination (S = 1), you can pay any amount corresponding to the natural number.

If the smallest denomination is reduced from 250 to 50, consumers can pay five times more than before. However, the 50 dinar notes can pay only 2% of the potential amount.

The second problem is to find how many ways there are to settle payment amounts by means of currency denominations. As a rigorous mathematical argument is given in Appendix 3.5, if the payable amount is 1,000, in Japan there are 248,908 ways to pay, given the currency denominations (1, 5, 10, 50, 100, 500 yen coins). In Iraq there are only 6 ways to pay, given the available currency denominations (50, 250, 1,000 dinar notes). Of course, this difference may reflect a difference in the economic value of 10,00 yen and 1,000 dinar. Even for a high value such as 100 thousands or 1 million, it is true that Japanese consumers have more ways to pay than do Iraqi consumers. In general, the more ways you have to settle payments, the less you are restricted on holding denominations in hand. Flexibility of consumers increases as a result. In the case of Iraq, consumers must hold many 50 dinar notes to settle payments, it would be quite inconvenient for consumers in Iraq.

Replacing the old 50 dinar by the new 1 dinar, all prices are adjusted to express in any natural number via the new 1 dinar; however, these are in fact multiples of the old 50 dinar.[12] Retail shops may set prices being equal to a multiple of 50 dinar, the utilities like electricity, water, and gas are priced according to the basic quantities (e.g., price for 1 kw, 1 L, and 1 m^3), and consumption of these quantities may not reflect the payable amounts (multiples of 50 dinar). For the same reason, financial transactions and taxation may not satisfy payable amounts. In the process of globalization, international trading increases and Iraq imports a huge quantities of foreign goods and services, international trade settlements, adjusted through the exchange rate and expressed in the Iraqi currency denomination, may not necessarily be payable amounts.

[12] It is a method of rescaling denominations. As discussed earlier, under an unstable currency value, after rescaling the dinar denominations, a rapid exchange rate appreciation might occur, and a rapid deflation could follow as a result. Thus, if the government introduces a rescaling of denominations, it is desirable to do so in the period of a stable exchange rate. As a matter of practice, if commercial trades in Iraq do not require any currency denomination below 50 dinars, it makes sense to introduce a rescaling denomination in Iraq. If, on the other hand, it requires currency denominations below 50 dinars, supplementary small denominations (coins and notes) need to be issued. It is virtually the same as issuing smaller dinar coins and notes (e.g., 1, 5, 10, 25).

3.3.2 Pricing Under the Allowance of Infeasible Payment Amounts

Suppose the payment amount of a trade is X_i and it is paid by currency denominations. The payment amount is determined independently from the currency denominations. In principle, payment must be made by the smallest number of denominations.

$$X_i = aQ_1 + bQ_2 + cQ_3 + dQ_4 + eQ_5 + fQ_6 + g_i \tag{3.2}$$

where $Q_1 > Q_2 > Q_3 > Q_4 > Q_5 > Q_6 > g_i \geq 0$ Q_i is a currency denomination and g_i is a remainder, that is, an infeasible payment amount.

Equation (3.2) indicates that the optimal payment pattern can be expressed by one of the most fundamental theorems, i.e., the division theorem, in which given any integer and any positive divisor, there is always a uniquely determined quotient and remainder. The principle is simple. Pay as much as possible (a) by the maximum denomination (Q_1), as much as possible of the remainder is paid (b) by the second-largest denomination (Q_2). Repeat the procedure in descending order of Q_i. The optimal payment pattern means that minimum numbers of denominations are used for payment. If $g_i = 0$, the minimum positive value for X_i is equal to gcd(Q_i) = Q_6, where gcd stands for the greatest common divisor (see Appendix 3.5). It is intuitively obvious when you think about a payment by means of currency denominations. That is, as $Q_6 = 50$ and there is no smaller denomination, the minimum payable amount must be 50.

The ratio of infeasible payment amount and total payment Z_i is defined as,

$$Z_i = \frac{g_i}{X_i} \tag{3.3}$$

where $0 \leq g_i < Q_6, 0 \leq Z_i < 1$

This ratio Z_i depends on the payment amount X_i. If X_i is a large amount, Z_i would be negligible. On the other hand, if X_i is a small amount, Z_i would be closer to 1. The problem occurs when the payment amount is relatively small, the ratio Z_i would not be negligible. For example, if the payment amount is 299 dinars and the infeasible payment amount is 49 dinars, then the ratio Z_i becomes 0.16388. If the merchant dismisses 49dinar from the payment, it implies that the consumer purchases goods or services with 16% discount.[13]

$$299 = 1 \times 250 + 49$$

$$Z_i = 49/299 = 0.16388 \tag{3.4}$$

[13] On the other hand, if the merchant raises the payment amount to 300 dinars, the consumer can pay the exact amount with 250 dinar and 50 dinar notes. The consumer loses 1/300 = 0.003 (0.3%).

Consumers may experience effective discounts and price increases (in case of raising the markup) within a certain range.

Let us think about a large payment amount. For example, to purchase 400 units of 299 dinar goods.

$$299 \times 400 = 478 \times 250 + 50 \times 2 \tag{3.5}$$

$$Z_i = 0/119600 = 0.00 \tag{3.6}$$

In this case, there is no infeasible payment amount, thus no discount.[14] As 50 dinars is the minimum denomination, if all trading units are fixed at 50, then all payment amounts can be payable without any remainder.

If the merchant dismisses the remainder or raises the price (marking up) or bunches goods just to avoid the remainder, the consumer may decide whichever payment method is most beneficial. For example, in the above dismissal case, the merchant dismisses 49 dinars for the payment of 299 dinars. The consumer is better off to purchase goods separately in the minimum units of 299 dinars, even if he or she wants 400 units. The consumer earns benefits of 19,600 dinars ($= 49 \times 400$) by dividing the purchases. If there is no uncertainty about shortage of goods and the consumer is allowed to trade repeatedly, a large trade tends to be divided into many small trades.

However, this may not happen in the real world. That is, the merchant may notice that the same consumer purchases the same goods many times, so he or she reduces a discount rate or raises a price to eliminate benefits from the small trading.[15] The advantage in the bilateral transaction like those of the bazaar in the Middle East rests on observability of customers and the possibility of price and quantity bargaining between merchants and customers, as contrasted with anonymous trade in the supermarket. Economically, this bilateral trade is a system in which merchants would not lose or gain one-sidedly.

The merchant's price-setting behavior can be theoretically described as flows,

with long-time customers (N_α) and one-time shoppers (N_β). There is only one merchant and one set of goods (x) for consumers (N_α, N_β). Of course, there are many merchants and consumers in the markets. Neither monopoly nor monopsony exists. The long-time customers' demand for x is x_α, and his or her price elasticity of demand is ε_α. The one-time shoppers' demand for x is x_β and his or her price elasticity of demand is ε_β.

The consumer maximizes his or her consumption, given the budget constraint. We ignore substitutional consumption and prices of other goods.

$$\max U_i(x) s.t. y_i = p_i x_i \leftrightarrow x_i = V_i(p, y) i = \alpha, \beta \tag{3.7}$$

[14] Even for a small payment, if small payment amounts are booked and settled after a certain period, the case would be similar to a large payment case. This will be discussed later.

[15] As we will discuss below, the merchant treats long-time customers and the one-time shopper differently.

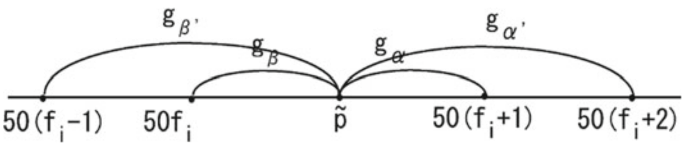

Fig. 3.1 Schedule for payable price

where $V_i(p, y)$ is an indirect utility function.

The long-time customer continues shopping at the same retail shop while many shops are available. In this case, the price elasticity of demand is lower than that of the one-time shopper ($\varepsilon_\alpha < \varepsilon_\beta$).

The self-employed merchant maximizes his/her profit, assuming that there is no other cost than the purchasing cost of goods.

$$\max\{\sum_{i\in\alpha,\beta} p_i x_i N_i - \tilde{p}\overline{X}\} \tag{3.8}$$

where $\tilde{p}$ is the wholesale price (producer price). The marginal cost is the average cost.[16] $\overline{X}$ is the total amount of goods x_i. The merchant knows the demand functions of both the long-time customer and the one-time shopper.

We need to consider an additional constraint, that is, the payable amount is restricted to a multiple of 50 dinars that is the minimum denomination.

Let us restrict the argument for small transactions in which we consider the wholesale price $\tilde{p}$ is divided into a multiple of 50 dinars (f_i) and a remainder part (g_i). i.e., $\tilde{p} = 50f_i + g_i$ where $f_i \epsilon \{0, 1, 2, 3, \ldots\}$, $g_i \epsilon \{0, 1, 2, 3, 4, \ldots .49\}$. The minimum payable amount above the wholesale price is $p = 50(f_i + 1)$ and the maximum payable amount below the wholesale price is $p = 50f_i$. See Fig. 3.1 for illustration.

Under perfect competition without any payment restrictions, the first best equilibrium price would be marginal cost = purchasing cost (wholesale price) $\tilde{p}$; all goods would be sold at that price. However, because of payment restrictions, the first best equilibrium price cannot be used; the merchant must set price $p = 50(f_i + 1)$ or higher for at least a part of his or her sales in order to avoid making losses.

It is inefficient, from the consumer surplus point of view, to sell goods at the same price when demand functions differ between the long-time customers and the one-time shoppers. Furthermore, other competitive merchants would try to steal some customers by selling goods at the lower price, $p = 50f_i$ which is below the wholesale price $\tilde{p}$. Under this circumstance, given two different demand functions, thus two different price elasticities of demand, the merchant sets prices following the Ramsey rule, in which the price is determined by a proportion to the inverse of price elasticity of demand ($\varepsilon_\alpha$and$\varepsilon_\beta$respectively) and a number of consumers

[16] According to Iwasaki (2004), the wholesale price (producer price) in the textile industry in Teheran, Iran, was so competitive that the merchants faced, more or less, the same price. We assume the wholesale price in Bagdad, Iraq, is the same for all merchants.

(N_α and $N_\beta$respectively) and by setting the price to level or to just exceed the purchasing cost (i.e., $\tilde{p} \times x_i$). That is, if the market is reasonably competitive and the wholesale price $\tilde{p}$ cannot be payable by denominations, the two or more payable prices above (e.g., $p = 50(f_i + 1)$) and below (e.g., $p = 50 f_i$) the wholesale price $\tilde{p}$ are used. This is an example of non-convergent to one price for one good under the competitive market.

In this example, the long-time customers have to pay a price higher than the marginal cost. How high would the price level be ? It depends on the price elasticity of the long-time customers and the number of them and also depends the lower price level set for the one-time shoppers which, in turn, depends on the price elasticity and the number of them. Figure 3.2 illustrates this case. The demand curve of the long-time customers is V_α, which implies a low price elasticity (the slope of the demand curve is steep). It implies that demand of the long-time customers would not lose (X_α) as much, therefore the deadweight loss would not be as large (pink triangle in Fig. 3.2), even if the price p_α is set above the wholesale (equilibrium) price $\tilde{p}$. The demand curve of the one-time shoppers V_β implies a high price elasticity (a flatter demand curve) and thus the deadweight loss would be very large if the price is set at p_α. On the other hand, if the price is set at p_β below the wholesale price $\tilde{p}$, the one-time shopper's demand (X_β) expands substantially. As is clear from the illustration in Fig. 3.2, it is better to allocate p_α for the long-time customers and p_β for the one-time shoppers. The market competition continues until the merchant's marginal profit reaches zero. To accomplish this, we can imagine that the merchant gains revenues by raising the price to $p_{\alpha,}$ and by losses from demand shrinkage, the net gain/loss from long-time customers is calculated. By the same token, as the merchant loses by reducing the price to p_β and gains from demand increases, the net loss/gain from one-time shoppers is calculated. These two-sided gains and losses will cancel out each other and reach zero net profit. However, because of price discontinuity, the payable price may not coincide with the price at which the marginal profit reaches zero from Eq. (3.8).

We assume that the demand function take a standard functional form,

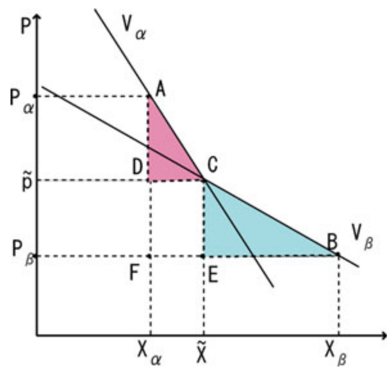

Fig. 3.2 Price setting under two demand curves

$$V_i = x_i = k_i p_i^{-\varepsilon_i} \tag{3.9}$$

where k_i is a parameter that determines the size of demand, and ε_i is the price elasticity of demand.

The Ramsey price corresponding to this demand function is defined as,

$$p_i = \frac{\varepsilon_i \tilde{p}}{\varepsilon_i - \lambda_i} \tag{3.10}$$

To solve (3.10) for λ_i,

$$\lambda_i = \frac{p_i - \tilde{p}}{p_i} \varepsilon_i i = \alpha, \beta \tag{3.11}$$

In general, $\frac{p_i - \tilde{p}}{p_i}$ is known as the markup rate.[17] This is equivalent to the ratio of infeasible payment and total payment,Z_i[18] as shown in Eq. (3.3). Inserting Z_i and rewriting Eq. (3.11),

$$\lambda_i = Z_i \varepsilon_i i = \alpha, \beta \tag{3.12}$$

Equation (3.12) implies that λ_i[19] is uniquely determined by Z_i andε_i. Furthermore, with payable price p_i as a multiple of 50 dinars, the wholesale price $\tilde{p}$, and price elasticity ε_i, λ_i is determined by Eq. (3.11). That is to say, by changing λ_i we can express any payable price that satisfies the Ramsey price. Here we divide consumers into two groups. If we could further divide them into more groups, we could assign the Ramsey prices respectively.[20] It is certainly efficient to introduce two prices p_α and $p_{\beta,}$ above and below $\tilde{p}$ the wholesale price (i.e., $p_\alpha > \tilde{p} > p_\beta$) in the sense that an increase in the consumer surplus is expected to exceed a reduction of the producer surplus under the two-price market, compared with the case in which the payable price is fixed at $p_i = 50(f_i + 1)$ and apply this price for all consumers. In other words, assuming many merchants sell the same goods and new merchants enter the market as far as the marginal profit remains positive, the merchant would not survive in the market competition if it applied the one good to one price.

We can observe the price-setting behavior of the merchant from a different perspective. Under the one good–two (many) prices mechanism, it is essential to

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17 According to Iqbal (1997), in Islamic society, interest income is prohibited in commercial trading. As a result, most profits are generated by markups,

18 Note that in Eq. (3.2), we restrict case $X_i = f Q_6 + g_i$.

19 In general, the Ramsey rule assumes λ_i is constant for the same goods. But as we treat different prices for different consumers for the same goods, we assume $\lambda_\alpha \neq \lambda_\beta$.

20 To set the price of individual goods as a multiple of 50 dinars is a sufficient condition for total payment (after aggregating several goods) becoming a multiple of 50 dinars but not a necessary condition. Given the possibility of purchasing a single good, all goods are priced as a multiples of 50 dinars. In fact, it was said that most goods in Iraq were priced in multiples of 50 dinars.

distinguish consumers with different demand functions. From our viewpoint, one of the most suitable verifying mechanisms is to use the bilateral (face to face) trade in the bazaar. In Arabic and Persian societies, the main trades take place in the bazaars, and the trading styles and divisions of labor are well defined.[21]

The advantage of bilateral trade in the bazaar is its ability to accumulate customer-specific information. Asymmetric information between the merchant and long-time customers would be substantially reduced by the repeated bilateral trades. In so doing, the merchant can set two (many) prices according to the types of consumers. This result reflects functioning in the bazaar as stated above. In short, the merchant cannot make profits one-sidedly in the bazaar; some consumers can purchase goods at prices lower than the wholesale price.

Another familiar retail trading system is the supermarket style. In this system, mass consumers buy "one good, one price" without any differentiation among different consumers. In order to make positive profits, the merchant (supermarket manager) must set prices at least, $\mathrm{p} = 50(f_i + 1)$ for all consumers. This trade is profitable for the merchant, but consumers face the higher average price than that in the bazaar. In this trade, consumers are price takers regardless of demand amounts, there is no information feedback from consumers to the merchant (supermarket manager).[22]

In general, the merchant cannot capture the entire demand information of individual consumers, thus asymmetric information between them remains somehow. However, consumers can be classified according to the amount of consumption. In the literature of industrial organization, in general and in public utility pricing in particular, for example, a residential electricity tariff or telephone calls or mail services often use efficient pricing such that, given a regulated firm that must break even and which serves M markets, the efficient set of prices $p_\alpha, p_\beta, p_\gamma, \ldots p_M$ is that set which maximizes total surplus subject to the constraint that the firm earns zero profit (Brown and Sibley, 1986, p. 37) (see Fig. 3.3).[23]

In case of electricity use or telephone calls, the amount of consumption (number $\times$ time of calls) and firms and individuals are identified, the public utility firms can differentiate individuals and firms, even if specific individuals who use electricity or telephone services may not be known. In other words, it is justifiable to differentiate consumers according to amount of consumption if and only if the amount of consumption and types of consumers correspond one to one. However, in the case of general consumption goods, such a relationship does not hold. For example, if the types of consumers are categorized not by price elasticity but by the amount of consumption, the large-scale consumers would be provided the lower price. We can easily imagine that small-scale consumers or firms can collusively form a cartel (e.g., residents in a large condominium or firms using a high-rise office building) and

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[21] Iwasaki (2004) reports detailed functioning and divisions of labor in the large-scale bazaar in Teheran, Iran. Similar trade systems are found in Istanbul(Turkey), Cairo (Egypt), and Bagdad (Iraq), among others.

[22] In principle, e-commerce tries to capture detailed information on individual customers by means of expending trading platforms that play an equivalent role of the bazaar.

[23] See Brown and Sibley (1986) and Shy (1995, Chap. 13).

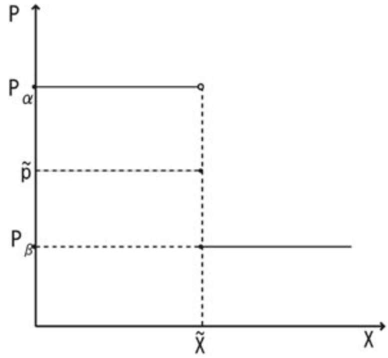

Fig. 3.3 Nonuniform price schedule

obtain goods and services at a discounted price. Without monitoring in person (e.g., bilateral trade in the bazaar), it is almost impossible to differentiate consumers by the amount of consumption in which time and places of individual consumption are not identified.

3.3.3 Pricing Under Sales on Credit and Risks on Settlement

As discussed above, it is possible to make an average payment price closer to the wholesale price (marginal cost) by pooling payment amounts and reducing the ratio of infeasible payment amount and total payment. However, if payment amounts are pooled indiscriminately, default risks at the time of settlement would increase as a result. That is, by pooling payments, distortion due to the limited denominations is eliminated to some extent; however, the merchant would certainly face higher risks. How can these two elements compromise each other?

In general, the merchant decides the timing of settlement according to the observable behavior of consumers, in other words, the length of credit is determined by the degree of default risk of consumers. In case of the one-time shopper, the merchant requires on-the-spot (real-time) payment. In case of long-time customers, for example, the other merchants in the same bazaar or the customers with stable income, the merchant can fix the time of settlement according to the merchant's payment schedule without a high default risk. Even if the customers come to the retail shop often, as long as the merchant judges the customers' income flows are not secure or stable, then he might ask for on-the-spot settlements. The merchant allows such a payment style according to creditworthiness (i.e., allows it from lower-risk consumers). He does not accept credit for risky individuals.[24]

[24] Iwasaki (2004) argues that in Iran, the producer and the wholesaler, the wholesaler and the retailer usually use a consignment sales format and they also use a few months' commercial bill rather than cash for settlement. In order to properly function, the sales on credit between the merchants and the customers and among the merchants, the long-time relationship between the merchants and the producers with consignment sales agreement is also very important. In a sense, the merchant plays

Suppose the average prices of a part of long-time customers are reduced by sales on credit to maintain the constraint in Eq. (3.8). The average prices of the rest of the long-time customers (e.g., salaried workers, elderly people, widows) must rise. In this circumstance, we may face "one good, three (more) prices",[25] namely, one-time shoppers pay a lower price than the wholesale price, then the long-time customers with stable income pay the second-lower price, and finally the customers with unstable income pay the highest price. It is a paradoxical result.[26]

Finally, let us consider the payment risks in society, rather than individual payment risks for individual goods—that is to say, the ratio between infeasible payment amounts and socially aggregate total payment for various goods with various prices. The socially infeasible payment ratio is given as follows,

$$W_i = \frac{\sum g_i}{\sum X_i} \tag{3.13}$$

For example, of all total payments between 1 and 1,000, the socially infeasible payment amounts consist of the numbers with the last two digits take 1–49.[27] To calculate this example, $\sum g_i = 24{,}500$, $\sum X_i = 500{,}500$, then $W_i = 0.04895$. The socially infeasible payment ratio is 4.9%. If we increase the total payment amount, this ratio will drop. This is not an example of aggregate small payments of the same goods to yield a large payment. We could interpret it to mean that individual small payments for different goods are aggregated to a large payment. With this method, the socially infeasible payment ratio can drop. However, in reality, this type of payment aggregation seems difficult to implement because usually commercial trades are decentralized by firms, retail shops, and merchants and it may not be feasible to aggregate individual payments socially in practice.

As we see, if a payment contains a remainder part, the merchant decides payment amounts by changing prices according to the customers' financial positions. This seemingly inefficient procedure prevents the merchant or the consumers from losing profits one-sidedly. This type of bilateral trade is allowed only to a limited extent. As the economy is modernized and expands its size, it is necessary to remove the payment and accounting obstacles in which the currency denomination system generates infeasible payment amounts.

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the role of intermediary, like a bank, as a financial intermediary. It is noteworthy that commercial bills and checks were an innovation of Islamic merchants in the tenth century. The English word "check" originally came from the Arabic word *cekk*.

[25] If the long-time customers know that the prices they face may differ according to creditworthiness or credit history, some of them may not be happy with this consignment sales format. From the customers' point of view, price differences between those who pay on-the-spot and those who pay on credit may not be observable, so it will not be a big problem.

[26] It may resemble the optimal consumption tax result in which the low price elasticity of demand for necessity goods such as food charges a higher tax rate and the high price elasticity of demand for luxury goods charges a lower rate.

[27] In case of the last two digits 51–99, as 50 dinara can be paid, the remaining amount would be 1–49, which is truly an infeasible amount.

3.4 Conclusion

The future of Iraq depends on how the government steadily constructs or reconstructs economic and social institutions that citizens will accept. Finance is no exception. The most important tasks are to maintain a credible central bank and, in turn, the central bank maintains the currency value and stability of the financial system. As long as achieving a stable currency value and keeping a low inflation rate, citizens will start saving and investing in the financial institutions and as a result, the financial market will induce further financial development. As the economy normalizes, Iraq will start exporting oil and importing various goods and services. The profits from oil would be invested abroad. In such a case, the government must face questions about how to introduce an acceptable foreign exchange law and how to maintain the foreign exchange system. It is an important policy question as to whether the government implements the exchange rate agreement with neighboring countries or the free-floating exchange system or the currency board system.

In this chapter, we have not discussed a broad financial market design as such. In a sense, the issue we discussed here is more profound and is highly relevant to the digital economy. At the moment, the modern theory of optimal pricing of goods and services uses the marginal cost and the price elasticity and assumes any price can be expressed in a set of currency denominations. In Iraq, however, this pricing method does not work because of limited denominations. As a practical reaction to this institutional obstacle, the merchants seem to use bilateral trade and prevent the market participants from losing profits one-sidedly. Nonetheless, the limited denominations prevent holding the "one good, one price" principle and cause many inconveniences in economic contracts that restrict payment feasibility greatly. In short, this currency system stands squarely in the way of the modernization of the Iraqi economy.

It may seem trivial to provide a 1 dinar denomination as the basic unit of accounting in their currency system in Iraq. As the production cost exceeds the face value at the moment, no one would dare to provide a 1 dinar denomination. As is discussed extensively in this chapter, a potential demand for small denominations would not be negligible.

As an aid for Iraq's reconstruction, the Japanese government can consider not only providing finance projects and their personnel, but also assisting in printing small-denomination notes such as 1 dinar, 5 dinar, and 10 dinar notes by the National Printing Bureau of Japan.[28] As the production cost exceeds the face value, there is no incentive to make counterfeits, so it is safe to use. As they are banknotes rather than coins, there is no risk of melting them and selling the materials. From the viewpoint of institutional complementarity, to improve the currency system and to make the commercial trading system more efficient, they will complement reconstruction of Iraqi society. It may not be a bad form of assistance, considering the next 20 or 30 years as our time horizon.

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[28] We all know the British De La Rue is the private company printing Iraqi banknotes for decades. Printing small denominations for Iraq may not make economic sense from the private sector's viewpoint.

3.5 Appendix: Solution for an Indeterminate Equation and Its Application

According to Yamamoto (2003, p. 17) and Kitamura (1999, pp. 35–36), a set of integers $(x_1, x_2, x_3, x_4, \ldots .x_n)$ is a solution for the algebra equation with integer coefficients f$(Q_1, Q_2, Q_3, \ldots .Q_n)$ where $Q_1, Q_2, Q_3, \ldots Q_n$ are variables. This type of algebra equation is called an indeterminate equation. Finding all solutions for integer coefficients is described as solving an indeterminate equation.

Theorem 3.1 Suppose d is the greatest common divisor (gcd) for integers $Q_1 > Q_2 > Q_3 > Q_4 > \ldots.., > Q_n$, then an indeterminate equation, $x_1Q_1 + x_2Q_2 + x_3Q_3 + x_4Q_4 + \ldots + x_nQ_n = kd$ having a solution. It is equivalent to finding d as the greatest common divisor for integers $Q_1, Q_2, Q_3, Q_4, \ldots, Q_n$.

Proof 3.1 It is obvious that an indeterminate equation kd has a solution; then the greatest common divisor is d. On the other hand, if d is the greatest common divisor for kd, given a set of solution for an indeterminate equation, $x'_1Q_1 + x'_2Q_2 + x'_3Q_3 + x'_4Q_4 + \ldots. + x'_nQ_n = d$, then k times the solution $(kx'_1, kx'_2, kx'_3, \ldots k'x_n)$ is also a solution. (Q.E.D.)

The general solution for an indeterminate equation is given as below. First, two largest variables, Q_1 and Q_2. Suppose the greatest common divisor of Q_1 and Q_2 is defined as $d_1(Q_1,Q_2)$, an indeterminate equation $x_1Q_1 = x_2Q_2 = md_1 = kd\text{-}x_3Q_3\text{-}x_4Q_4\text{-}\ldots\text{-}x_nQ_n$ (where m is an integer) has a solution $(\widehat{x_1}, \widehat{x_2})$, the general solution can be defined as such,

$$x_1 = \widehat{x_1} + Q_2t \tag{3.14}$$

$$x_2 = \widehat{x_2} + Q_1t \text{where t} = 0, \pm 1, \pm 2 \tag{3.15}$$

As $x_1Q_1 = x_2Q_2 = md_1$, this indeterminate equation can be rewritten as:
$md_1 + x_3Q_3 + x_4Q_4 + \ldots + x_nQ_n = kd$.

Second, suppose the greatest common divisor of d_1 and Q_3 is $d_2(d_1, Q_3)$,
$md_1 + x_3Q_3 = sd2 = kd\text{-}x_4Q_4 - \ldots - x_nQ_n$ (where s is an integer) has a solution, the general solution can be defined as such,

$$m = \hat{m} + Q_3t \tag{3.16}$$

$$x_3 = \widehat{x_3} - d_1t \text{where t} = 0, \pm 1, \pm 2 \tag{3.17}$$

Third, repeat the same procedures and finally reach the greatest common divisor of $d_{n\text{-}2,}$ and Q_n is d, the indeterminate equation $zd_{n\text{-}2} + x_nQ_n = kd$ has the general solution such that

$$z = \hat{z} + Q_nt \tag{3.18}$$

$$x_n = \widehat{x_n} - d_{n-2}t \text{where t} = 0, \pm 1, \pm 2 \tag{3.19}$$

We have obtained the general solution for $x_1, x_2, x_3, x_4, \ldots x_n$..

If we take $Q_1, Q_2, Q_3, Q_4, \ldots. Q_n$ as currency denominations and kd as the payment amount, it is the same as figuring out how to pay the amount by means (integer) of different denominations. Suppose we have six denominations, that is, $Q_1 = 25{,}000$, $Q_2 = 10{,}000$, $Q_3 = 5{,}000$, $Q_4 = 1{,}000$, $Q_5 = 250$, and $Q_6 = 50$. The greatest common divisor in this currency system is $\gcd(Q_i) = 50$.

Equation (3.2) is rewritten as,

$$X_i - g_i = aQ_1 + bQ_2 + cQ_3 + dQ_4 + eQ_5 + fQ_6$$

$$= \gcd(Q_6) \times \{500a + 200b + 100c + 20d + 5e + f\} \tag{3.20}$$

The minimum positive value for X_i is obtained when a = b = c = d = e = 0, and f = 1, $X_i - g_i = \gcd(Q_6) \times 1 = 50$. This result can be interpreted to mean that economic value X_i is converted into payable monetary value $X_i - g_i$ by discounting g_i. From our daily experience, there are many ways to pay a certain amount by currency denominations, allowing changes.

Let us think of the 6-element first-order indeterminate equation,

$$X_i - g_i = 50f_i + 250e_i + 1000d_i + 5000c_i + 10000b_i + 25000a_i \tag{3.21}$$

Payable monetary value $X_i - g_i$ is restricted to the case of multiples of 50, i.e., $50n$. For simplicity, we ignore the case of making changes.[29] How many ways can we pay, given all available currency denominations?[30]

Let us consider the small denominations first, contrary to the general solution for indeterminate equations:

(1) To pay 250 m dinar with 50 and 250 dinar notes[31] (i.e., two denominations).
$250\, m = 50f_i + 250e_i$

$$f_i = 5m - 5e_i = 5(m - e_i) \tag{3.22}$$

Define $N_2(m)$ as the number of solutions (0 and positive integers) for e and f in the indeterminate equation (A9). As f_i is a multiple of 5, put $f_i = 5k$ and

[29] Allowing changes implies negative integer solutions in the indeterminate equation. The number of solutions increase as a result, but mathematical fundamentals remain the same.

[30] This problem is basically the same as discussed in Yamamoto (2003, p. 22, 354–355). See also the original contribution in this problem by Polya (1956) and its modern treatment by Graham et al. (1994, Chap. 7).

[31] The payable amount is set at 250 m because this amount can be paid either by 250 dinar notes or multiples of 50 dinar notes (250mk × 50n). In the following, the payable amount is considered the multiple of the maximum denomination.

then $e_i = m - k$. From $e_i \geq 0$, $f_i \geq 0$, $0 \leq k \leq m$. If k is fixed, then $f_i ande_i$ are automatically determined. The number of solutions in this case is $m + 1$.

$$N_2(m) = m + 1 \tag{3.23}$$

(2) To pay 1,000 m dinars with 50, 250, and 1,000 dinar notes (i.e., three denominations).
$1000\ \mathrm{m} = 50 f_i + 250 e_i + 1,000 d_i$

$$f_i + 5e_i = 20(m - d_i) = 20k \tag{3.24}$$

Define $N_3(m)$ as the number of solutions, $d_i = m - k$. The same as the two-denomination case (1) above, $0 \leq k \leq m$. For this amount (1,000 m) in the case of two denominations, the number of solutions are $N_2(4\ k)$.

$$N_3(m) = \sum_{k=0}^{m} N_2(4k) = \sum_{k=0}^{m} (4k + 1) = (2m + 1)(m + 1) \tag{3.25}$$

(3) To pay 5,000 m dinars with 50, 250, 1000 and 5,000 dinar notes (i.e., four denominations),

$$5000m = 50 f_i + 250 e_i + 1000 d_i + 5000 c_i$$

$$f_i + 5e_i + 20d_i = 100k \tag{3.26}$$

From (3.26), $c_i = m - k$. The same logic follows $0 \leq k \leq m$. For this amount (5,000 m) in case of three denominations, the number of solutions is $N_3(5\ k)$

$$\begin{aligned} N_4(m) &= \sum_{k=0}^{m} N_3(5k) = \sum_{k=0}^{m} \{(10k + 1)(5k + 1)\} \\ &= 50 \sum_{k=0}^{m} k^2 + 15 \sum_{k=0}^{m} k + (m + 1) \\ &= \frac{1}{6}(m + 1)\left(100m^2 + 95m + 6\right) \end{aligned} \tag{3.27}$$

(4) To pay 10,000 dinars with 50, 250, 1,000, 5,000, 10,000 dinar notes (i.e., 5 denominations).
$10000\ m = 50 f_i + 250 e_i + 1,000 d_i + 5,000 c_i + 10,000 b_i$

$$f_i + 5e_i + 20d_i + 100c_i = 200k \tag{3.28}$$

From (3.28), $b_i = m - k$. The same logic follows, $0 \leq k \leq m$. For this amount (10,000 m) in case of three denominations, the number of solutions is

$N_4(2\,k)$.

$$N_5(m) = \sum_{k=0}^{m} N_4(2k) = \sum_{k=0}^{m} \frac{1}{6}\left(800k^3 + 780k^2 + 202k + 6\right)$$

$$= \frac{400}{3}\sum_{k=0}^{m} k^3 + 130\sum_{k=0}^{m} k^2 + \frac{101}{3}\sum_{k=0}^{m} k + (m+1)$$

$$= \frac{1}{6}(m+1)\left(200m^3 + 460m^2 + 231m + 6\right) \tag{3.29}$$

(5) To pay 25,000 dinars with 50, 250, 1,000, 5,000, 10,000, and 25,000 dinar notes (i.e., 6 denominations).

25,000 $m = 50f_i + 250e_i + 1,000d_i + 5,000c_i + 10,000b_i + 25,000a_i$

$$f_i + 5e_i + 20d_i + 100c_i + 200b_i = 500k \tag{3.30}$$

From (3.30), $a_i = m - k$. The same logic follows, $0 \le k \le m$. For this amount (25,000 m) in case of three denominations, the number of solutions is $N_5(\frac{5}{2}\,k)$.

$$N_6(m) = \sum_{k=0}^{m} N_5\left(\frac{5}{2}k\right) = \sum_{k=0}^{m} \frac{1}{6}\left(\frac{5}{2}k + 1\right)\{200(\frac{5}{2}k)^3 + 460(\frac{5}{2}k)^2 + 231\left(\frac{5}{2}k\right) + 6\}$$

$$= \frac{400}{3}\sum_{k=0}^{m} k^3 + 130\sum_{k=0}^{m} k^2 + \frac{101}{3}\sum_{k=0}^{m} k + (m+1)$$

$$= \frac{1}{48}(m+1)\left(125000m^4 + 39375m^3 + 34225m^2 + 6045m + 48\right) \tag{3.31}$$

When $m = 1$, $N_6(1) = 3841$, $m = 2$, $N_6(2) = 41{,}502$.

In case of six denominations in Japan (1, 5, 10, 50, 100, 500 yen coins), to pay 500n, we have a similar solution such that

$$N_6(n) = \frac{1}{6}(n+1)\left(12500n^4 + 29372n^3 + 15800n^2 - 195n + 6\right) \tag{3.32}$$

When n $= 1$, $N_6(1) = 19,162$, $n = 2$, $N_6(2) = 248{,}908$. With the same six denominations, although denomination units are different between Iraq and Japan, there is the same face-number 500—in the case of Iraq, $N_2(2) = 3$, while in Japan it is $N_6(1) = 19,162$. To pay 25,000, in Japan with many small denominations, $n = 50$, $N_6(50) = 695{,}609{,}104{,}676$. In Iraq it is $m = 1$, $N_6(1) = 3841$. It is obvious that with many small denominations, Japan has overwhelmingly many ways to pay the same number (amount).

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