Designing Allocation Rules in Economic Problems

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Abstract

How should we divide a resource among a group of agents who have conflicting claims for it? A typical example is the bankruptcy problem: The liquidation value of a bankrupt firm must be divided among creditors. An alternative example with the same mathematical structure is the taxation problem: The cost of some public projects must be collected by a government through taxing incomes. An awards vector determines the division of the resource among the agents. An allocation rule, or simply a rule, is a function that associates an awards vector to each problem of this kind. Our goal is to construct "good" rules. By that, we intend to learn how to achieve one of the Sustainable Development Goals (SDGs) (Goal 16: Peace, Justice and Strong Institutions). In our study of rules, we follow the axiomatic approach. That is, we (i) formulate rules (mathematically), (ii) introduce "desirable" properties of rules (called axioms), and (iii) study those implications (e.g., identify rules that satisfy those properties). We present some well-known characterization results based on the properties analyzed.

Keywords

Economic problems · Claims problems · Allocation rules · Axiomatic approach

14.1 Introduction

The problem of allocating scarce resources among agents is called an "economic problem." The resource could be a variety of goods, including natural resources, foods, clothes, houses, etc. It could also be economic services, money, time, or human resources. An "allocation rule," or simply a rule, is a systematic way to determine an answer (or a recommendation) to such an economic problem. More precisely, it is a mapping that associates an allocation with each economic problem. Our goal is to study how to design "good" rules. By that, we intend to learn how to achieve Goal 16 (Peace, Justice and Strong Institutions) of the Sustainable Development Goals (SDGs). Designing good rules for economic problems makes each nation wealthy and achieves world peace. Fairness (equity) is the key to designing good rules in this chapter. However, how can we construct a fair rule? How can we evaluate whether a rule is fair? We adopt an axiomatic approach. That is, we (i) formulate rules (mathematically), (ii) introduce "desirable" properties of rules (called axioms), and (iii) study

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those implications (e.g., identify rules that satisfy those properties).

The value of the designed rule depends on the types of economic problems that we face. Among the many economic problems, we consider one of the simplest economic problems called the "claims problem" (O'Neill 1982), described as follows.[1] There are a set of agents and a resource to divide among them. Each agent has a claim over the resource, but the quantity of the resource is not sufficient to cover all claims. We assume that the resource and each agent's claim are non-negative real numbers. The question is how to allocate the resource to the agents. For instance, suppose that there are three agents: agents 1, 2, and 3. Let $c_1 = 100$, $c_2 = 200$, $c_3 = 300$ be the claims of agent 1, agent 2, and agent 3, respectively, and $E = 250$ be a resource to divide. As $c_1 + c_2 + c_3 > E$, the resource is clearly not sufficient to cover all claims. However, we still have to divide $E = 250$ among the agents. A typical example of this is bankruptcy: The liquidation value of a bankrupt firm has to be divided among creditors. Another example with the same mathematical structure is taxation, where the cost of some public projects must be collected by a government through taxing incomes. Here, each agent's claim is considered the agent's pre-tax income, while the resource corresponds to the amount of money needed to implement the project. However, we will use language based on the bankruptcy interpretation.

When we allocate a resource to the agents, we require that each agent receives a non-negative amount that is at most as large as their claim. Additionally, the resource is entirely divided among the agents (i.e., the total amount that agents receive is equal to the resource). In our previous three-agent example, let x_1, x_2, and x_3 be the amounts agents 1, 2, and 3 receive, respectively. Then, the above requirements can be expressed as $0 \leq x_1 \leq c_1, 0 \leq x_2 \leq c_2, 0 \leq x_3 \leq c_3$ and $x_1 + x_2 + x_3 = E$. We call such (x_1, x_2, x_3) an awards vector. A rule is a function that associates with each claims problem an awards vector. As mentioned before, we adopt an axiomatic approach to analyze the rules. We study the implications of several important properties of rules and present some well-known characterization results based on these properties.

[1] See Thomson (2003, 2015, 2019) for an extensive survey on this subject.

14.2 Model

Here, we introduce a formal model. There is an infinite set of "potential" agents indexed by the natural numbers $\mathbb{N}$. For each case, a finite number of them are present. Let $\mathcal{N}$ be a set of finite subsets of $\mathbb{N}$. The set $N \in \mathcal{N}$ is called the **set of agents**. For instance, $N = \{1, 2, 3, 4\}$. There is a **resource** $E \in \mathbb{R}_+$ that needs to be divided among agents.[2] Each agent $i \in N$ has a **claim** over the resource. Let $c_i \in \mathbb{R}_+$ be the claim of agent i. Let $c = (c_i)_{i \in N}$ be the claims vector. If $N = \{1, 2, 3, 4\}$, $c = (c_1, c_2, c_3, c_4)$. We assume that the resource is not sufficient to honor all claims, that is, $\sum_{i \in N} c_i \geq E$.[3] We allow the equality $\sum_{i \in N} c_i = E$ for convenience. A claims problem, or simply a **problem**, with agent set $N \in \mathcal{N}$ is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$.[4] Let $\mathcal{C}^N$ be the set of all claims problems. If $N = \{1, 2, 3, 4\}$, $c = (5, 10, 20, 30)$, and $E = 50$, then $(c, E) = (5, 10, 20, 30; 50) \in \mathcal{C}^N$ since $\sum_{i \in N} c_i \geq E$. On the other hand, given the same N and c, if $E = 70$, then $(c, E) = (5, 10, 20, 30; 70) \notin \mathcal{C}^N$ since $\sum_{i \in N} c_i < E$.

We impose the following restrictions on the amount that the agents can receive: Each agent should not receive a negative amount; each agent should not receive more than their claim; and the total amount that agents receive should be equal to the resource. A list of amounts satisfying these requirements is called an awards vector. Formally, an **awards vector** for $(c, E) \in \mathcal{C}^N$ is a

[2] By $\mathbb{R}_+$, we mean the set of non-negative real numbers.

[3] If $N = \{1, 2, 3, 4\}$, $\sum_{i \in N} c_i = c_1 + c_2 + c_3 + c_4$.

[4] By $\mathbb{R}^N_+$, we mean the Cartesian product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N.

vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ such that (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$.[5] Let $X(c, E)$ be the set of awards vectors for $(c, E) \in \mathcal{C}^N$. If $N = \{1, 2, 3, 4\}$, $c = (5, 10, 20, 30)$, $E = 50$, and $x = (3, 7, 13, 27)$, then $x \in X(c, E)$. On the other hand, under the same N, c, and E, if $x = (3, 12, 15, 20)$, then $x \notin X(c, E)$ since agent 2 receives more than her claim ($x_2 > c_2$). Again, under the same N, c, and E, if $x = (3, 7, 10, 15)$, then $x \notin X(c, E)$ since the resource is not entirely allocated ($\sum_{i \in N} x_i < E$). A **rule**, denoted generically φ, is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a vector $x \in X(c, E)$. We introduce several examples of these rules in the next section.

The **bankruptcy problem** is a typical application of the situation we have considered. Suppose that a firm goes bankrupt. Let N be the set of creditors of that firm, c_i be creditor i's amount of claim, and E be the liquidation value of the firm. Naturally, $\sum_{i \in N} c_i \geq E$. Then, a rule determines how to allocate the liquidation value to the creditors. Another application with the same mathematical structure is the **taxation problem**. Consider the case in which a government has to collect money from the people in the nation by taxing their incomes. Let N be the set of people in the nation, c_i be the person i's pre-tax income, and E be the amount of money the government has to collect. Since the government cannot ask for more than the total pre-tax income, it is natural to assume that $\sum_{i \in N} c_i \geq E$. Then, a rule determines how to tax income.

We also refer to two problems that appear in the Talmud. One is the **marriage contract problem** (Thomson 2019). This is described as follows:

> If a man who was married to three wives died and the kethubah of one was a maneh (100 zuz), of the other two hundred zuz, and of the third three hundred zuz, and the estate (was worth) only one maneh (one hundred zuz), the (the sum) is divided equally. If the estate (was worth) two hundred zuz (the claimant) of the maneh (one hundred zuz) receives fifty zuz (and the claimants respectively) of the two hundred and the three hundred zuz (receive each) three gold denarii (seventy-five zuz). If the estate (was worth) three hundred zuz (the claimant) of the maneh receives fifty zuz and (the claimant) of the two hundred zuz (receives) a maneh (one hundred zuz) while (the claimant) of the three hundred zuz (receives) six gold denarii (one hundred and fifty zuz). Similarly if three persons contributed to a joint fund and they had made a loss or a profit they share in the same manner.
>
> (O'Neill 1982, p. 370)

The above problem can be described using our notations as follows. Let N be the set of wives, c_i be the claim of each wife $i \in N$, and E be the worth of the estate. In addition, let x_i be the amount that wife $i \in N$ receives. This problem comprises three situations. For each situation, $N = \{1, 2, 3\}$, $c_1 = 100$, $c_2 = 200$, and $c_3 = 300$. For the first situation, $E = 100$, and the Talmud suggests $x_1 = x_2 = x_3 = 33\frac{1}{3}$. For the second situation, $E = 200$, and the recommendation is $x_1 = 50$ and $x_2 = x_3 = 75$. For the third situation, $E = 300$, $x_1 = 50$, $x_2 = 100$, and $x_3 = 150$.

The other problem is called the **contested garment problem** (Thomson 2019), which is described as follows:

> Two hold a garment… if one of them says, "It is all mine" and the other says, "Half of it is mine",… the former then receives three quarters and the latter receives one quarter.
>
> (O'Neill 1982, page 346)

Let the worth of the garment be 100. Then, we can describe the above problem by $N = \{1, 2\}$, $c_1 = 50$, $c_2 = 100$, and $E = 100$. The Talmud suggests $x_1 = 25$ and $x_2 = 75$, where x_i denotes the amount that agent $i \in N$ receives.

Table 14.1 summarizes examples that appear in the marriage contract and contested garment problems. What kind of rule will generate numbers that the Talmud recommends for each problem in Table 14.1? The answer does not seem to be straightforward. For instance, in (A), the Talmud seems to suggest "equal division," while in (C), it seems to insist on "proportional division." In the next section, we answer the

[5] By $\mathbb{R}$, we mean the set of real numbers. We define $\mathbb{R}^N$ similar to footnote 4.

above question by exploring examples of rules. Hereafter, unless specified, we use language based on bankruptcy application.

14.3 Rules

14.3.1 CEA Rule

We introduce four central rules proposed in the literature. In our first rule, we try to divide a resource equally among the agents. However, if we simply divide the resource equally, some agents may receive more than their claims. Thus, we divide the resource equally, subject to no agent receiving more than their claim. For instance, let $N = \{1, 2, 3, 4\}$, $c = (20, 30, 50, 60)$, and $E = 140$. If we divide E equally among the four agents, agents 1 and 2 will receive more than their claims. In that case, we first determine that agent 1 (who has the smallest claim) receives 20. Next, we divide the rest $140 - 20 = 120$ to agents 2, 3, and 4. Now, if we divide the rest equally among them, agent 2 will receive more than her claim. Thus, we decide to give 30 to agent 2. The rest is $120 - 30 = 90$. We can divide this amount equally to agents 3 and 4 because neither of them receives more than their claim. Thus, each agent obtains 45. The formal definition is as follows:

Constrained Equal Awards rule (CEA rule): For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $\text{CEA}_i(c, E) = \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}$ is chosen such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Example 14.1 Let $N = \{1, 2, 3\}$, $c = (50, 80, 100)$, and $E = 170$. Then, $\text{CEA}(c, E) = (50, 60, 60)$. ◀

Example 14.2 Let $N = \{1, 2, 3, 4, 5\}$, $c = (20, 40, 40, 70, 90)$, and $E = 200$. Then, $\text{CEA}(c, E) = (20, 40, 40, 50, 50)$. ◀

The reader can check that $\lambda = 60$ and $\lambda = 50$ in Examples 14.1 and 14.2, respectively. In Table 14.1, the CEA rule generates the numbers recommended by the Talmud only in problem (A). Under the CEA rule, even an agent with a small claim can receive some amount. In this sense, smaller claimants favor this rule.

14.3.2 CEL Rule

The next rule focuses on the losses that agents face and the total loss (the difference $\sum_{i \in N} c_i - E$). We try to divide the total loss equally among agents so that each agent bears the same loss. However, some agents may receive a negative amount if we do so. Thus, we divide the total loss equally, subject to no agent receiving a negative amount. For instance, let $N = \{1, 2, 3, 4\}$, $c = (20, 35, 55, 60)$, and $E = 40$. Then, the total loss is $(20 + 35 + 55 + 60) - 40 = 130$. If we divide this amount equally among the four agents ($130/4 = 32.5$) and ask each agent to give up that amount, agent 1 ends up receiving a negative amount. Then, we determine that agent 1 receives 0 (we ask agent 1 only to give up 20). Note that the final awards vector is not $(0, 35 - 32.5, 55 - 32.5, 60 - 32.5) = (0, 2.5, 22.5, 27.5)$ because the total award is not equal to $E = 40$. Thus, the total loss must be revised. Since agent 1 only gave up 20, the revised total loss among the remaining agents 2, 3, and 4 is $130 - 20 = 110$. If we divide this amount equally among them, agent 2 receives a negative amount. Thus, we set agent 2 to receive 0 (agent 2 only gives up 35). The revised total loss among agents 3 and 4 is now $110 - 35 = 75$. We can ask each of them to give up $75/2 = 37.5$ since neither of them ends up receiving a negative amount. This results in agents 3 and 4 receiving 17.5 and 22.5, respectively. Thus, the formal definition is as follows:

Constrained Equal Losses rule (CEL rule): For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $\text{CEL}_i(c, E) = \max\{0, c_i - \lambda\}$, where $\lambda \in \mathbb{R}$ is chosen such that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

Example 14.3 Let $N = \{1, 2, 3\}$, $c = (50, 80, 100)$, and $E = 50$. Then, $\text{CEL}(c, E) = (0, 15, 35)$. ◀

Table 14.1 Examples in the Talmud

Problem in the Talmud	Recommendation by the Talmud
(A) $c_1 = 100, c_2 = 200, c_3 = 300, E = 100$	$x_1 = 33\frac{1}{3}, x_2 = 33\frac{1}{3}, x_3 = 33\frac{1}{3}$
(B) $c_1 = 100, c_2 = 200, c_3 = 300, E = 200$	$x_1 = 50, x_2 = 75, x_3 = 75$
(C) $c_1 = 100, c_2 = 200, c_3 = 300, E = 300$	$x_1 = 50, x_2 = 100, x_3 = 150$
(D) $c_1 = 50, c_2 = 100, E = 100$	$x_1 = 25, x_2 = 75$

Example 14.4 Let $N = \{1, 2, 3, 4, 5\}$, $c = (20, 40, 40, 70, 90)$, and $E = 70$. Then, $\text{CEL}(c, E) = (0, 0, 0, 25, 45)$. ◀

The reader can verify that $\lambda = 65$ in Example 14.3. Thus, all agents either give up "their entire claim" (if their individual claim is less than λ) or "λ" (if their individual claim is at least as large as λ). Similarly, $\lambda = 45$ in Example 14.4. In Table 14.1, the CEL rule generates recommendations by the Talmud only in problem (D). Under the CEL rule, an agent with a small claim can easily receive nothing. Since each agent, regardless of the claim size, is asked to give up the same amount (or the entire claim), an agent with a larger claim gives up a relatively small part of their claim. In this sense, larger claimants are more likely to favor this rule.[6]

14.3.3 Proportional Rule

The next rule divides the resource proportional to the claims. Let $N = \{1, 2, 3, 4\}$, $c = (10, 30, 50, 60)$, and $E = 120$. Then, for instance, agents 1 and 2 receive $\frac{10}{10+30+50+60} \times 120 = 8$ and $\frac{30}{10+30+50+60} \times 120 = 24$, respectively. Note that

$$\frac{10}{10+30+50+60} \times 120 = 10 \times \frac{120}{10+30+50+60} = 10 \times \frac{4}{5} = 8 \text{ and}$$
$$\frac{30}{10+30+50+60} \times 120 = 30 \times \frac{120}{10+30+50+60} = 30 \times \frac{4}{5} = 24.$$

[6] The CEA and CEL rules are called "dual" rules. See Aumann and Maschler (1985), Herrero (2003), and Thomson and Yeh (2008) for details regarding the notion of duality in this context.

Thus, agents 1 and 2 receive $\frac{4}{5}$ of their individual claims. A similar calculation can also be performed for agents 3 and 4. Therefore, we can claim that each agent obtains the same "ratio" of their claim under this rule. This is denoted by λ in the following formal definition:

Proportional rule: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $P_i(c, E) = \lambda c_i$, where $\lambda \in \mathbb{R}$ is chosen such that $\sum_{i \in N} \lambda c_i = E$.

Example 14.5 Let $N = \{1, 2, 3\}$, $c = (20, 40, 60)$, and $E = 90$. Then, $P(c, E) = (15, 30, 45)$. ◀

Example 14.6 Let $N = \{1, 2, 3, 4, 5\}$, $c = (20, 40, 40, 70, 70)$, and $E = 60$. Then, $P(c, E) = (5, 10, 10, 17.5, 17.5)$. ◀

The reader can check that $\lambda = \frac{3}{4}$ and $\lambda = \frac{1}{4}$ in Examples 14.5 and 14.6, respectively. In Table 14.1, the proportional rule gives the same recommendation as the Talmud only in problem (C).

14.3.4 Talmud Rule

The final rule generates numbers recommended by the Talmud for each problem in Table 14.1 (Aumann and Maschler 1985). The rule is described as follows. We distinguish two cases: when the quantity of the resource is less than or equal to the half-sum of the claims ($\sum_{i \in N} \frac{c_i}{2} \geq E$); and when the resource is more than the half-sum of the claims ($\sum_{i \in N} \frac{c_i}{2} < E$). In the first case, we apply the CEA rule to the problem $\left(\frac{c}{2}, E\right)$. We reduce each agent's claim by half since even if each agent only claims half of their claim, the resource is not sufficient (except for the case when $\sum_{i\in N}\frac{c_i}{2}=E$). In the second case, we first give each agent half of their individual claim. After that, we still have the amount $E-\sum_{i\in N}\frac{c_i}{2}$ left to divide. We then apply the CEL rule to divide this amount. Since the agents have already given half of their claims, we let them assert only half of their claims when applying the CEL rule. To summarize, after giving each agent half of their claim, we use the CEL rule to the problem $\left(\frac{c}{2},E-\sum_{i\in N}\frac{c_i}{2}\right)$. For example, let $N=\{1,2,3,4\}$, $c=(20,30,40,60)$, and $E=65$. Since $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(20+30+40+60)=75>E=65$, we apply the CEA rule to the problem $\left(\frac{c}{2},E\right)$. Thus, the awards vector is $\text{CEA}\left(\frac{c}{2},E\right)=\text{CEA}(10,15,20,30;65)=(10,15,20,20)$. Next, under the same N and c, let $E=90$. Since $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(20+30+40+60)=75<E=90$, we give each agent half of their claim and apply the CEL rule to divide the rest $E-\sum_{i\in N}\frac{c_i}{2}=90-75=15$ under the claims vector $\frac{c}{2}=(10,15,20,30)$. Thus, the awards vector is $\frac{c}{2}+\text{CEL}\left(\frac{c}{2},E-\sum_{i\in N}\frac{c_i}{2}\right)=(10,15,20,30)+\text{CEL}(10,15,20,30;15)=(10,15,20,30)+(0,0,2.5,12.5)=(10,15,22.5,42.5)$. One can say that this rule is a "well-balanced" rule in the sense that it mixes the CEA rule (favored by smaller claimants) and the CEL rule (favored by larger claimants). The formal definition of this rule is as follows:

Talmud rule: For each $N\in\mathcal{N}$ and $(c,E)\in\mathcal{C}^N$,

$$\text{Tal}(c,E)=\begin{cases}\text{CEA}\left(\frac{c}{2},E\right)\text{if }\sum_{i\in N}\frac{c_i}{2}\geq E\\ \frac{c}{2}+\text{CEL}\left(\frac{c}{2},E-\sum_{i\in N}\frac{c_i}{2}\right)\text{if }\sum_{i\in N}\frac{c_i}{2}<E.\end{cases}$$ ◀

Example 14.7 Let $N=\{1,2,3,4,5\}$, $c=(10,30,40,80,100)$, and $E=100$. Note that $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(10+30+40+80+100)=130>E=100$. Thus, we have $\text{Tal}(c,E)=\text{CEA}\left(\frac{c}{2},E\right)=\text{CEA}(5,15,20,40,50;100)=(5,15,20,30,30)$. ◀

Example 14.8 Let $N=\{1,2,3,4,5\}$, $c=(10,30,40,80,100)$, and $E=170$. Note that $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(10+30+40+80+100)=130<E=170$. Thus, we have

$$\begin{aligned}\text{Tal}(c,E)&=\frac{c}{2}+\text{CEL}\left(\frac{c}{2},E-\sum_{i\in N}\frac{c_i}{2}\right)\\&=(5,15,20,40,50)+CEL(5,15,20,40,50;170-130)\\&=(5,15,20,40,50)+(0,0,0,15,25)\\&=(5,15,20,55,75).\end{aligned}$$

◀

Example 14.9 (Examples in Table 14.1):

(A) Note, $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(100+200+300)=300>E=100$. Thus, $\text{Tal}(c,E)=\text{CEA}\left(\frac{c}{2},E\right)=\text{CEA}(50,100,150;100)=\left(33\frac{1}{3},33\frac{1}{3},33\frac{1}{3}\right)$.

(B) Note, $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(100+200+300)=300>E=200$. Thus, $\text{Tal}(c,E)=\text{CEA}\left(\frac{c}{2},E\right)=\text{CEA}(50,100,150;200)=(50,75,75)$.

(C) Note, $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(100+200+300)=300\geq E=300$. Thus, $\text{Tal}(c,E)=\text{CEA}\left(\frac{c}{2},E\right)=\text{CEA}\ (50,100,150;300)=(50,100,150)$.

(D) Note, $\sum_{i\in N}\frac{c_i}{2}=\frac{1}{2}(50+100)=75<E=100$. Thus,

$$\begin{aligned}\text{Tal}(c,E)&=\frac{c}{2}+\text{CEL}\left(\frac{c}{2},E-\sum_{i\in N}\frac{c_i}{2}\right)\\&=(25,50)+CEL(25,50;100-75)\\&=(25,50)+(0,25)=(25,75).\end{aligned}$$

14.4 Axioms (Properties of Rules)

14.4.1 Equal Treatment of Equals

As previously mentioned, we adopt an axiomatic approach to evaluate the rules. We introduce the properties of rules, called **axioms**, and see whether the rules defined in the previous section satisfy those properties. Our first axiom is basic. We require agents with the same claim to receive the same amount. More precisely, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$, if two agents $\{i,j\} \subseteq N$ have the same claim ($c_i = c_j$), then, this axiom states that awards given to agents i and j under the rule φ ($\varphi_i(c, E)$ and $\varphi_j(c, E)$, respectively) should be the same. Thus, the formal definition is as follows:

Equal treatment of equals (ete): For each $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and each $\{i,j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i(c, E) = \varphi_j(c, E)$.

All four rules that have appeared in the previous section satisfy this axiom.

14.4.2 Minimal Rights First

Our next axiom is more demanding. First, we consider the minimal amount that each agent should receive. Let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 60$. In the case of agent 2, even if agents 1 and 3 are fully compensated, there is an amount of $E - (c_1 + c_3) = 60 - (10 + 40) = 10$ left to divide. Thus, we can say that agent 2 should receive at least 10. Similarly, as $E - (c_1 + c_2) = 60 - (10 + 30) = 20$, agent 3 should receive at least 20. However, if agents 2 and 3 are fully compensated, agent 1 receives a negative amount ($E - (c_2 + c_3) = 60 - (30 + 40) = -10$). In that case, agent 1's minimal amount is 0 (recall that each agent should end up receiving a non-negative amount). Thus, this is formally described as follows:

For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, let $m_i(c, E) = \max\left\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\right\}$ be **agent *i*'s minimal right** (minimal amount). Further, let $m(c, E) = (m_i(c, E))_{i \in N}$ be the **minimal rights vector**.

In our previous example, $m_1(c, E) = 0$, $m_2(c, E) = 10$, $m_3(c, E) = 20$, and $m(c, E) = (0, 10, 20)$. Now we identify each agent's minimal right in $(c, E) \in \mathcal{C}^N$. Next, consider dividing the resource E in two steps: first, give each agent their minimal right; second, divide the rest $E - \sum_{i \in N} m_i(c, E)$ under the claims vector $c - m(c, E)$. We use the claims vector $c - m(c, E)$ in the second step because the agents have been given their minimal rights in the first step. The next axiom requires that the awards vector chosen by the rule for $(c, E) \in \mathcal{C}^N$ is the same as that obtained by dividing E in the two steps described above. Thus, the formal definition is as follows:

Minimal rights first (mrf): For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi(c, E) = m(c, E) + \varphi(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$.

The CEL and Talmud rules satisfy *minimal rights first*. For instance, consider the CEL rule. Let $N = \{1, 2, 3\}$, $c = (5, 20, 35)$, and $E = 40$. Then, $CEL(c, E) = (0, 12.5, 27.5)$ (we choose "$\lambda = 7.5$" to calculate $CEL(c, E)$). Note that $m_1(c, E) = \max\{40 - (20 + 35), 0\} = 0$, $m_2(c, E) = \max\{40 - (5 + 35), 0\} = 0$, and $m_3(c, E) = \max\{40 - (5 + 20), 0\} = 15$. Thus, $m(c, E) = (0, 0, 15)$. If we first assign agents their minimal rights and apply the CEL rule to the remaining problem, the awards vector becomes

$$
\begin{aligned}
& m(c, E) + \mathrm{CEL}\left(c - m(c, E), E - \sum_{i \in N} m_i(c, E)\right) \\
& = (0, 0, 15) + \mathrm{CEL}(5 - 0, 20 - 0, 35 - 15; 40 - (0 + 0 + 15)) \\
& = (0, 0, 15) + (0, 12.5, 12.5) = (0, 12.5, 27.5).
\end{aligned}
$$

(Note that we choose "$\lambda = 7.5$" again when we calculate $\mathrm{CEL}(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$.) Thus, $\mathrm{CEL}(c, E) = m(c, E) + \mathrm{CEL}(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$ under this example. In fact, this equality holds for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$.

However, neither the CEA rule nor the proportional rule satisfies this axiom, as shown below.

Proposition 14.1 The CEA rule does not satisfy *minimal rights first*.

Proof Let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 60$. Then, $\text{CEA}(c, E) = (10, 25, 25)$. On the other hand,

$$m(c,E) + \text{CEA}\left(c - m(c,E), E - \sum_{i \in N} m_i(c,E)\right)$$
$$= (0, 10, 20) + \text{CEA}$$
$$(10 - 0, 30 - 10, 40 - 20; 60 - (0 + 10 + 20))$$
$$= (0, 10, 20) + (10, 10, 10) = (10, 20, 30).$$

Thus, $\text{CEA}(c, E) \neq m(c, E) + \text{CEA}(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$, in violation of *minimal rights first*. ∎

Proposition 14.2 The proportional rule does not satisfy *minimal rights first*.

Proof Let $N = \{1, 2\}$, $c = (40, 60)$, and $E = 50$. Then, $P(c, E) = (20, 30)$. Also,

$$m(c,E) + P\left(c - m(c,E), E - \sum_{i \in N} m_i(c,E)\right)$$
$$= (0, 10) + P(40 - 0, 60 - 10; 50 - (0 + 10))$$
$$= (0, 10) + \left(\frac{160}{9}, \frac{200}{9}\right) = \left(\frac{160}{9}, \frac{290}{9}\right).$$

Thus, $P(c, E) \neq m(c, E) + P(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$, in violation of *minimal rights first*. ∎

14.4.3 Claims Truncation Invariance

Next, we consider truncating some claims. For example, let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 25$. For agent 2, the claim is greater than the resource ($c_2 = 30 > 25 = E$). In this case, we truncate agent 2's claim by the resource amount (implying that agent 2's "relevant" claim is 25). Similarly, since agent 3's claim is also greater than the resource ($c_3 = 40 > 25 = E$), we truncate this claim by the resource amount too (agent 3's relevant claim is 25). Since agent 1's claim does not exceed the resource quantity ($c_1 = 10 < 25 = E$), there is no need to truncate the claim (agent 1's relevant claim is 10). Thus, this is formally described as follows.

For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, let $t_i(c, E) = \min\{c_i, E\}$ be **agent *i*'s truncated claim**. Further, let $t(c, E) = (t_i(c, E))_{i \in N}$ be the **truncated claims vector**.

In our previous example, $t_1(c, E) = 10$, $t_2(c, E) = 25$, $t_3(c, E) = 25$, and $t(c, E) = (10, 25, 25)$. The following axiom requires that the awards vector given by applying the rule to $(c, E) \in \mathcal{C}^N$ is the same as that obtained by using the truncated claims vector. Thus, the formal definition is as follows:

Claims truncation invariance (ctinv):

For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi(c, E) = \varphi(t(c, E), E)$.

The CEA and Talmud rules satisfy *claims truncation invariance*. For instance, consider the CEA rule. Let $N = \{1, 2, 3\}$, $c = (5, 20, 35)$, and $E = 25$. Then, $\text{CEA}(c, E) = (5, 10, 10)$ (we choose "$\lambda = 10$" to calculate $CEA(c, E)$). Note that $t_1(c, E) = \min\{5, 25\} = 5$, $t_2(c, E) = \min\{20, 25\} = 20$, $t_3(c, E) = \min\{35, 25\} = 25$, and $t(c, E) = (5, 20, 25)$. If we apply the CEA rule under the truncated claims vector to divide $E = 25$, the awards vector becomes $\text{CEA}(t(c, E), E) = \text{CEA}(5, 20, 25; 25) = (5, 10, 10)$ (we choose "$\lambda = 10$" again when we calculate $\text{CEA}(t(c, E), E)$). Thus, $\text{CEA}(c, E) = \text{CEA}(t(c, E), E)$ under this example. In fact, this equality holds for each $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$.

However, the CEL and proportional rules do not satisfy this axiom, as shown below.

Proposition 14.3 The CEL rule does not satisfy *claims truncation invariance*.

Proof Let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 25$. Then, $\text{CEL}(c, E) = (0, 7.5, 17.5)$ and

$\text{CEL}(t(c,E),E) = \text{CEL}(10,25,25;25) = (0,12.5,12.5)$. Thus, $\text{CEL}(c,E) \neq \text{CEL}(t(c,E),E)$, in violation of *claims truncation invariance*. ∎

Proposition 14.4 The proportional rule does not satisfy *claims truncation invariance*.

Proof Let $N = \{1,2,3\}$, $c = (20,30)$, and $E = 25$. Then, $P(c,E) = (10,15)$ and $P(t(c,E),E) = P(20,25;25) = \left(\frac{100}{9},\frac{125}{9}\right)$, in violation of *claims truncation invariance*. ∎

14.4.4 Consistency

Next, we introduce an axiom broadly used in the context of resource allocation. Given $N \in \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, and the awards vector $x = \varphi(c,E)$ chosen by the rule φ, suppose that some agents leave the scene with their awards. The reduced problem consists of the remaining agents $N' \subset N$, their claims $c_{N'}$, and the resource reduced by the awards given to the leaving agents, $E - \sum_{i \in N \setminus N'} x_i$.[7] For instance, let $N = \{1,2,3,4,5\}$. Suppose that agents 2 and 4 leave with their awards x_2 and x_4, respectively. Then, the reduced problem consists of agents $N' = \{1,3,5\}$, claims vector $c_{N'} = (c_1,c_3,c_5)$, and the resource $E - (x_2 + x_4)$. The axiom requires that starting with the awards vector chosen by the rule, if some agents leave the situation with their awards and apply the rule to the reduced problem, each remaining agent still receives the same amount as before. In our previous example, this means that $(x_1,x_3,x_5) = \varphi(c_1,c_3,c_5;E-(x_2+x_4))$. The formal definition is as follows[8]:

Consistency (cons): For each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x = \varphi(c,E)$, then $x_{N'} = \varphi\left(c_{N'}, E - \sum_{i \in N \setminus N'} x_i\right)$.[9]

All four rules defined in the previous section satisfy *consistency*.[10] In fact, for each of the four rules, the same "λ" (appearing in their definition) is chosen in both the initial and reduced problems.[11] For example, let $N = \{1,2,3,4,5\}$, $c = (10,20,40,60,70)$, and $E = 120$. Let us consider the CEA rule. Then, we choose $\lambda = 30$ to obtain $\text{CEA}(c,E) = (10,20,30,30,30)$. Now suppose that agents 2 and 4 leave with their awards. Let $N' = \{1,3,5\}$. Then, the reduced problem is $\left(c_{N'}, E - \sum_{i \in N \setminus N'} x_i\right) = (10,40,70;120-20-30)$. If we apply the CEA rule to this reduced problem, we again choose $\lambda = 30$ to obtain $CEA(10,40,70;70) = (10,30,30)$. Since we choose the same λ in both initial and reduced problems, the awards given to the remaining agents do not change.

Table 14.2 summarizes the findings from this section. This also includes other results that we study in Sect. 14.5.2. The abbreviations "ete," "mrf," "ctinv," "cons," "cp-up," "cp-down," and "nat" stand for "*equal treatment of equals*," "*minimal rights first*," "*claims truncation invariance*," "*consistency*," "*composition up*," "*composition down*," and "*no advantageous transfer*," respectively. The symbol "+" (respectively, "−") means the corresponding rule satisfies (respectively, does not satisfy) the corresponding axiom.

14.5 Characterizations

14.5.1 Characterization of the Talmud Rule

We provide characterizations of the rules based on the properties (axioms) studied in the previous section. Before that, we introduce the following

[7] We denote by $c_{N'}$ the restriction of c to N'.

[8] See Thomson (2011a) for a survey regarding this axiom. Also, see Thomson (2012) for interpreting this axiom.

[9] The notation $x_{N'}$ means the restriction of x to N'.

[10] See Young (1987) for a wide family of rules satisfying *consistency*.

[11] For the Talmud rule, we chose λ when we calculate $\text{CEA}\left(\frac{c}{2},E\right)$ or $\text{CEL}\left(\frac{c}{2}, E - \sum_{i \in N} \frac{c_i}{2}\right)$.

Table 14.2 Rules which satisfy the axioms

Rules	Axioms						
	ete	mrf	ctinv	cons	cp-up	cp-down	nat
CEA	+	−	+	+	+	+	−
CEL	+	+	−	+	+	+	−
Proportional	+	−	−	+	+	+	+
Talmud	+	+	+	+	−	−	−

lemma that describes the awards vector given by the Talmud rule for the two-agent case.

Lemma 14.1 Let $N = \{1,2\}$ and $(c,E) \in \mathcal{C}^N$ be such that $c_1 < c_2$. Then,

$$\text{Tal}(c,E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } 0 \le E \le c_1 \\ \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right) & \text{if } c_1 < E < c_2 \\ \left(\frac{c_1 - c_2 + E}{2}, \frac{-c_1 + c_2 + E}{2}\right) & \text{if } c_2 \le E \le c_1 + c_2. \end{cases}$$

The reader can verify this lemma by applying the definition of the Talmud rule provided in Sect. 14.3.4. For instance, if $c_2 \le E \le c_1 + c_2$, since $c_1 < c_2$, we have $\frac{c_1}{2} + \frac{c_2}{2} < \frac{c_2}{2} + \frac{c_2}{2} = c_2 \le E$. Thus, the awards vector is given by $\left(\frac{c_1}{2}, \frac{c_2}{2}\right) + CEL\left(\frac{c_1}{2}, \frac{c_2}{2}; E - \frac{c_1}{2} - \frac{c_2}{2}\right)$. The total loss in the problem $\left(\frac{c_1}{2}, \frac{c_2}{2}; E - \frac{c_1}{2} - \frac{c_2}{2}\right)$ is $\frac{c_1}{2} + \frac{c_2}{2} - \left(E - \frac{c_1}{2} - \frac{c_2}{2}\right) = c_1 + c_2 - E$. Half of the total loss is $\frac{c_1 + c_2 - E}{2} \le \frac{c_1}{2}$ (where the inequality holds by $c_2 \le E$). Thus, $CEL\left(\frac{c_1}{2}, \frac{c_2}{2}; E - \frac{c_1}{2} - \frac{c_2}{2}\right) = \left(\frac{c_1}{2} - \frac{c_1 + c_2 - E}{2}, \frac{c_2}{2} - \frac{c_1 + c_2 - E}{2}\right)$. This leads to the desired conclusion.

Suppose we want a rule to satisfy *equal treatment of equals*, *minimal rights first*, and *claims truncation invariance*. Among the four rules studied in the previous section, only the Talmud rule satisfies all of them. The following theorem states that, among any rules, the Talmud rule is the only one that satisfies the three axioms for the two-agent case.

Theorem 14.1 (Dagan 1996): Let $N \in \mathcal{N}$ with $|N| = 2$. The Talmud rule is the only rule satisfying *equal treatment of equals*, *minimal rights first*, and *claims truncation invariance*.

Proof We prove only the uniqueness. Let $N = \{1,2\}$ and $(c,E) \in \mathcal{C}^N$. Let φ be a rule that satisfies the three axioms listed in the theorem. We want to show that $\varphi(c,E) = Tal(c,E)$. If $c_1 = c_2$, because both φ and the Talmud rule satisfy *equal treatment of equals*, $\varphi(c,E) = Tal(c,E) = \left(\frac{E}{2}, \frac{E}{2}\right)$. Without loss of generality, let $c_1 < c_2$. We distinguish the three cases.

Case 1: $0 \le E \le c_1$.

Note that $t(c,E) = (E,E)$. Since φ satisfies *claims truncation invariance*,

$$\varphi(c,E) = \varphi(t(c,E),E) = \varphi(E,E;E). \quad (14.1)$$

Since φ satisfies *equal treatment of equals*,

$$\varphi(E,E,E) = \left(\frac{E}{2}, \frac{E}{2}\right). \quad (14.2)$$

By (14.1) and (14.2), $\varphi(c,E) = \left(\frac{E}{2}, \frac{E}{2}\right)$.

Case 2: $c_1 < E < c_2$.

Note that $m(c,E) = (0, E - c_1)$. Since φ satisfies *minimal rights first*,

$$\begin{aligned} \varphi(c,E) &= m(c,E) + \varphi\left(c - m(c,E), E - \sum_{i \in N} m_i(c,E)\right) \\ &= (0, E - c_1) + \varphi(c_1 - 0, c_2 - (E - c_1); E - 0 - (E - c_1)) \\ &= (0, E - c_1) + \varphi(c_1, c_1 + c_2 - E; c_1). \end{aligned} \quad (14.3)$$

Since $c_2 - E > 0$, $c_1 + c_2 - E > c_1$. In addition, because φ satisfies *claims truncation invariance* (corresponding to the first equality below) and *equal treatment of equals* (corresponding to the second equality below),

$$\varphi(c_1, c_1 + c_2 - E; c_1) = \varphi(c_1, c_1; c_1) = \left(\frac{c_1}{2}, \frac{c_1}{2}\right). \quad (14.4)$$

By (14.3) and (14.4), $\varphi(c, E) = (0, E - c_1) + \left(\frac{c_1}{2}, \frac{c_1}{2}\right) = \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right)$.

Case 3: $c_2 \leq E \leq c_1 + c_2$.

Note that $m(c, E) = (E - c_2, E - c_1)$. Since φ satisfies *minimal rights first*,

$$\begin{aligned}\varphi(c, E) &= (E - c_2, E - c_1) \\ &\quad + \varphi(c_1 - (E - c_2), c_2 - (E - c_1); \\ &\quad E - (E - c_2) - (E - c_1)) \\ &= (E - c_2, E - c_1) + \varphi(c_1 + c_2 - E, \\ &\quad c_1 + c_2 - E; c_1 + c_2 - E)\end{aligned} \quad (14.5)$$

Since φ satisfies *equal treatment of equals*,

$$\varphi(c_1 + c_2 - E, c_1 + c_2 - E; c_1 + c_2 - E) = \left(\frac{c_1 + c_2 - E}{2}, \frac{c_1 + c_2 - E}{2}\right) \quad (14.6)$$

By (14.5) and (14.6),

$$\begin{aligned}\varphi(c, E) &= (E - c_2, E - c_1) \\ &\quad + \left(\frac{c_1 + c_2 - E}{2}, \frac{c_1 + c_2 - E}{2}\right) \\ &= \left(\frac{c_1 - c_2 + E}{2}, \frac{-c_1 + c_2 + E}{2}\right).\end{aligned}$$

In each cases, by Lemma 14.1, we have $\varphi(c, E) = Tal(c, E)$. ∎

For more than two agents, the Talmud rule is not the only the rule satisfying *equal treatment of equals*, *minimal rights first*, and *claims truncation invariance*. However, if we additionally require a rule to be *consistent*, then the Talmud rule becomes the only one.

Theorem 14.2 (Aumann and Maschler 1985; Dagan 1996): The Talmud rule is the only rule satisfying *equal treatment of equals*, *minimal rights first*, *claims truncation invariance*, and *consistency*.

We provide a sketch of the proof of Theorem 14.2. Before doing so, we introduce two lemmata. The first lemma, known as the "Elevator Lemma," relates the result obtained in the two-agent case to more than two agents by applying two axioms: *consistency* and "converse consistency."[12]

Lemma 14.2 (The Elevator Lemma) (Thomson 2011a): Let φ and $\overline{\varphi}$ be two rules. If (i) φ coincides with $\overline{\varphi}$ in the two-agent cases, (ii) φ is *consistent*, and (iii) $\overline{\varphi}$ is *conversely consistent*, then φ coincides with $\overline{\varphi}$ for any number of agents.[13]

The second lemma states that if a rule is *consistent* and satisfies a fundamental requirement called "resource monotonicity," the rule is *conversely consistent*.[14]

Lemma 14.3 (Chun 1999): *Resource monotonicity* and *consistency* together imply *converse consistency*.

Now, we explain a sketch of the proof of Theorem 14.2. We show only the uniqueness part. Let rule φ satisfy *equal treatment of equals*,

---

[12] A rule φ satisfies *converse consistency* if for each $N \in \mathcal{N}$ with $|N| \geq 3$, each $(c, E) \in \mathcal{C}^N$, and each $x \in \mathbb{R}^N_+$ such that $\sum_{i \in N} x_i = E$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i)$, then $x = \varphi(c, E)$. In words, *converse consistency* requires the following. Let $N \in \mathcal{N}$ with $|N| \geq 3$ and $(c, E) \in \mathcal{C}^N$ be given. Suppose that there is an awards vector $x = (x_i)_{i \in N}$ for (c, E) such that for each two-agent group $N' \subset N$, the restriction of x to the group N' is chosen by the rule φ for the problem of dividing $\sum_{i \in N'} x_i$ (the total awards of the group N') under the claims vector $c_{N'} = (c_i)_{i \in N'}$ (claims of agents in the group N'), that is, $x_{N'} = \varphi\left(c_{N'}, \sum_{i \in N'} x_i\right)$. Then, x should be chosen by φ for (c, E).

[13] In fact, the Elevator Lemma holds in many other economic problems. See Thomson (2011a) for details.

[14] A rule φ satisfies *resource monotonicity* if for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $E' > E$ such that $\sum_{i \in N} c_i \geq E'$, we have $\varphi_i(c, E') \geq \varphi_i(c, E)$. In words, *resource monotonicity* requires that if the amount to divide increases, no agent receives less than before. Thus, each agent becomes at least as well off as before by the increment of the resource (if each agent prefers to receive more).

minimal rights first, *claims truncation invariance*, and *consistency*. Let rule $\overline{\varphi}$ be the Talmud rule. We want to show that φ coincides with $\overline{\varphi}$ for any number of agents. According to Theorem 14.1, φ coincides with $\overline{\varphi}$ in the two-agent cases. Based on the assumption, φ is *consistent*. It is easy to see that the Talmud rule satisfies *resource monotonicity*. Then, $\overline{\varphi}$ is *resource monotonic* and *consistent*. By Lemma 14.3, $\overline{\varphi}$ is *conversely consistent*. Therefore, by Lemma 14.2, we obtain the desired conclusion.

14.5.2 Other Characterizations

14.5.2.1 Characterization of the CEA Rule

We introduce other axioms and provide other characterizations. Consider the following situation. Given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$, suppose, we initially obtain the awards vector $\varphi(c, E)$ by applying rule φ. However, following this, we found that there is more of the resource to divide. In our bankruptcy application, this could happen after re-evaluating the bankrupt firm's assets. Let $E' \in \mathbb{R}_+$ be the new resource ($E' > E$). We assume that $\sum_{i \in N} c_i \geq E'$. Now we have to divide E' among the agents. There may be two ways to achieve this. One way is to forget about the initial awards given to the agents and apply rule φ directly to the new problem (c, E'). Thus, the resulting awards vector becomes $\varphi(c, E')$. Another way is to give each agent their initial award and divide the increment $E' - E$ under the claims vector $c - \varphi(c, E)$ (we reduce each agent's claim by their initial award). The resulting awards vector is $\varphi(c, E) + \varphi(c - \varphi(c, E), E' - E)$. The reader can verify that $\sum_{i \in N} c_i - \sum_{i\ N} \varphi_i(c, E) \geq E' - E$. Thus, the problem $(c - \varphi(c, E), E' - E)$ is well defined. The following axiom requires that both methods yield the same awards vector. This solves any disagreement among agents regarding the way to choose. The formal definition is as follows:

Composition up (cp-up): For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$ such that $\sum_{i \in N} c_i \geq E'$, we have $\varphi(c, E') = \varphi(c, E) + \varphi(c - \varphi(c, E), E' - E)$.

The CEA, CEL, and proportional rules satisfy *composition up*. For instance, consider the CEL rule. Let $N = \{1, 2, 3\}$, $c = (5, 20, 35)$, and $E = 40$. Then, $\text{CEL}(c, E) = (0, 12.5, 27.5)$. Suppose, after re-evaluating the resource, the resource becomes larger. Let the new resource be $E' = 51$. If we forget about the initial awards and apply the CEL rule to the new problem, we have $\text{CEL}(c, E') = \text{CEL}(5, 20, 35; 51) = (2, 17, 32)$ (note that we choose "$\lambda = 3$" to calculate $\text{CEL}(c, E')$). On the other hand, if we first assign agents their initial awards for $E = 40$ and divide the remaining $E' - E = 51 - 40 = 11$ under the claims vector where each agent's claim is reduced by their initial award, we have

$$
\begin{aligned}
&\text{CEL}(c, E) + \text{CEL}(c - CEL(c, E), E' - E) \\
&= \text{CEL}(5, 20, 35; 40) \\
&\quad + \text{CEL}((5, 20, 35) - \text{CEL}(5, 20, 35; 40), 51 - 40) \\
&= (0, 12.5, 27.5) + \text{CEL}((5, 7.5, 7.5), 11) \\
&= (0, 12.5, 27.5) + (2, 4.5, 4.5) = (2, 17, 32).
\end{aligned}
$$

(Note that we choose "$\lambda = 3$" again to calculate $\text{CEL}(c - \text{CEL}(C, E), E' - E)$). Thus, $\text{CEL}(c, E') = \text{CEL}(c, E) + \text{CEL}(c - \text{CEL}(c, E), E' - E)$ for this example. In fact, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$ such that $\sum_{i \in N} c_i \geq E'$, we obtain the above equality.

However, the Talmud rule does not satisfy *composition up*, as shown below.

Proposition 14.5 The Talmud rule does not satisfy *composition up*.

Proof Let $N = \{1, 2\}$, $c = (20, 30)$, $E = 20$, and $E' = 30$. Then, $\text{Tal}(c, E') = (10, 15) + \text{CEL}(10, 15; 30 - 25) = (10, 15) + (0, 5) = (10, 20)$. Note that $\text{Tal}(c, E) = \text{CEA}(10, 15; 20) = (10, 10)$. Thus, $\text{Tal}(c, E) + \text{Tal}(c - \text{Tal}(c, E), E' - E) = (10, 10) + \text{Tal}(20 - 10, 30 - 10; 10) = (10, 10) + \text{CEA}(5, 10; 10) = (10, 10) +$

$(5,5) = (15,15)$. Therefore, $\mathrm{Tal}(c,E') \neq \mathrm{Tal}(c,E) + \mathrm{Tal}(c - \mathrm{Tal}(c,E), E' - E)$, in violation of *composition up*. ∎

If we want a rule to satisfy *equal treatment of equal*, *claims truncation invariance*, and *composition up*, the next theorem states that there is only one rule that does so.

Theorem 14.3 (Dagan 1996): The CEA rule is the only rule satisfying *equal treatment of equals*, *claims truncation invariance*, and *composition up*.

We will not prove Theorem 14.3, but see how the three axioms listed in the theorem identify the awards vector for some problem. Let $N = \{1,2,3\}$, $c = (36,44,50)$, and $E = 75$. Let a rule φ satisfy *equal treatment of equals*, *claims truncation invariance*, and *composition up*. We cannot identify $\varphi(c,E)$ by directly applying these three axioms. However, as φ satisfies *equal treatment of equals* and *claims truncation invariance*, we know how to divide $\hat{E} = 36$ $(= \min\{c_1, c_2, c_3\})$ under c, that is, $\varphi(c,\hat{E}) = \varphi(36,44,50;36) = \varphi(36,36,36;36) = (12,12,12)$. Suppose that initially, the resource was $\hat{E} = 36$, but after re-evaluating the resource, the resource becomes $E = 75$. Since φ satisfies *composition up*,

$$\begin{aligned}\varphi(c,E) &= \varphi(c,\hat{E}) + \varphi(c - \varphi(c,\hat{E}), E - \hat{E}) \\ &= \varphi(36,44,50;36) \\ &\quad + \varphi((36,44,50) - \varphi(36,44,50;36); 75 - 36) \\ &= (12,12,12) + \varphi(24,32,38;39).\end{aligned}$$

Again, we cannot identify $\varphi(24,32,38;39)$ directly, but as φ satisfies *equal treatment of equals* and *claims truncation invariance*, we know how to divide $\tilde{E} = 24$ $(= \min\{24,32,38\})$ under the claims vector $(24,32,38)$, that is, $\varphi(24,32,38;24) = \varphi(24,24,24;24) = (8,8,8)$. Now suppose that after re-evaluating the resource, the resource increases from $\tilde{E} = 24$ to $E - \hat{E} = 39$. As φ satisfies *composition up*,

$$\begin{aligned}\varphi(24,32,38;39) &= \varphi(24,32,38;24) \\ &\quad + \varphi((24,32,38) - \varphi(24,32,38;24); \\ &\quad 39 - 24) \\ &= (8,8,8) + \varphi(16,24,30;15).\end{aligned}$$

As φ satisfies *equal treatment of equals* and *claims truncation invariance*, $\varphi(16,24,30;15) = \varphi(15,15,15;15) = (5,5,5)$. Overall, $\varphi(c,E) = (12,12,12) + (8,8,8) + (5,5,5) = (25,25,25)$. Thus, $\varphi(c,E) = \mathrm{CEA}(c,E)$ under this example.

14.5.2.2 Characterization of the CEL Rule

Let $N \in \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, and the awards vector $\varphi(c,E)$ obtained by applying rule φ be given. Suppose that, in contrast to the situation in the previous axiom, after re-evaluating E, we find that there is less of the resource to divide. Let $E' \in \mathbb{R}_+$ be the new resource $(E' < E)$. There may be two ways to divide E'. One way is to forget about the initial awards given to the agents and apply rule φ to the new problem (c,E'). Thus, the resulting awards vector is $\varphi(c,E')$. Another way is to think that each agent claims the initial award and apply rule φ to divide E'. The resulting awards vector is $\varphi(\varphi(c,E),E')$. Since $\sum_{i \in N} \varphi_i(c,E) > E'$, the problem $(\varphi(c,E),E')$ is well defined. The following axiom requires (as with the previous axiom) that each agent receives the same amount in both ways. The formal definition is as follows:

Composition down (cp-down): For each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{C}^N$, and each $E' < E$ such that $0 \leq E'$, we have $\varphi(c,E') = \varphi(\varphi(c,E),E')$.

The CEA, CEL, and proportional rules satisfy *composition down*. For instance, consider the CEA rule. Let $N = \{1,2,3\}$, $c = (5,20,35)$, and $E = 40$. Then, $\mathrm{CEA}(c,E) = (5,17.5,17.5)$. Suppose, after re-evaluating the resource, it becomes $E' = 30$. If we apply the CEA rule to (c,E'), we have $\mathrm{CEA}(c,E') = CEA(5,20,35;30) = (5,12.5,12.5)$ (note that we choose

$\lambda = 12.5$ to calculate $\text{CEA}(c, E')$). On the other hand, if we think that each agent's claim is their initial award and apply the CEA rule to divide E', we have $\text{CEA}(\text{CEA}(c, E), E') = \text{CEA}(5, 17.5, 17, 5; 30) = (5, 12.5, 12.5)$ (note that we choose $\lambda = 12.5$ again to calculate CEA $(\text{CEA}(c, E), E')$). Thus, $\text{CEA}(c, E') = \text{CEA}(\text{CEA}\ (c, E), E')$ under this example. In fact, this equality holds for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' < E$ such that $0 \leq E'$.

The Talmud rule, however, violates *composition down*, as shown below.

Proposition 14.6 The Talmud rule does not satisfy *composition down*.

Proof Let $N = \{1, 2\}$, $c = (20, 30)$, $E = 30$ and $E' = 20$. Then, $\text{Tal}(c, E') = \text{CEA}(10, 15; 20) = (10, 10)$. Note that $\text{Tal}(c, E) = (10, 15) + \text{CEL}(10, 15; 30 - 25) = (10, 15) + (0, 5) = (10, 20)$. Thus, $\text{Tal}(\text{Tal}(c, E), E') = \text{Tal}(\text{Tal}(20, 30; 30), 20) = \text{Tal}(10, 20; 20) = (5, 10) + \text{CEL}(5, 10; 20 - 15) = (5, 10) + (0, 5) = (5, 15)$. Thus, Tal $(c, E') \neq \text{Tal}(\text{Tal}(c, E), E')$, in violation of *composition down*. ∎

The following theorem states that only one rule satisfies *equal treatment of equals*, *minimal rights first*, and *composition down*.

Theorem 14.4 (Herrero 2003): The CEL rule is the only rule satisfying *equal treatment of equals*, *minimal rights first*, and *composition down*.

As for the previous theorem, we will not prove Theorem 14.4 but explain how the axioms listed in the theorem determine the awards vector for some problem. Let $N = \{1, 2, 3\}$, $c = (36, 44, 50)$, and $E = 55$. Let φ be a rule that satisfies *equal treatment of equals*, *minimal rights first*, and *composition down*. Note that $m(c, E) = (0, 0, 0)$. Thus, we cannot identify $\varphi(c, E)$ directly by applying the above three axioms. However, if we consider dividing $\hat{E} = 94\ (= c_2 + c_3)$ under c, the minimal rights vector is $m(c, \hat{E}) = (0, 8, 14)$. As φ satisfies *equal treatment of equals* and *minimal rights first*,

$$\begin{aligned}\varphi(c, \hat{E}) &= m(c, \hat{E}) + \varphi\left(c - m(c, \hat{E}), \hat{E} - \sum_{i \in N} m_i(c, \hat{E})\right) \\ &= (0, 8, 14) + \varphi(36, 36, 36; 72) \\ &= (0, 8, 14) + (24, 24, 24) = (24, 32, 38).\end{aligned}$$

Suppose that initially, the resource was $\hat{E} = 94$, but after re-evaluating the resource, it becomes $E = 55$. As φ satisfies *composition down*,

$$\varphi(c, E) = \varphi\big(\varphi\big(c, \hat{E}\big), E\big) = \varphi(24, 32, 38; 55).$$

Note that $m(24, 32, 38; 55) = (0, 0, 0)$. Thus, we cannot identify $\varphi(24, 32, 38; 55)$ directly by applying the three axioms. But if we consider dividing $\tilde{E} = 70\ (= 32 + 38)$ under the claims vector $(24, 32, 38)$, we have $m(24, 32, 38; 70) = (0, 8, 14)$. As φ satisfies *equal treatment of equals* and *minimal rights first*,

$$\begin{aligned}\varphi(24, 32, 38; 70) &= (0, 8, 14) + \varphi(24, 24, 24; 48) \\ &= (0, 8, 14) + (16, 16, 16) \\ &= (16, 24, 30).\end{aligned}$$

Now, suppose that after re-evaluating the resource, it decreases from $\tilde{E} = 70$ to $E = 55$. As φ satisfies *composition down*,

$$\begin{aligned}\varphi(24, 32, 38; 55) &= \varphi(\varphi(24, 32, 38; 70); 55) \\ &= \varphi(16, 24, 30; 55).\end{aligned}$$

Note that $m(16, 24, 30; 55) = (1, 9, 15)$. As φ satisfies *equal treatment of equals* and *minimal rights first*,

$$\begin{aligned}\varphi(16, 24, 30; 55) &= (1, 9, 15) + \varphi(15, 15, 15; 30) \\ &= (1, 9, 15) + (10, 10, 10) \\ &= (11, 19, 25).\end{aligned}$$

Overall, $\varphi(c, E) = \varphi(24, 32, 38; 55) = \varphi(16, 24, 30; 55) = (11, 19, 25)$. Thus, $\varphi(c, E) = \text{CEL}(c, E)$ under this example.

14.5.2.3 Characterization of the Proportional Rule

Suppose that some agents transfer their claims among group members. For instance, let $N = \{1, 2, 3, 4, 5\}$, $c = (10, 20, 50, 60, 80)$, and $E = 160$. Consider the group $M = \{2, 3, 5\} \subset N$. Suppose, agent 3 transfers 10 units of her claim to agent 2 and 5 units of her claim to agent 5. Then, the new claims vector is $c' = (10, 30, 35, 60, 85)$. Note that $\sum_{i \in M} c_i = 20 + 50 + 80 = \sum_{i \in M} c'_i = 30 + 35 + 85 = 150.$ Thus, the claims are transferred among agents in M. The next axiom requires no group of agents to benefit from transferring their claims. More precisely, it says that the total amount given to agents who transfer the claims among themselves does not change. Thus, it intends to avoid this kind of strategic behavior caused by a group of agents. In our previous example, under the rule φ, it means that $\sum_{i \in M} \varphi_i(c, E) = \sum_{i \in M} \varphi_i(c', E)$, or $\varphi_2(c, E) + \varphi_3(c, E) + \varphi_5(c, E) = \varphi_2(c', E) + \varphi_3(c', E) + \varphi_5(c', E)$. The formal definition is as follows:

No advantageous transfer (nat): For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $M \subset N$, and each $(c'_i)_{i \in M} \in \mathbb{R}^M_+$, if $\sum_{i \in M} c_i = \sum_{i \in M} c'_i$, then $\sum_{i \in M} \varphi_i(c, E) = \sum_{i \in M} \varphi_i(c', E)$ where $c' = \left((c'_i)_{i \in M}, c_{N \setminus M}\right)$.

As shown below, none of the CEA, CEL, and Talmud rules satisfy this property.

Proposition 14.7 The CEA rule does not satisfy *no advantageous transfer*.

Proof Let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 36$. Then, $\text{CEA}(c, E) = (10, 13, 13)$. Consider the group $M = \{1, 2\} \subset N$. If agent 2 transfers 10 units of her claim to agent 1, the new claims vector will be $c' = (20, 20, 40)$. Note that $c_1 + c_2 = c'_1 + c'_2 = 40$. Since $\text{CEA}(c', E) = (12, 12, 12)$, group M benefits by transferring their claims, that is, $\text{CEA}_1(c, E) + \text{CEA}_2(c, E) = 23 < \text{CEA}_1(c', E) + \text{CEA}_2(c', E) = 24$. ∎

Proposition 14.8 The CEL rule does not satisfy *no advantageous transfer*.

Proof Let $N = \{1, 2, 3\}$, $c = (10, 30, 40)$, and $E = 30$. Then, $\text{CEL}(c, E) = (0, 10, 20)$. Consider the group $M = \{1, 2\} \subset N$. If agent 1 transfers 10 units of her claim to agent 2, the new claims vector will be $c' = (0, 40, 40)$. Note that $c_1 + c_2 = c'_1 + c'_2 = 40$. Since $\text{CEL}(c', E) = (0, 15, 15)$, the group M benefits by transferring their claims, that is, $\text{CEL}_1(c, E) + \text{CEL}_2(c, E) = 10 < \text{CEL}_1\ (c', E) + \text{CEL}_2(c', E) = 15$. ∎

Proposition 14.9 The Talmud rule does not satisfy *no advantageous transfer*.

Proof Let $N = \{1, 2, 3\}$, $c = (10, 50, 70)$, and $E = 45$. Then, $\text{Tal}(c, E) = (5, 20, 20)$. Let $M = \{1, 2\} \subset N$ and $c' = (30, 30, 70)$. Note that $c_1 + c_2 = c'_1 + c'_2 = 60$. Since $\text{Tal}(c', E) = (15, 15, 15)$, the group M benefits by transferring their claims, that is, $\text{Tal}_1(c, E) + \text{Tal}_2(c, E) = 25 < \text{Tal}_1\ (c', E) + \text{Tal}_2(c', E) = 30$. ∎

However, the proportional rule satisfies *no advantageous transfer*. For instance, let $N = \{1, 2, 3, 4, 5\}$, $c = (c_1, c_2, c_3, c_4, c_5) \in \mathbb{R}^5_+$, and $E \in \mathbb{R}_+$ with $\sum_{i \in N} c_i \geq E$. Consider the group $M = \{2, 3, 5\}$. Under the proportional rule, this group in total receives

$$\begin{aligned} &P_2(c, E) + P_3(c, E) + P_5(c, E) \\ &= \frac{c_2}{\sum_{i \in N} c_i} \times E + \frac{c_3}{\sum_{i \in N} c_i} \times E + \frac{c_5}{\sum_{i \in N} c_i} \times E \\ &= \frac{c_2 + c_3 + c_5}{\sum_{i \in N} c_i} \times E. \end{aligned}$$

Now suppose that agents in the group M transfer claims among themselves. Let $c'_M = (c'_2, c'_3, c'_5) \in \mathbb{R}^3_+$ be their claims. Note that $c_2 + c_3 + c_5 = c'_2 + c'_3 + c'_5$. Let $c' = (c_1, c'_2, c'_3, c_4, c'_5)$. Then, after agents in the group M transfer their claims, this group in total receives

$$P_2(c',E)+P_3(c',E)+P_5(c',E) = \frac{c_2'}{\sum_{i\in N} c_i'}\times E+\frac{c_3'}{\sum_{i\in N} c_i'}\times E+\frac{c_5'}{\sum_{i\in N} c_i'}\times E = \frac{c_2'+c_3'+c_5'}{\sum_{i\in N} c_i'}\times E.$$

Since $c_2+c_3+c_5=c_2'+c_3'+c_5'$ and $\sum_{i\in N} c_i = \sum_{i\in N} c_i'$, $P_2(c,E)+P_3(c,E)+P_5(c,E)=P_2(c',E)+P_3(c',E)+P_5(c',E)$. Thus, the group M cannot receive more by transferring their claims under the proportional rule. A similar analysis can be made for each $N\in\mathcal{N}$, each $(c,E)\in\mathcal{C}^N$, each $M\subset N$, and each $(c_i')_{i\in M}\in\mathbb{R}_+^M$ such that $\sum_{i\in M} c_i=\sum_{i\in M} c_i'$ (unless $\sum_{i\in N} c_i=0$). Thus, the proportional rule satisfies *no advantageous transfer*.

The following theorem states that the proportional rule is the only one that satisfies this axiom for more than two agents.

Theorem 14.5 (Moulin 1985a, b; Chun 1988; Ju et al. 2007): The proportional rule is the only rule satisfying *no advantageous transfer* for more than two agents.

14.6 Concluding Remarks

This chapter studies the basics of designing a good (fair) allocation rule in economic problems by analyzing the so-called claims problems. In Sect. 14.5.1, we learn that the Talmud rule is the only rule satisfying *equal treatment of equals*, *minimal rights first*, *claims truncation invariance*, and *consistency*. Thus, if society agrees with the above four properties of rules (axioms), we can recommend adopting the Talmud rule. However, some other societies may disagree with one or more of these properties. Alternatively, these properties may be deemed acceptable or not depending on the situation (application) they face. Thus, providing other characterizations as in Sect. 14.5.2 would be very meaningful for selecting a good rule. This logic applies not only to claims problems but also to other economic problems. In fact, extensive studies (providing characterizations) have been conducted in the literature.[15]

To achieve Goal 16 (Peace, Justice and Strong Institutions) of the SDGs, especially when designing rules for economic problems, it becomes necessary to have this kind of precise analysis (mathematical analysis). As mentioned in Sect. 14.2, the problem studied in this chapter can also be interpreted as a taxation problem. In view of constructing a desirable tax system or correcting inequalities among people in other economic systems, our analysis in this chapter is related to Goal 10 (Reduced Inequalities) of the SDGs. Further, since our focus is on economic problems, our analysis is also closely related to Goal 8 (Decent Work and Economic Growth) of the SDGs. When we try to design good rules in economic problems, it is crucial to take into account a variety of strategic behaviors carried out by people. We studied one such behavior in Sect. 14.5.2 (recall the axiom of *no advantageous transfer*). In fact, extensive analyses have been performed in the literature.[16] If we successfully design an economic system that is immune to strategic behavior, such a system would motivate people to work and promote the nation's economic growth.

When constructing a building, we draw a plan in detail. The same thing must be done when designing a rule for economic problems (as we did for the claims problem in this chapter). By doing so, we may achieve several goals in the SDGs and make the world better in the future.

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