Circular L(j,k)-Labeling Numbers of Cartesian Product of Three Paths

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Abstract. The circular $L(j,k)$-labeling problem with $k \geq j$ arose from the code assignment in the wireless network of computers. Given a graph G and positive numbers j, k, σ, and a circular σ-$L(j,k)$-labeling of a graph G is an assignment f from $[0, \sigma)$ to the vertices of G, for any two vertices u and v, such that $|f(u) - f(v)|_\sigma \geq j$ if $uv \in E(G)$, and $|f(u) - f(v)|_\sigma \geq k$ if u and v are distance two apart, where $|f(u) - f(v)|_\sigma = min\{|f(u) - f(v)|, \sigma - |f(u) - f(v)|\}$. The minimum σ such that graph G has a circular σ-$L(j,k)$-labeling of a graph G, which is called the circular $L(j,k)$-labeling number of graph G and is denoted by $\sigma_{j,k}(G)$. In this paper, we determine the circular $L(j,k)$-labeling numbers of Cartesian product of three paths, where $k \geq 2j$.

Keywords: Code assignment · Circular-*L(j,k)*-labeling · Cartesian product

1 Introduction

The rapid growth of wireless networks causes the scarcity of available codes for communication in Multihop *Packet Radio Network* (PRN) which was studied in 1969 at University of Hawaii [1] firstly. In a multihop PRN, it is an important design consideration to assign transmission codes to network nodes. Because of the finite number of transmission codes, the number of network nodes may be larger than the number of transmission codes. It may take place that the time overlap of two or more packet receptions at the destination station. That is called *interference* or *collision*. For example, there exist two types of interference in a PRN using code division multiple access (CDMA). *Direct* interference occurs when two adjacent stations transmitting to each other directly. *Hidden terminal* interference is due to two stations at distance two communicate with the same receiving station at the same time.

Two stations are *adjacent* if they can transmit to each other directly. If two stations are called *at distance two* if two stations are nonadjacent but they are adjacent to one common station.

The wireless network can be modeled as an undirected graph $G = (V, E)$, such that the set of stations are represented as a set of *vertices* $V = \{v_0, v_1, \cdots, v_{n-1}\}$, and two vertices are joined by an undirected *edge* in E if and only if their corresponding stations can communicate directly.

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Since the interference (or collision) lowers the system throughput and increases the packets delay at destination, it is necessary to investigate the problem of code assignment for interference avoidance in Multi-hop PRN. Bertossi and Bonuccelli [2] introduced a type of code assignment for the network whose direct interference is so weak that we can ignore it, that is, only two distance-two stations are required to transmit by different codes to avoid the hidden terminal interference. By abstracting codes as labels, the above problem is equivalent to an $L(0, 1)$-labeling problem. That is, the distance-two vertices should be labeled numbers with difference at least 1.

In the real world, the direct interference cannot be ignored. In order to avoid the direct interference and hidden terminal interference, the code assignment problem was generalized to $L(j, k)$-labeling problem by Jin and Yeh [3], where $j \leq k$. That is, to avoid direct interference, any two adjacent stations must be assigned codes with difference at least j, then any two distance-two apart stations are required to be assigned larger code differences to avoid hidden terminal interference, as well as to avoid direct interference.

For two positive real numbers j and$k$, an $L(j, k)$-labeling f of G is a mapping of numbers to vertices of G such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if u, v are at distance two, where $|a-b|$ is called *linear difference*. The *$L(j, k)$-labeling number* of G is denoted by $\lambda_{j,k}(G)$, where $\lambda_{j,k}(G) = \min_{f} \max_{u,v \in V(G)} \{|f(u)-f(v)|\}$. For $j \leq k$, there exist some results on the $L(j, k)$-labeling of graphs. For example, Wu introduced the $L(j, k)$-labeling numbers of generalized Petersen graphs [4] and Cactus graphs [5], Shiu and Wu investigated the $L(j, k)$-labeling numbers of direct product of path and cycles [6, 7], Wu, Shiu and Sun [8] determined the $L(j, k)$-labeling numbers of Cartesian product of path and cycle..

For any $x \in \mathbb{R}$,$[x]_\sigma \in [0, \sigma)$ denotes the remainder of x upon division by σ. The *circular difference* of two points p and q is defined as $|p-q|_\sigma = min\{|p-q|, \sigma-|p-q|\}$.

Heuvel, Leese and Shepherd [9] used the circular difference to replace the linear difference in the definition of $L(j, k)$-labeling, and obtained the definition of circular $L(j, k)$-labeling as follows.

Given G and positive real numbers j and k, a circular σ- $L(j, k)$-labeling of G is a function $f: V(G) \to [0, \sigma)$ satisfying $|f(u)-f(v)|_\sigma \geq j$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq k$ if $d(u, v) = 2$. The minimum σ is called the *circular $L(j, k)$-labeling number* of G, denoted by $\sigma_{j,k}(G)$. For $j \leq k$, this problem was rarely investigated. For instance, Wu and Lin [10] introduced the circular $L(j, k)$- labeling numbers of trees and products of graphs. Wu, Shiu and Sun [11] determined the circular $L(j, k)$-labeling numbers of direct product of path and cycle. Furthermore, Wu and Shiu [12] investigated the circular $L(j, k)$-labeling numbers of square of paths.

Two labels are *t-separated* if the circular difference between them is at least t.

The *Cartesian product* of three graphs G, H and K, denoted by $G\Box H\Box K$, is the graph with vertices set $V(G\Box H\Box K) = V(G) \times V(H) \times V(K)$, and two vertices $v_{u,v,w}, v_{u',v',w'} \in V(G\Box H\Box K)$ are adjacent if $v_u = v_{u'}, v_v = v_{v'}$ and $(v_w, v_{w'}) \in E(K)$, or $v_u = v_{u'}, v_w = v_{w'}$ and $(v_v, v_{v'}) \in E(H)$, or $v_w = v_{w'}, v_v = v_{v'}$ and $(v_u, v_{u'}) \in E(G)$. For convenience, the Cartesian product of three paths P_l, P_m and P_n is denoted by $G_{l,m,n}$. For any vertex $v_{x,y,z} \in V\big(G_{l,m,n}\big)$,$x, y, z$ are called *subindex* of vertex. If two vertices with one different subindex are called *at the same row*. For instance, $v_{a,y,z}$ and $v_{b,y,z}$ are at the same row, where $a \neq b$, $0 \leq a, b \leq l-1$, $0 \leq y \leq m-1$, and $0 \leq z \leq n-1$.

All notations not defined in this thesis can be found in the book [13].

2 Circular $L(j, k)$-Labeling Numbers of Cartesian Product of Three Paths

Lemma 2.1 [10]. Let j and k be two positive numbers with $j \leq k$. Suppose H is an induced subgraph of graph G. Then $\sigma_{j,k}(G) \geq \sigma_{j,k}(H)$.

Note that Lemma 2.1 is not true if H is not an induced subgraph of G. For example, $\sigma_{1,2}(K_{1,3}) = 6 > 4 = \sigma_{1,2}(K_4)$, where $K_{1,3}$ is a subgraph of K_4 instead of an induced subgraph.

Lemma 2.2 [5]. Let a, b and σ be three positive real numbers, then $|[a]_\sigma - [b]_\sigma|$ equals to $[a-b]_\sigma$ or $\sigma - [a-b]_\sigma$.

Lemma 2.3. Let a, b and σ be three positive real numbers with $0 \leq a < \sigma$, then $[a+b]_\sigma - [b]_\sigma = a$ or $a - \sigma$.

Proof: The conclusion can be obtained as following cases.

a) If $0 \leq a+b < \sigma$ and $0 \leq b < \sigma$, then $[a+b]_\sigma - [b]_\sigma = a+b-b = a$.
b) If $\sigma \leq a+b < 2\sigma$ and $0 \leq b < \sigma$, then $[a+b]_\sigma - [b]_\sigma = a+b-\sigma-b = a-\sigma$.
c) If $\sigma \leq b$, let $b = r + k\sigma$, where $0 \leq r < \sigma$ and $k \in \mathbb{Z}^+$, according to the above two cases, we have $[a+b]_\sigma - [b]_\sigma = [a+r]_\sigma - [r]_\sigma = a$ or $a - \sigma$.

Hence, the lemma is proved.

2.1 Circular $L(j, k)$-labeling Numbers of Graph $G_{2,m,n}$

This subsection introduces the circular $L(j, k)$-labeling numbers of $G_{2,m,n}$ for $m, n \geq 2$ and $k \geq 2j$.

Theorem 2.1.1 Let j and k be two positive numbers with $k \geq 2j$. For $n \geq 2$, Then $\sigma_{j,k}(G_{2,2,n}) = 4k$.

Proof: Given a circular labeling f for $G_{2,2,n}$ as follows:

$$f(v_{0,0,z}) = \left[\frac{zk}{2}\right]_{2k}, f(v_{1,0,z}) = \left[\frac{(z+3)k}{2}\right]_{2k} + 2k,$$
$$f(v_{0,1,z}) = \left[\frac{(z+1)k}{2}\right]_{2k} + 2k, f(v_{1,1,z}) = \left[\frac{(z+2)k}{2}\right]_{2k},$$

where $0 \leq z \leq n-1$.

Note that the labels of two adjacent vertices at the same row are $\frac{k}{2}$-separated ($k \geq 2j$), and the labels of distance-two vertices at the same row are k-separated. Let $\sigma = 4k$. For an arbitrary vertex $v_{x,y,z} \in V(G_{2,2,n})$, according to the symmetry of the graph $G_{2,2,n}$, we need to check the differences between the labels of vertices $v_{x,y,z}$ and $v_{1-x,1-y,z}$, $v_{x,1-y,z\pm1}$, $v_{1-x,y,z\pm1}$(if they exist). That is, we need to make sure that f satisfies the following cases.

a) $\left|f\left(v_{1-x,y,z+1}\right) - f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, where $x, y \in \{0, 1\}, 0 \leq z \leq n-1$.
By Lemma 2.3 and the definition of circular difference, we have the following four subcases.

$$\left|f\left(v_{1,0,z+1}\right) - f\left(v_{0,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+4)k}{2}\right]_{2k} + 2k - \left[\frac{zk}{2}\right]_{2k}\right|_{4k}$$
$$= |2k|_{4k} \geq k.$$
$$\left|f\left(v_{0,0,z+1}\right) - f\left(v_{1,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+1)k}{2}\right]_{2k} - \left(\left[\frac{(z+3)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-3k|_{4k} \text{ or } |-k|_{4k} \geq k.$$
$$\left|f\left(v_{1,1,z+1}\right) - f\left(v_{0,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z+3)k}{2}\right]_{2k}\right) - \left(\left[\frac{(z+1)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-k|_{4k} \text{ or } |-3k|_{4k} \geq k.$$
$$\left|f\left(v_{0,1,z+1}\right) - f\left(v_{1,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z+2)k}{2}\right]_{2k} + 2k\right) - \left(\left[\frac{(z+2)k}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |2k|_{4k} \geq k.$$

Thus, $\left|f\left(v_{1-x,y,z+1}\right) - f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, for $x, y \in \{0, 1\}$, $0 \leq z \leq n-1$.

b) $\left|f\left(v_{1-x,y,z-1}\right) - f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, where $x, y \in \{0, 1\}, 0 \leq z \leq n-1$.
By Lemma 2.3 and the definition of circular difference, we have the following four subcases.

$$\left|f\left(v_{1,0,z-1}\right) - f\left(v_{0,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+2)k}{2}\right]_{2k} + 2k - \left[\frac{zk}{2}\right]_{2k}\right|_{4k}$$
$$= |3k|_{4k} \text{ or } |k|_{4k} \geq k.$$
$$\left|f\left(v_{0,0,z-1}\right) - f\left(v_{1,0,z}\right)\right|_{4k} = \left|\left[\frac{(z-1)k}{2}\right]_{2k} - \left(\left[\frac{(z+3)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-2k|_{4k} \geq k.$$
$$\left|f\left(v_{1,1,z-1}\right) - f\left(v_{0,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z+1)k}{2}\right]_{2k}\right) - \left(\left[\frac{(z+1)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-2k|_{4k} \geq k.$$
$$\left|f\left(v_{0,1,z-1}\right) - f\left(v_{1,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{zk}{2}\right]_{2k} + 2k\right) - \left(\left[\frac{(z+2)k}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |3k|_{4k} \text{ or } |k|_{4k} \geq k.$$

Thus, $\left|f\left(v_{1-x,y,z-1}\right) - f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, for $x, y \in \{0, 1\}$, $0 \leq z \leq n-1$.

c) $.\left|f\left(v_{x,1-y,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, where $x, y \in \{0, 1\}, 0 \leq z \leq n-1$.
By Lemma 2.3 and the definition of circular difference, we have the following four subcases.

$$\left|f\left(v_{1,1,z+1}\right)-f\left(v_{1,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+3)k}{2}\right]_{2k} - \left[\frac{(z+3)k}{2}\right]_{2k} - 2k\right|_{4k}$$
$$= |-2k|_{4k} \geq k.$$
$$\left|f\left(v_{1,0,z+1}\right)-f\left(v_{1,1,z}\right)\right|_{4k} = \left|\left[\frac{(z+4)k}{2}\right]_{2k} + 2k - \left(\left[\frac{(z+2)k}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |3k|_{4k} \text{ or } |k|_{4k} \geq k.$$
$$\left|f\left(v_{0,1,z+1}\right)-f\left(v_{0,0,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z+2)k}{2}\right]_{2k} + 2k\right) - \left(\left[\frac{zk}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |3k|_{4k} \text{ or } |k|_{4k} \geq k.$$
$$\left|f\left(v_{0,0,z+1}\right)-f\left(v_{0,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z+1)k}{2}\right]_{2k}\right) - \left(\left[\frac{(z+1)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-2k|_{4k} \geq k.$$

Thus, $\left|f\left(v_{x,1-y,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, for $x, y \in \{0, 1\}$, $0 \leq z \leq n-1$.

d) $\left|f\left(v_{x,1-y,z-1}\right)-f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, where $x, y \in \{0, 1\}, 0 \leq z \leq n-1$.
By Lemma 2.3 and the definition of circular difference, we have

$$\left|f\left(v_{1,1,z-1}\right)-f\left(v_{1,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+1)k}{2}\right]_{2k} - \left[\frac{(z+3)k}{2}\right]_{2k} - 2k\right|_{4k}$$
$$= |-3k|_{4k} \text{ or } |-k|_{4k} \geq k.$$
$$\left|f\left(v_{1,0,z-1}\right)-f\left(v_{1,1,z}\right)\right|_{4k} = \left|\left[\frac{(z+2)k}{2}\right]_{2k} + 2k - \left(\left[\frac{(z+2)k}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |2k|_{4k} \geq k.$$
$$\left|f\left(v_{0,1,z-1}\right)-f\left(v_{0,0,z}\right)\right|_{4k} = \left|\left(\left[\frac{zk}{2}\right]_{2k} + 2k\right) - \left(\left[\frac{zk}{2}\right]_{2k}\right)\right|_{4k}$$
$$= |2k|_{4k} \geq k.$$
$$\left|f\left(v_{0,0,z-1}\right)-f\left(v_{0,1,z}\right)\right|_{4k} = \left|\left(\left[\frac{(z-1)k}{2}\right]_{2k}\right) - \left(\left[\frac{(z+1)k}{2}\right]_{2k} + 2k\right)\right|_{4k}$$
$$= |-3k|_{4k} \text{ or } |-k|_{4k} \geq k.$$

Thus, $\left|f\left(v_{x,1-y,z-1}\right)-f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, for $x, y \in \{0, 1\}$, $0 \leq z \leq n-1$.

e) $\left|f\left(v_{1-x,1-y,z}\right)-f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, where $x, y \in \{0, 1\}, 0 \leq z \leq n-1$.
By Lemma 2.3 and the definition of circular difference, we have the following four subcases.

$$\left|f\left(v_{1,1,z}\right)-f\left(v_{0,0,z}\right)\right|_{4k} = \left|\left[\frac{(z+2)k}{2}\right]_{2k} - \left[\frac{zk}{2}\right]_{2k}\right|_{4k} = |-k|_{4k} \text{ or } |k|_{4k} \geq k.$$
$$\left|f\left(v_{1,0,z}\right)-f\left(v_{0,1,z}\right)\right|_{4k}$$

$$= \left| \left[\frac{(z+3)k}{2} \right]_{2k} + 2k - \left(\left[\frac{(z+1)k}{2} \right]_{2k} + 2k \right) \right|_{4k} = |k|_{4k} \text{ or } |-k|_{4k} \geq k.$$

Thus, $\left|f\left(v_{1-x,1-y,z}\right) - f\left(v_{x,y,z}\right)\right|_{4k} \geq k$, for $x, y \in \{0, 1\}$, $0 \leq z \leq n-1$.

Hence, f is a circular $4k$-$L(j, k)$-labeling of graph $G_{2,2,n}$, it means that $\sigma_{j,k}\left(G_{2,2,n}\right) \leq 4k$ for $n \geq 2$ and $k \geq 2j$.

Figure 1 shows a circular $4k$-$L(j, k)$-labeling of graph $G_{2,2,8}$.

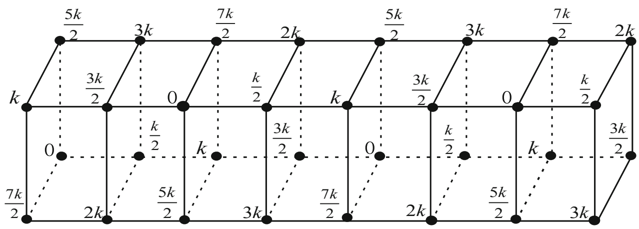

Fig. 1. A circular 4k-L(j, k)-labeling of graph $G_{2,2,8}$

On the other hand, the vertices $v_{0,0,0}$, $v_{1,0,1}$, $v_{0,1,1}$, and $v_{1,1,0}$ are distance two apart mutually, the circular difference among their labels should be at least k, it implies that $\sigma_{j,k}\left(G_{2,2,n}\right) \geq 4k$ for $n \geq 2$.
Hence, $\sigma_{j,k}\left(G_{2,2,n}\right) = 4k$ for $n \geq 2$ and $k \geq 2j$.

Theorem 2.1.2. Let j and k be two positive real numbers with $k \geq 2j$. For $m, n \geq 3$, $\sigma_{j,k}\left(G_{2,m,n}\right) = 5k$.

Proof: Defined a circular labeling f for graph $G_{2,m,n}$ as follows:

$$f\left(v_{x,y,z}\right) = \left[\frac{(5x+y+3z)k}{2} \right]_{5k},$$

where $x = 0, 1$, $0 \leq y \leq m-1$ and $0 \leq z \leq n-1$.

Note that the labels of adjacent vertices at the same row are $\frac{k}{2}$-separated ($k \geq 2j$) and the labels of vertices with distance two apart at the same row are k-separated. Let $\sigma = 5k$. For an arbitrary vertex $v_{x,y,z} \in V(G_{2,m,n})$, according to the symmetry of the graph $G_{2,m,n}$, it is sufficient to verify the circular differences between $v_{x,y,z}$ and $v_{1-x,y+1,z}$, $v_{1-x,y,z+1}$, $v_{x,y+1,z\pm1}$(If they exist) are k-separated, respectively, where $x \in \{0, 1\}$,$0 \leq y \leq m-1$ and $0 \leq z \leq n-1$. By Lemma 2.3 and the definition of circular difference, we have the following results.

a)
$$\begin{aligned}
&\left|f\left(v_{1-x,y+1,z}\right)-f\left(v_{x,y,z}\right)\right|_{5k}\\
&=\left|\left[\frac{[5(1-x)+y+1+3z]k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=\left|\left[\frac{(6-5x+y+3z)k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=2k\geq k.
\end{aligned}$$

b)
$$\begin{aligned}
&\left|f\left(v_{1-x,y,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{5k}\\
&=\left|\left[\frac{[5(1-x)+y+3(z+1)]k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=\left|\left[\frac{(8-5x+y+3z)k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=k\geq k.
\end{aligned}$$

c)
$$\begin{aligned}
&\left|f\left(v_{x,y+1,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{5k}\\
&=\left|\left[\frac{[5x+y+1+3(z+1)]k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=\left|\left[\frac{(4+5x+y+3z)k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=2k\geq k.
\end{aligned}$$

d)
$$\begin{aligned}
&\left|f\left(v_{x,y+1,z-1}\right)-f\left(v_{x,y,z}\right)\right|_{5k}\\
&=\left|\left[\frac{[5x+y+1+3(z-1)]k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=\left|\left[\frac{(5x+y+3z-2)k}{2}\right]_{5k}-\left[\frac{(5x+y+3z)k}{2}\right]_{5k}\right|_{5k}\\
&=k\geq k.
\end{aligned}$$

Hence, f is a circular $5k$-$L(j,k)$-labeling of graph $G_{2,m,n}$, it means that $\sigma_{j,k}\left(G_{2,m,n}\right)\leq 5k$ for $m,n\geq 3$ and $k\geq 2j$.

For example, Fig. 2 is a circular $5k$-$L(j,k)$-labeling of graph $G_{2,3,3}$.

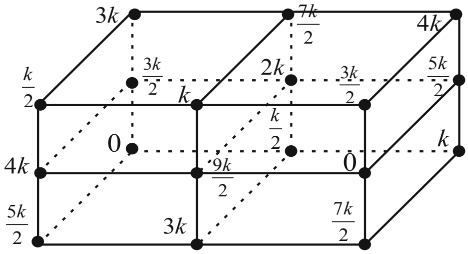

Fig. 2. A circular 5k-L(j, k)-labeling of graph $G_{2,3,3}$.

On the other hand, the vertices $v_{0,0,1}, v_{0,1,2}, v_{0,1,0}, v_{0,2,1}$ and $v_{1,1,1}$ are at distance two from each other, the circular difference among their labels should be at least k, it implies that $\sigma_{j,k}(G_{2,m,n}) \geq 5k$ for m,$n \geq 3$.

Hence, $\sigma_{j,k}(G_{2,m,n}) = 5k$ for $m, n \geq 3$ and $k \geq 2j$.

2.2 Circular L(j, k)-Labeling Numbers of Graph $G_{2,m,n}$

This subsection introduces the general results on the circular $L(j,k)$-labeling numbers of $G_{l,m,n}$ for $l, m, n \geq 3$ and $k \geq 2j$.

Theorem 2.2.1. Let j and k be three positive real numbers with $k \geq 2j$. For $l, m, n \geq 3$, $\sigma_{j,k}(G_{l,m,n}) = 6k$.

Proof: Given a circular labeling f for $G_{l,m,n}$ as follows:

$$f(v_{x,y,z}) = \left[\frac{(3x+y+5z)k}{2}\right]_{6k},$$

where $0 \leq x \leq l-1, 0 \leq y \leq m-1$ and $0 \leq z \leq n-1$.

Note that the labels of adjacent vertices at the same row are $\frac{k}{2}$-separated ($k \geq 2j$) and the labels of distance-two vertices at the same row are k-separated. Let $\sigma = 6k$. For an arbitrary vertex $v_{x,y,z} \in V(G_{l,m,n})$, according to the symmetry of the graph $G_{l,m,n}$, it is sufficient to verify the circular differences between $v_{x,y,z}$ and $v_{x+1,y\pm1,z}$, $v_{x+1,y,z\pm1}$, $v_{x,y+1,z\pm1}$(If they exist) are k-separated, respectively, where $0 \leq x \leq l-1, 0 \leq y \leq m-1$ and $0 \leq z \leq n-1$. By Lemma 2.3 and the definition of circular difference, we have the following results.

a)
$$\begin{aligned}
&|f(v_{x+1,y+1,z}) - f(v_{x,y,z})|_{6k} \\
&= \left|\left[\frac{[3(x+1)+(y+1)+5z]k}{2}\right]_{6k} - \left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k} \\
&= \left|\left[\frac{(4+3x+y+3z)k}{2}\right]_{6k} - \left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k} \\
&= 2k \geq k.
\end{aligned}$$

b)
$$\begin{aligned}
&\left|f\left(v_{x+1,y-1,z}\right)-f\left(v_{x,y,z}\right)\right|_{6k}\\
&=\left|\left[\frac{\left[3(x+1)+(y-1)+5z\right]k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=\left|\left[\frac{(2+3x+y+3z)k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=k\geq k.
\end{aligned}$$

c)
$$\begin{aligned}
&\left|f\left(v_{x+1,y,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{6k}\\
&=\left|\left[\frac{\left[3(x+1)+y+5(z+1)\right]k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=\left|\left[\frac{(8+3x+y+3z)k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=2k\geq k.
\end{aligned}$$

d)
$$\begin{aligned}
&\left|f\left(v_{x+1,y,z-1}\right)-f\left(v_{x,y,z}\right)\right|_{6k}\\
&=\left|\left[\frac{\left[3(x+1)+y+5(z-1)\right]k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=\left|\left[\frac{(3x+y+3z-2)k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=k\geq k.
\end{aligned}$$

e)
$$\begin{aligned}
&\left|f\left(v_{x,y+1,z+1}\right)-f\left(v_{x,y,z}\right)\right|_{6k}\\
&=\left|\left[\frac{\left[3x+y+1+5(z+1)\right]k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=\left|\left[\frac{(6+3x+y+3z)k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=3k\geq k.
\end{aligned}$$

f)
$$\begin{aligned}
&\left|f\left(v_{x,y+1,z-1}\right)-f\left(v_{x,y,z}\right)\right|_{6k}\\
&=\left|\left[\frac{\left[3x+y+1+5(z-1)\right]k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=\left|\left[\frac{(3x+y+3z-4)k}{2}\right]_{6k}-\left[\frac{(3x+y+5z)k}{2}\right]_{6k}\right|_{6k}\\
&=2k\geq k.
\end{aligned}$$

Hence, f is a circular $6k$-$L(j,k)$-labeling of graph $G_{l,m,n}$, it means that $\sigma_{j,k}\left(G_{l,m,n}\right)\leq 6k$ for $l,m,n\geq 3$ and $k\geq 2j$.

For example, Fig. 3 is a circular $6k$-$L(j,k)$-labeling of graph $G_{3,3,3}$.

On the other hand, the vertices $v_{1,0,1}, v_{0,1,1}, v_{1,2,1}, v_{2,1,1}, v_{1,1,2}$ and $v_{1,1,0}$ are at distance two from each other, the circular difference among their labels should be at least k, this means $\sigma_{j,k}\left(G_{l,m,n}\right)\geq 6k$ for $l,m,n\geq 3$.

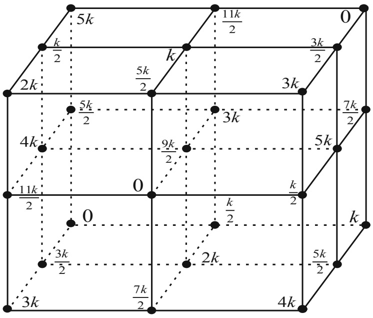

Fig. 3. A circular 6k-L(j, k)-labeling of graph $G_{3,3,3}$.

Hence, $\sigma_{j,k}(G_{l,m,n}) = 6k$ for $l, m, n \geq 3$ and $k \geq 2j$.

3 Conclusion

In this paper, we investigate the circular $L(j, k)$-labeling number of Cartesian product of three paths which arose from the code assignment of interference avoidance in the PRN. For $k \geq 2j$, we obtain that

$$\sigma_{j,k}(G_{l,m,n}) = \begin{cases} 4k, \ if\ l, m = 2\ and\ n \geq 2, \\ 5k, \ if\ l = 2\ and\ m, n \geq 3, \\ 6k, \qquad if\ l, m, n \geq 2. \end{cases}$$

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