THE CSMP STAFF

THE CSMP DEVELOPMENT OF GEOMETRY

Of all the decisions one must make in a curriculum development project with respect to choice of content usually the most controversial and least defensible is the decision about geometry. Pressures shaping such decisions in our time range from "Euclid is dead; teach them linear algebra", to "Teach them a rigorous Euclidean geometry with Euclid's deficiencies corrected".

The approaches to geometry that will finally be adopted by CSMP should be designed to give varied and rich experiences to students at several levels, rather than a single approach at one level. It is not a coincidence that this International Conference on the Teaching of Geometry is convened at a time when CSMP is making its final decisions about the kinds of geometries it will teach at three levels of its program. The following is a description of CSMP's current planning; certainly this conference will prove a deciding factor in revisions of these plans.

ELEMENTARY SCHOOL: FIRST EXPERIENCES

The CSMP elementary geometry program provides initial geometric experiences and will eventually dove-tail with the geometry of the secondary program. Such a program must not only satisfy the prerequisites for an anticipated secondary program, but also miss no opportunity to exploit each child's intuition with respect to the geometry of the world of two and three dimensions. Because of the non-standard nature of the content of the elementary program, it is described here in some detail.

In K through 2, the average child does not read fluently, and so all his geometrical experiences must necessarily be gained via concrete situations. These take the form of recognizing and naming simple planar and space figures, making distinctions between the inside and the outside of simple closed curves, and experimenting with tesselations and reflective symmetries. The vehicles for these experiences consist of teacher-directed play and games involving attribute blocks (both flat and solid), mazes, figure fitting, mirror cards, paper folding, etc.

When the child can read, he is allowed to proceed independently at his own pace and at his own depth of comprehension. At this stage most of his initial observations of geometrical situations are over and he begins to explore the

H.-G. Steiner (ed.), *The Teaching of Geometry at the Pre-College Level*

relations between the things he observes. He considers congruence of planar figures by deciding whether or not they "match." He is introduced to translations as slide mappings of the plane, and reflections are introduced as fold mappings of the plane. He decides whether certain planar figures are reflective symmetric by means of folding or tracing. Rotations are seen as turn mappings of the plane, and the child is able to find images of points and figures under any of these transformations, or under a composition of such transformations.

This knowledge of the isometries of the plane is then exploited in the definitions of parallel and perpendicular, and in describing various planar figures such as parallelogram, square, rhombus. He is introduced to parallel projection as a first experience of a mapping of the plane which is not one-to-one. His knowledge of parallel projections is also extended to three dimensions by means of activities dealing with shadows of planar figures as well as three-dimensional shapes. His first experience of similarity of planar figures is in terms of homotheties, which are seen as combinations of translations and dilatations.

After the child is led by experimentation to discover Euler's formula for a polyhedron, he is introduced to solids with holes in them; he works out their Euler characteristics and sees a pattern. Thus, he will be able to predict the Euler characteristic of a solid, given the number of holes.

The earlier notions of symmetry are now described in terms of invariant sets of isometries of the plane. He is introduced to the properties of a group by consideration of the group of symmetries of an equilateral triangle. The group concept is then further developed when the set of translations is shown to be a group. A central symmetry or halfturn is introduced, and it is shown that the set of central symmetries generates the group of all central symmetries and translations.

Vectors are introduced in terms of translations and are used to show some standard results in the plane.

The area of a rectangle is found by means of counting unit squares. Using the fact that translations preserve area, it is shown that parallelograms between the same parallels and on the same base are equal in area. As a corollary, the child can find the area of any triangle. Volume is introduced by an extension of the techniques used in finding areas.

The child completes his intuitive knowledge of similarity by considering similarities which involve rotations and/or reflections as well as translations and dilatations. He is introduced to the idea of orientation in the plane, and he reconsiders the isometries of the plane in terms of their orientation-preserving or reversing properties. Having done this, the child has a brief look at orientation in space.

The elementary program culminates in the introduction of angle measure and various construction methods.

Having completed this program, students will have covered, in an intuitive and experimental way, the background necessary for an appropriate secondary program in which such first experiences are formalized. Every student will have experienced geometric activities commensurate with his ability and interest.

MIDDLE SCHOOL: EXPLOITATION OF GEOMETRIC INTUITION

It is planned that the CSMP curriculum will some day extend from K through 12. Currently, the primary and secondary programs are being developed disjointly. The present need is for a geometry course preceding the secondary program which provides the intuitive experiences in geometry on which the more formal treatments in the secondary program can be built.

This intermediate course in geometry serves two purposes. For the time being it will be a substitute for the geometric background anticipated in the projected CSMP primary program. In the future it will serve as a bridge for entry into the CSMP secondary program by students who have not experienced the CSMP primary program.

The course, called *Intuitive Geometry*, begins with descriptions of incidence properties of points, lines, and planes that are suggested by physical experiences. When intuition fails to support the necessary notions of completeness of points in a line, agreements and definitions are made which replace intuition and which lead to a naive introduction to real numbers and distance mappings.

As vehicles for the development of congruence, symmetry, similarity and classification of figures, the primitives *line reflection* and *dilatation* are selected.

After descriptions are given of mappings and composites of mappings in general, the intuition concerning motion of figures is exploited to introduce mappings of the plane and space. From the primitive (mirror) reflection mapping, all the isometries of the plane are constructed as composites of reflections. Certain properties extend to all composites of reflections, giving characterizing properties of all mappings in the group of isometries.

Congruence of figures is then defined in terms of isometries, and properties of congruence stem from properties of isometries. Much of the usual classification of figures is then done in terms of reflective symmetries. Similarity of figures is described and studied in terms of composites of dilatations and isometries. In particular, right triangles are sorted into similarity classes, and the trigonometric ratios are defined for each class. Occasional extensions are made of reflections and their composites from the plane to space.

Vectors are introduced in terms of translations, and the course terminates with some elementary consequences of the properties of a plane vector space. Coordinate frames are used occasionally to provide an algebraic representation.

This course is in an intermediate position in several ways. It lies between the experimental probing of geometric experiences in the primary school and the formal deduction of the secondary program. As such it utilizes all the paraphernalia of tracings, mirrors, paper folding to arrive at geometric conjectures, and then makes heuristic arguments about the conjectures. It also begins to perfect the enormously powerful device of mappings in searching for geometric truths, a process to be formalized in the secondary program.

SECONDARY SCHOOL: FORMAL GEOMETRY

Two books in the Elements of Mathematics series, CSMP's secondary program, are devoted to Geometry. Book 10, with the tentative title *Elements of Geometry*, develops the synthetic approach to geometry, while Book 12, *Linear Geometry with Trigonometry*, is planned to utilize the linear algebra approach.

Book 10 will not develop all of Euclidean geometry. On the contrary, Book 10 is planned as a short book that will develop only a fragment of geometry, but that fragment rigorously.

Since no history has as yet been included in the *Elements of Mathematics* series, and since the history of geometry is very rich, introductory and concluding historical chapters will be included in Book 10. The first chapter will investigate the origins of mathematics (not just geometry), both in the East and in the West, and will culminate with Euclid.

The middle chapters of the book will develop the formal theory. Affine geometry in 2- and 3-space will be studied with considerable emphasis given to characterizing the finite affine planes. Hilbert's system will be mentioned, but Euclidean geometry will not formally be developed in Book 10 beyond ordered affine geometry. Rather, projective planes will be investigated, particularly finite projective planes. This will be an easy task, as the usual correspondence between affine and projective planes will be established. Also, of course, duality will be stressed: given the logical background of the students, duality should be an extremely meaningful and interesting topic.

The concluding chapter of Book 10 will pick up the history of the parallel postulate and it should serve as an informal introduction to non-Euclidean geometry for any student who wishes to read further in that subject.

Book 12 is in a very preliminary planning stage; thus, only a rough outline can be given. The main objective of the book should be a thorough investigation of plane and space geometry using the language and tools of linear algebra. Thus, emphasis will be on the development of two- and three-dimensional vector spaces over the field R of real numbers and the use of these vector spaces to study geometry. CSMP students have studied fields other than the field of real numbers; therefore vector spaces can be defined over arbitrary fields with no restriction on the dimension. Examples and exercises might well be given about vector spaces over the field of complex numbers, the field of rational numbers and finite fields; nevertheless, the main investigation should be restricted to two- and three-dimensional vector spaces over the field of real numbers. If students obtain a thorough knowledge and working facility in these dimensions, the transition to higher dimensions is almost trivial. Similar remarks hold for the study of higher dimensional geometry.

The proposed content of Book 12 is divided into the six 'chapters' listed below.

1. Vector Spaces and Linear Transformations
2. Affine Spaces and Affine Transformations
3. Bilinear Functions (Inner Products)
4. Euclidean Geometry and Euclidean Transformations
5. Trigonometry
6. Further Topics from Linear Algebra and Geometry.

After the conference these topics will be developed into a more detailed outline.