A Computable Solution to Partee's Temperature Puzzle

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Abstract. This paper presents a computable solution to Partee's temperature puzzle which uses one of the standard tools of mathematics and the exact sciences: countable approximation. Our solution improves upon the standard Montagovian solution to the puzzle (i) by providing computable natural language interpretations for this solution, (ii) by lowering the complexity of the types in the puzzle's interpretation, and (iii) by acknowledging the role of linguistic and communicative context in this interpretation. These improvements are made possible by interpreting natural language in a model that is inspired by the Kleene-Kreisel model of countable-continuous functionals. In this model, continuous functionals are represented by lower-type objects, called the *associates* of these functionals, which only contain countable information.

Keywords: Temperature puzzle · Individual concepts · Associates · Continuous functionals · Computability

1 Partee's Puzzle and Montague's Solution

Partee's temperature puzzle [33, p. 267] is a touchstone for any formal semantics for natural language. This puzzle regards the incompatibility of our intuitions about the validity of the inference from (1) (i.e. *in*valid) with predictions about the validity of this inference in extensional semantics (cf. [8,32]) (i.e. valid).

a. The temperature is ninety.
b. The temperature rises. (1)
/////////////////////
c. Ninety rises.

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$$\frac{\exists c^{se}\big(\forall c_1^{se}[\text{TEMP}^{(se)t}(c_1) \leftrightarrow c = c_1] \wedge c(@^s) = \text{NINETY}^e\big) \quad \exists c^{se}\big(\forall c_1^{se}[\text{TEMP}^{(se)t}(c_1) \leftrightarrow c = c_1] \wedge \text{RISE}^{(se)t}(c)\big)}{\text{RISE}^{(se)t}(\mathbf{ninety}^{se})} \quad \text{(invalid)} \tag{2}$$

Montague-style formal semantics (e.g. [13,17,29,33]) solve this puzzle by distinguishing two readings of the DP the temperature: a function-reading (cf. (1b)), on which the DP is interpreted as an *individual concept* (i.e. as a function from indices/world-time pairs to individuals; type[1] *se*), and a value-reading (cf. (1a)), on which the DP is interpreted as the extension of this concept at the current index, @ (i.e. as an *individual*; type *e*). The different readings prevent the replacement of the occurrence of the DP the temperature from (1b) by the name ninety (s.t. the conclusion of (1) cannot be derived from the premises) (cf. (2)).[2]

2 Problems with Montague's Solution

Montague's solution to the temperature puzzle is inspired by Carnap's theory of intensions (cf. [7]) and is supported by the fact that Montague semantics already uses indices in the semantic analysis of declarative sentences, which are interpreted as functions from indices to truth-values (cf. also [26]). Because of its ready availability, Montague's solution has been adopted by many contemporary theories of formal semantics.[3] However, there are a number of problems with this solution. These include the non-computability of natural language interpretations in this solution, (ii) the high type-complexity of natural language interpretations in this solution, and (iii) the disregard of relevant contextual parameters in this solution. The latter are described below:

2.1 Problem 1: Non-Computability of NL Interpretations

Intensional (or 'possible world') semantics – which include Montague-style formal semantics – fail to provide computable (or 'effective') interpretations of natural language expressions. This is due to the non-computability of models of possible world semantics and the impossibility of finitely describing the set of possible worlds that provides the meaning of a sentence in the absence of the sentence's translating/intermediate formula (cf. [34]). As a result of these facts, intensional

[1] For brevity, we use a short notation for types, where *se* corresponds to the arrow type $s \rightarrow e$ and to Montague's type $\langle s, e\rangle$. We will hereafter indicate types in superscript.

[2] In (2), we assume that **ninety** is s.t. $\forall i^s(\mathbf{ninety}(i) = \text{NINETY})$.

[3] These theories include hyperintensional theories (e.g. [16,39]), which do not adopt an atomic type for indices, and relational theories (e.g. [35,48]), which only accept non-atomic types with range Bool. To accommodate the intensionality of DPs like the temperature in (1b), hyperintensional theories introduce an atomic type for individual concepts. Relational theories code individual concepts as binary relations between indices and individuals.

semantics are unable to compute the semantic representation of a given sentence. However, given the need to explain the human ability to form and understand new complex expressions (cf. [9,15,37,44]), such an effective semantics is clearly desirable.

2.2 Problem 2: High-Rank Typing

The interpretation of DPs as individual concepts increases the complexity of the types of natural language interpretations. On Montague's interpretation, proper names and common nouns are expressions of rank 1 (i.e. se) resp. 2 ($(se)t$), rather than of rank 0 (e) resp. 1 (et), as in extensional semantics. Montague semantics even interprets transitive verbs – which have rank 3 (i.e. $((et)t)(et)$) in extensional semantics – in rank **4** (i.e. $(((se)t)t)((se)t)$). But this complicates the type of the interpretations of linguistic expressions analogously to the (much-criticized) treatment of referential DPs as generalized quantifiers (cf. [19,27,38]). Further, while formal semanticists and theoretical computer scientists are used to working with rank-4 (or higher-rank) objects, such objects are highly uncommon in the natural sciences and even in most parts of mathematics.

2.3 Problem 3: Context-Invariance

Montague's solution further neglects the salient role of context in the interpretation of the verb rise (cf. [10]): Intuitively, for different DPs, rise will assert the DP referent's rising *over different-length intervals*. Thus, in (1b), rise will be interpreted with respect to a shorter interval (e.g. minutes, or hours) than in the CP The oil price rises (e.g. weeks, or months). Even when applied to the same DP, rise is often interpreted with respect to different-length intervals. For example, in the context of global climate development, (1b) will be taken to make a claim about a longer interval than in the context of the local weather forecast. Since Montague semantics analyzes intensional intransitive verbs as characteristic functions of sets of individual concepts (which send all occurrences of a DP to the same truth-value), it does not capture this context-sensitivity.

3 Solving the Problems

We solve the above problems by interpreting natural language in a model[4] that is inspired by the *Kleene-Kreisel model of countable-continuous functionals* [21,25] (cf. [30, Ch. 2.3.1]). In this model, continuous functionals are represented by lower-type objects called *associates*.

Following Kleene [21] and Kreisel [25], we hereafter use *finite types* over the natural numbers. The latter are the smallest set of strings that contains the type for natural numbers, 0, and the types for function spaces over natural numbers,

[4] To enable a compositional interpretation of the sentences from (1) (cf. Sect. 4), this model extends the Kleene-Kreisel model (which only contains natural numbers and functions over natural numbers) to objects of higher type.

$(\rho \to \tau)$ (with ρ, τ finite types) (cf. [36]). To ease notation, we abbreviate the type for functions over natural numbers, $(0 \to 0)$, as '1', abbreviate the type for *functionals* over sequences of natural numbers, $((0 \to 0) \to 0)$ $(\equiv (1 \to 0))$, as '2', and abbreviate $(n \to 0)$ as '$n + 1$'. Our considerations will make special use of *coded* finite sequences of natural numbers (type 0). To distinguish natural numbers which *do* from natural numbers which do *not* code such sequences, we denote the former by '0^*'.

Our solution to the temperature puzzle briefly works as follows: By representing the DP the temperature from (1b) as (a code for) a finite sequence of natural numbers (type 0^*) and by approximating the continuous functional denoted by rise by an associate of type $1 \equiv (0^* \to 0)$, we 'lower' the types of many expressions from (1) (cf. Problem 2). In particular, our solution interprets the DP's occurrence from (1a) as a natural number (type 0) and the DP's occurrence from (1b) as a (coded) sequence of natural numbers (type 0^*). Since distinguishing between types 0 and 0^* is *decidable*, we obtain a *computable* solution to the temperature puzzle (cf. Problem 1). Because associates are introduced through the use of a context-dependent variable, the domain of application of the verb rise is restricted to a specific, contextually salient, temporal interval (cf. Problem 3). As to the computability of our solution, it suffices for now to point out that the Kleene-Kreisel model can be defined inside Martin-Löf type theory and has been implemented in the associated programming language Agda [14,45–47].

Note the integrative nature of our solution to the above problems: Since associates are *computable*, *lower-type* representations of continuous functionals that approximate these functionals *with regard to a contextually determined parameter*, our solution(s) to the above problems are all sides of the same (three-sided) coin. This contrasts with other solutions to the temperature puzzle (e.g. [3,20,27,41]) which still assume more complex types, are not effective, and/or rely on the use of other methods to render the interpretation of the sentences from (1) context-sensitive.

We describe our solution in some detail below. To this end, we first show how the Montagovian interpretation of the verb rise corresponds to a continuous functional (in Sect. 3.1). Following the informal introduction of associates (in Sect. 3.2), we then outline our *associates*-approach to the temperature puzzle (in Sect. 3.3). This approach receives a compositional implementation in Sect. 4. The empirical domain of our *associates*-approach and the computational properties of associates are discussed in Sects. 5 and 3.4.

3.1 Continuity and the Temperature Puzzle

Our solution to the temperature puzzle starts from the observation that the interpretation of RISE from (2) corresponds to a continuous functional, φ_{rise}, in the space $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. The correspondence between RISE and φ_{rise} is based on the possibility of representing individual concepts as sequences over natural numbers (assuming a fixed starting index/world-time pair $\langle w, t \rangle$ and a discrete unit of time measurement; cf. [27]). The latter enables the representation of the individual concept 'the temperature' from (3) as the sequence from (4), and the representation of sets of individual concepts as sets of such sequences.

$$\langle w, t_0\rangle \mapsto 89, \langle w, t_1\rangle \mapsto 90, \langle w, t_2\rangle \mapsto 91, \dots, \langle w, t_n\rangle \mapsto 89+n \tag{3}$$

$$89, 90, 91 \dots, 89+n \tag{4}$$

With this representation in mind, the temperature as given by $\gamma^1 = (T_0, T_1, \dots)$ (where T_0, T_1, etc. are the values of some temperature measurement) rises, i.e. RISE(γ), iff $\varphi_{\text{rise}}(\gamma) = 1$. The temperature as given by γ does not rise iff $\varphi_{\text{rise}}(\gamma) = 0$.

The *continuity* of the functional φ_{rise} is suggested by (i) the 'finite relevance' of input sequences for φ_{rise} and (ii) the equivalence of sequences which are identical up to some point in time.

Ad (i): Intuitively, after having observed a rise in the values of some temperature measurement *for a certain finite period of time*, even the most ardent skeptic will agree that the values are, in fact, rising. Thus, if the temperature as given by $\gamma = (T_0, T_1, \dots)$ is rising, i.e. if $\varphi_{\text{rise}}(\gamma) = 1$, we will agree to this fact after having observed the temperature up to some point in time n, i.e. by considering $(T_0, \dots, T_n)$.

Ad (ii): If the temperature as given by the values of some other measurement $\beta = (T'_0, T'_1, \dots)$ is further exactly γ up to the point in time n, we will agree that $\varphi_{\text{rise}}(\beta) = 1$, i.e. that the temperature as given by β is also rising. The functional φ_{rise} is thus continuous in the usual mathematical sense (cf. [30, Ch. 2.3.1]).

Continuity is defined below:

Definition 1 (Continuity of type-2 functionals). *A type-2 functional φ is* continuous (*on the Baire*[5] *space*) *if*

$$\forall\gamma^1\, \exists n^0\, \forall\beta^1 \big(\overline{\gamma}n = \overline{\beta}n \rightarrow \varphi(\gamma) = \varphi(\beta)\big), \tag{5}$$

where $\overline{\gamma}n = (T_0, T_1, \dots, T_n)$ and $\overline{\beta}n = (T'_0, T'_1, \dots, T'_n)$ (both type 0^*) are the initial segments (up to n) of γ and β.

Above, the point n (for φ_{rise}: a point in time at which everyone agrees that the temperature is rising) is called a *point of continuity* of φ (at γ). Obviously, this point may be different for different sequences. We will use this fact in Sect. 3.2 to explain the dependence of interpretations on the expressions' linguistic context.

The correspondence of the interpretation of RISE to the continuous functional φ_{rise} gives rise to the following 'continuous functional'-version of (2):

$$\begin{array}{c}\exists\gamma^1\big(\forall\beta^1[\mathit{temp}^2(\beta) \leftrightarrow \gamma = \beta] \wedge \mathit{now}^2(\gamma) = \mathit{ninety}^0\big)\\ \exists\gamma^1\big(\forall\beta^1[\mathit{temp}^2(\beta) \leftrightarrow \gamma = \beta] \wedge \varphi^2_{\text{rise}}(\gamma) = 1\big)\\ \hline /\,/\,/\,/\,/\,/ \qquad \varphi^2_{\text{rise}}(\boldsymbol{ninety}^1) = 1 \qquad /\,/\,/\,/\,/\,/\end{array} \tag{6}$$

[5] The Baire space is usually defined as the set of all infinite sequences of natural numbers with a certain topology. This space has many alternative characterisations (up to isomorphism) as explored in, e.g., [31, Ch. I].

In (6), ***ninety*** denotes the sequence which is constant *ninety* (s.t. ***ninety*** serves the function of $\mathbf{ninety}^{se}$ from (2)). The constant *now* denotes a functional that takes as input non-coded sequences of natural numbers (type 1) and produces as output the value-at-@ in these sequences. The introduction of this constant is made necessary by the absence of indices in (the variant of) our preferred model of countable-continuous functionals (cf. Sect. 4) in which we interpret Partee's temperature puzzle.

We close this section with a remark on the 'coding' of finite sequences as is done in mathematics and computer science (cf. e.g. [6, p. 92]):

Remark 1 (Coding). Finite sequences of natural numbers can be represented (or 'coded') by a single natural number using *pairing functions*. The most widely known of these functions, due to Cantor, is defined as follows:

$$\pi(n,m) := \tfrac{1}{2}(n+m)(n+m+1)+m$$

Notably, not all natural numbers necessarily code finite sequences (given a certain fixed pairing function).

The coding and the associated decoding of finite sequences has been implemented in most of the common programming languages. In particular, there is a computable function $\mathsf{IsCodeForSeq}(n)$ of comparatively low complexity which outputs '1' if it is indeed the case that the input n codes some finite sequence $(T_0, T_1, \ldots, T_m)$, and '0' otherwise.

As is common in mathematics and computer science, we assume below that a particular coding and decoding function has been fixed (e.g. Gödel numbers as in [6, p. 92]). This assumption allows us to treat finite sequences (type 1) as natural numbers (type 0). We further assume that *ninety* from (6) is a number which does not[6] code a finite sequence. We will see below that this property of pairing functions is essential in our solution to Partee's temperature puzzle (in Sect. 3.3).

This completes our discussion of the interpretation of the verb rise as a continuous functional. We next introduce the notion of *associate* and discuss its role in our solution to the temperature puzzle.

3.2 Associates and the Temperature Puzzle

Intuitively, associates of continuous functionals are countable approximations (or representations) of these functionals which uniquely determine the value of these functionals for every (represented) argument. The Kleene-Kreisel model of countable-continuous functionals is defined in terms of associates (cf. [30, §8.2.1]). Associates are formally defined as follows:

Definition 2 (Associates [21,25]). *An* associate*, α_φ, of a continuous type-*2 *functional φ is a sequence of natural numbers (i.e. type $1 \equiv (0^* \to 0)$) such that*

$$\forall \gamma^1\, \exists n^0\, \forall N^0 \geq n\big[\alpha_\varphi(\overline{\gamma}N) = \varphi(\gamma)+1 \wedge (\forall i < n)\, \alpha_\varphi(\overline{\gamma}i) = 0\big]. \qquad (7)$$

[6] For the coding from [6, p. 92], there exist numbers which do not code finite sequences.

The associate α_φ thus enumerates[7] the values of φ at all $\overline{\gamma}n$, where n is a point of continuity for γ. In particular, the first conjunct of (7) identifies the value of the associate of φ *for any initial segment of* γ *up to at least* n (here: the value of $\alpha_\varphi(\overline{\gamma}N)$) with the value $+1$ of φ *for* γ. As a result of the identification of $\alpha_\varphi(\overline{\gamma}N)$ and $\varphi(\gamma) + 1$, a continuous functional and its associate contain the same information: Beyond the point of continuity n, φ remains constant, i.e. no new information can be learned.

The '+1' in the first conjunct of (7) expresses a kind of partiality: If the input sequence, $\overline{\gamma}k$, of α_φ is 'too short' (i.e. if k is less than the least point of continuity, n, for γ), $\alpha_\varphi(\overline{\gamma}k)$ cannot provide any information about $\varphi(\gamma)$. The second conjunct from (7) captures this possibility by returning the value 0, which is not a possible value for $\varphi(\gamma) + 1$.

The above yields the following intuitive picture for an associate, α_{rise}, of φ_{rise}. Below, γ denotes a temperature-representing sequence (type-1, as in Sect. 3.1); m is a natural number:

$$\alpha_{\text{rise}}(\overline{\gamma}m) = \begin{cases} 0 & \text{if } \overline{\gamma}m \text{ is too short to judge if the temperature is rising;} \\ 1 & \text{if } \varphi_{\text{rise}}(\gamma) = 0 \text{ by (7), i.e. the temperature is not rising;} \\ 2 & \text{if } \varphi_{\text{rise}}(\gamma) = 1 \text{ by (7), i.e. the temperature is rising.} \end{cases}$$

We close this section with an observation about associates and context-dependence:

The variation of the point n in (7) with different input sequences reflects the role of *linguistic* context in the interpretation of verbs like rise and fall: While some occurrences of these verbs only consider comparatively short initial segments of sequences in order to judge whether the sequence rises or falls, others consider longer (or even countably infinite) initial segments of these sequences. Consider the application of the *associates*-interpretation of fall to the type-0* interpretations of the DPs the water drop and the pitch drop: To confirm that the water drop is, in fact, falling, it suffices to observe its behavior for a short period of time (i.e. for a few (milli-)seconds). In contrast, to confirm that the pitch drop is falling, we need to observe its behavior for a rather long period of time (i.e. for several years).

The (possible) existence of multiple points of continuity for *the same* sequence – and the attendant need to choose a particular point up to which we consider this sequence – further reflects the dependence of the above verbs on the salient *communicative* context. For example, for the sentence The temperature rises (cf. (1b)), we will choose a larger n in the context of global climate development than in the context of the local weather forecast.

[7] Note that it is impossible to enumerate the space $\mathbb{N}^\mathbb{N}$. Since we can, thus, not enumerate the values of a discontinuous type-2 functional, our approach breaks down for *discontinuous* functionals. We will identify a promising solution to this problem in Sect. 7.

3.3 The *Associates*-Solution to the Temperature Puzzle

We are now ready to present our *associates*-solution to the temperature puzzle. In particular, we can reformulate (6) using the associate, α_{rise}, of φ_{rise} as follows:

$$\begin{array}{c} \exists\gamma^1\big(\forall\beta^1[\mathit{temp}^2(\beta)\leftrightarrow\beta=\gamma]\wedge \mathit{now}^2(\gamma)=\mathit{ninety}^0\big) \\ \exists\gamma^1\big(\forall\beta^1[\mathit{temp}^2(\beta)\leftrightarrow\beta=\gamma]\wedge\exists n^0[\alpha^1_{\text{rise}}(\overline{\gamma}n)=2]\big) \\ \hline \exists m^0\big(\alpha^1_{\text{rise}}(\overline{\boldsymbol{ninety}}\,m)=2\big)\end{array} \tag{8}$$

We next show that the inference from (8) indeed does not go through:

Montague semantics solves Partee's temperature puzzle by interpreting the occurrences of the DP the temperature from (1a) and (1b) as an *individual* (cf. the constant NINETY in (2)) resp. as an individual *concept* (cf. the variable c in (2)). Our solution works analogously, but – thanks to the presence of α_{rise} – with lower types. In our solution, the different occurrences of the DP from (1a) and (1b) are interpreted as a natural number which does *not* code a finite sequence of natural numbers (by the assumption following Remark 1) (cf. the constant *ninety* in (8)) and as a natural number, k, which *codes* the finite sequence $\overline{\gamma}n$ from (8). The information whether *ninety* and k do or do not code a sequence of natural numbers is obtained by applying the function $\mathsf{IsCodeForSeq}(n)$ from Remark 1. The different types of *ninety* and k (i.e. 0 resp. 0^*) – and the subsequent impossibility of replacing the occurrence of $\overline{\gamma}n$ in the second premise of (8) by the constant *ninety* – blocks the temperature puzzle.

In conclusion: the introduction of the associate, α_{rise}, of φ_{rise} allows us to block the inference from (8) while lowering the types of many expressions from (1).

3.4 Computability and the Temperature Puzzle

We have suggested in Sect. 2.1 that our *associates*-solution to the temperature puzzle is computable. To support this claim, we now discuss the computational properties of associates that are relevant for our solution.

An obvious conceptual question about associates is whether every continuous functional has an associate and, if this is the case, whether this associate is computable. We provide three partial answers to this question:

1. Kohlenbach has shown in [24, Sect. 4] that the statement *every continuous functional of type* $(1 \to 1)$ *has an associate* carries no significant logical strength. Thus, as a special case, we may safely assume the existence of an associate for every continuous type-2 functional.
2. In general, there is no *computable* functional which takes as input a continuous type-2 functional and produces as output an associate (cf. [24, 25]).
3. However, every primitive recursive functional (in the sense of Gödel's system T) has a *canonical* associate which can be computed via the procedure from [42, p. 139]. Since the class of primitive recursive functionals is rather large, it captures essentially any functional 'occurring in practice'.

A second question about associates regards the computability of the associate's point of continuity n. We here provide two partial answers:

1. There is no *computable* functional which returns a point of continuity on input a continuous type-2 functional and a sequence (cf. [25]).
2. However, the *fan functional* returns a point of (uniform) continuity on input a continuous type-2 functional and a sequence *in a fixed compact space*. The fan functional is present in the Kleene-Kreisel model and has a computable associate (cf. [30, Sect. 8], [47]).

Since temperature measurements come with upper and lower bounds dictated by physics (s.t. they are part of a compact space), a point of continuity of φ_{rise} can always be computed for α_{rise} and a sequence of temperature measurements γ.

This completes our presentation of the *associates*-approach to Partee's temperature puzzle. We next show that this approach can be implemented in a compositional semantics for natural language.

4 Compositional Implementation

To obtain our *associates*-solution to the temperature puzzle, we compositionally interpret natural language in a model, inspired by the Kleene-Kreisel model of countable-continuous functionals, which contains continuous functionals and their associates. This interpretation proceeds via the translation of the relevant subset of the linguistic fragment from [33] into the language of the simply typed lambda logic $\boldsymbol{\lambda}^0_{\rightarrow}$ ([8]; cf. [36, Ch. 1.1]). This is a logic with a single atomic type, 0, from which all other types are built up through the type constructor $\rightarrow$ (see the definition of *finite types* from Sect. 3). The language and models of $\boldsymbol{\lambda}^0_{\rightarrow}$ are specified in [2,8].

To identify the $\boldsymbol{\lambda}^0_{\rightarrow}$-interpretation of the sentences from (1), we first specify the particular language $\mathcal{L}^{\boldsymbol{\lambda}^0_{\rightarrow}}$ (abbreviated '$\mathcal{L}$') and frame $\mathcal{F}^{\boldsymbol{\lambda}^0_{\rightarrow}}$ (abbreviated '$\mathcal{F}$') whose elements translate resp. interpret the syntactic constituents of these sentences. The members of $\mathcal{L}$ are specified in Table 1. Our conventions for the use of $\boldsymbol{\lambda}^0_{\rightarrow}$ variables are introduced in Table 2.

In the list of non-logical $\boldsymbol{\lambda}^0_{\rightarrow}$ constants, α_{rise} enables the translation of the verb rise as an associate of the continuous functional denoted by *rise* (formerly, φ_{rise}).

Table 1. $\mathcal{L}$ constants.

Constant	$\boldsymbol{\lambda}^0_{\rightarrow}$ Type
ninety	0
ninety, α_{rise}	1
now, *temp*, *rise*	$1 \rightarrow 0$

Table 2. $\mathcal{L}$ variables.

Variable	$\boldsymbol{\lambda}^0_{\rightarrow}$ Type
m, n, N, x	0
β, γ	1
P, Q	$1 \rightarrow 0$

The interpretation function $\mathcal{I}_{\mathcal{F}} : \mathcal{L} \to \mathcal{F}$ respects the way in which different content words are conventionally related. Thus, this function identifies the interpretation of the generalized $\boldsymbol{\lambda}^0_{\to}$-translation, $\lambda P.P(\boldsymbol{ninety})$, of the DP ninety as a subset of the interpretation of the $\boldsymbol{\lambda}^0_{\to}$ translation, $\lambda P \exists \gamma.temp(\gamma) \wedge P(\gamma)$, of the DP a temperature (s.t. ninety is a temperature under this interpretation). To ensure the 'right' interpretation of the syntactic constituents of (1a) to (1c), we demand that the function $\mathcal{I}_{\mathcal{F}}$ further satisfies a number of semantic constraints.

Definition 3 (Constraints on $\mathcal{L}$ constants). *The function $\mathcal{I}_{\mathcal{F}}$ satisfies the following semantic constraints:*

(C1) $now(\boldsymbol{ninety}) = ninety;$

(C2) $\forall \gamma^1 \exists n^0 \forall \beta^1 \big(\overline{\gamma} n = \overline{\beta} n \to rise(\gamma) = rise(\beta)\big);$

(C3) $\forall \gamma^1 \exists n^0 \forall N^0 \geq n \big[\alpha_{\text{rise}}(\overline{\gamma} N) = rise(\gamma) + 1 \wedge (\forall i < n) \alpha_{\text{rise}}(\overline{\gamma} i) = 0\big]$

The constraint (C1) demands that the interpretation of the type-0 constant *ninety* be the output of the functional *now* on input $\boldsymbol{ninety}$ (cf. [33, rule T1.(d), MP1]). The constraints (C2) and (C3) demand that the constant *rise* be interpreted as a continuous functional (cf. (C2)) resp. that α_{rise} behaves as an associate of this functional (cf. (C3)).

Admittedly, (C2) and (C3) are additional requirements on our semantic models which are not postulated for the models of Montague's Intensional Logic (cf. [33]). However, since these requirements reflect natural assumptions about the domain of interpretation of the verb rise (cf. Sect. 3.1) – and since continuous functionals can be represented via their associates (cf. Sect. 3.2) –, these requirements are rather innocent.

This completes our specification of the interpretation function $\mathcal{I}_{\mathcal{F}}$. We next turn to the compositional translation of Partee's temperature puzzle: To enable this translation, we first translate the lexical elements of the sentences from (1). In these translations, $\leadsto$ is the smallest relation between syntactic trees and $\boldsymbol{\lambda}^0_{\to}$terms which conforms to the rules from [22]:

Definition 4 (Basic $\boldsymbol{\lambda}^0_{\to}$translations). *The lexical elements of (1a) to (1c) are translated into the following $\boldsymbol{\lambda}^0_{\to}$terms:*

$$\begin{aligned}
\text{ninety} &\leadsto \boldsymbol{ninety}\\
\text{temperature} &\leadsto temp\\
\text{rise} &\leadsto \lambda\beta \exists n \big(\alpha_{\text{rise}}(\overline{\beta} n) = 2\big)\\
\text{is} &\leadsto \lambda\beta\lambda\gamma\big(now(\gamma) = now(\beta)\big)\\
\text{the} &\leadsto \lambda Q \lambda P \exists \gamma \big(\forall \beta [Q(\beta) \leftrightarrow \gamma = \beta] \wedge P(\gamma)\big)
\end{aligned}$$

As expected, Definition 4 specifies the translation of the verb rise as an associate of the continuous functional denoted by the $\boldsymbol{\lambda}^0_{\to}$constant *rise* (cf. (C2), (C3)). The translations of the copula is, of the DP ninety, and of the definite determiner follow the translations of these expressions from [33, cf. rules T1.(b), (d), T2].[8] In particular, our translation of is follows Montague's translation of the copula as the designator of a relation between the *extensions* of (generalized quantifiers over) individual concepts (here: as the designator of a relation between natural numbers, rather than between sequences of numbers).

The above translations enable the compositional $\boldsymbol{\lambda}^0_{\rightarrow}$translation of the sentences from (1). We start with the translation of (1a):

$$\begin{aligned} &1.\quad [_{\mathrm{VP}}[_{\mathrm{CP}}\mathrm{is}][_{\mathrm{DP}}\mathrm{ninety}]] \rightsquigarrow \lambda\gamma\big(now(\gamma) = now(\boldsymbol{ninety})\big) \\ &\qquad = \lambda\gamma(now(\gamma) = ninety) \\ &2.\quad [_{\mathrm{DP}}[_{\mathrm{DET}}\mathrm{the}][_{\mathrm{N}}\mathrm{temperature}]] \rightsquigarrow \lambda P\exists\gamma\big(\forall\beta\,[temp(\beta) \leftrightarrow \gamma = \beta] \wedge P(\gamma)\big) \\ &3.\quad [_{\mathrm{S}}[_{\mathrm{DP}}[_{\mathrm{DET}}\mathrm{the}][_{\mathrm{N}}\mathrm{temperature}]][_{\mathrm{VP}}[_{\mathrm{CP}}\mathrm{is}][_{\mathrm{DP}}\mathrm{ninety}]]] \\ &\qquad \rightsquigarrow \exists\gamma\big(\forall\beta\,[temp(\beta) \leftrightarrow \gamma = \beta] \wedge (now(\gamma) = ninety)\big) \end{aligned} \tag{9}$$

Sentences (1b) and (1c) are translated as follows:

$$\begin{aligned} &[_{\mathrm{S}}[_{\mathrm{DP}}[_{\mathrm{DET}}\mathrm{the}][_{\mathrm{N}}\mathrm{temperature}]][_{\mathrm{VP}}[_{\mathrm{IV}}\mathrm{rises}]]] \\ &\rightsquigarrow \lambda P\exists\gamma\big(\forall\beta\,[temp(\beta) \leftrightarrow \gamma = \beta] \wedge P(\gamma)\big)\big(\lambda\delta\,\exists n[\alpha_{\mathrm{rise}}(\overline{\delta}n) = 2]\big) \\ &= \exists\gamma\big(\forall\beta[temp(\beta) \leftrightarrow \gamma = \beta] \wedge \exists n[\alpha_{\mathrm{rise}}(\overline{\gamma}n) = 2]\big) \end{aligned} \tag{10}$$

$$[_{\mathrm{S}}[_{\mathrm{DP}}\mathrm{ninety}][_{\mathrm{IV}}\mathrm{rises}]] \rightsquigarrow \exists m\big(\alpha_{\mathrm{rise}}(\overline{\boldsymbol{ninety}}\ m) = 2\big) \tag{11}$$

The resulting $\boldsymbol{\lambda}^0_{\rightarrow}$formulas are exactly the formulas from (8).

We next discuss the empirical scope of our *associates*-approach and the relation of this approach to other solutions to the temperature puzzle.

5 Domain and Scope

Our previous discussion has been restricted to the example of the verb rise. However, the *associates*-approach generalizes to all degree achievement verbs and change-of-state verbs ([28]; cf. [1,5,11]) whose interpretation corresponds to a continuous functional. The latter constitute a sizable[9] class of verbs with the following members:

1. *verbs of continuous calibratable change of state* (cf. [28, pp. 247–248]): decline, drop, grow, increase, plummet, plunge, rocket, rise, soar, surge, ...
2. *verbs of entity-specific continuous change of state* (cf. [28, pp. 246–247]): blush, blossom, burn, ferment, molt, rust, sprout, swell, ...
3. *other verbs of continuous state-change* (cf. [28, pp. 240–246]): abate, advance, age, clog, compress, condense, degrade, distend, mature; in particular:

[8] We simplify Montague's translation of the copula to a translation that takes as its first argument the designator of a type-1 object (instead of a *generalized quantifier* over type-1 objects).

[9] For example, Levin [28] lists 369 members of classes 1 to 4.

(a) break-/bend-*verbs:* crack, shatter, split, tear; crumple, fold, wrinkle, ...
(b) *adjective-related verbs:* blunt, clear, cool, dry, empty, narrow, quiet, ...
(c) *change-of-color verbs:* blacken, brown, gray, redden, tan, whiten, ...
(d) -en *verbs:* darken, flatten, harden, ripen, sharpen, strengthen, ...
(e) -ify *verbs:* acidify, humidify, magnify, nitrify, petrify, purify, solidify, ...
(f) -ize *verbs:* crystallize, fossilize, pressurize, pulverize, stabilize, ...
(g) -ate *verbs:* accelerate, coagulate, degenerate, detonate, evaporate, ...

4. *(continuous) directed motion verbs* (cf. [28, pp. 263–264]): arrive, ascend, descend, drop, enter, fall, pass, rise, ...
5. *accomplishment verbs* (cf. [43]): run a mile, draw a circle, build a house, eat a sandwich, play a game of go; grow up, recover from illness, ...

The above-listed verbs all take individual concepts as their arguments (i.e. they are co-classified with the verb rise) (cf. [10]). The intensional interpretation of these verbs is motivated by their particular, non-instantaneous, evaluation procedure: To judge whether John is blushing (cf. class 2), it does not suffice to observe his red face at a particular point in time.[10] Instead, we need to observe John's facial complexion at different neighboring points in time. We can only conclude that John is blushing if he has a normal (non-red) skin color at the earliest observed time-point and an increasingly redder complexion at the later time-points (cf. [27]).

Note that, in contrast to their counterparts from class 1, the 'continuous functional'-interpretations of the verbs from classes 2 to 5 are not restricted to input sequences of *natural numbers* (see *blush*), may describe non-temporal change [10,18] (see the extent reading of verbs like *narrow* and *darken*)[11], and do not presuppose an *established* scale or unit of measurement (i.e. they describe non-discrete change). For example, in contrast to rising, blushing and narrowing are not properties of sequences of numbers, but of sequences of *temporal states of an individual* (viz. of his/her face) resp. of *spatial states of an object*. Further, there is no established unit of measurement of a person's facial redness (or of a window cracking, a storm arriving, a person recovering from illness, etc.).

The above-described absence of a numerical/measurement structure does not compromise the applicability of our *associates*-approach to the verbs from classes 2 to 5. This is due to the possibility of labelling temporal stages of individuals (or of other physical objects) by natural numbers, of identifying a *contextually salient* unit and scale (here: dominant wavelength or visible change in hue) for the measurement of the relevant property, and of selecting the value of the measurement (under the selected scale and unit of measurement) of the individual's relevant attribute for that property. In particular, the continuous functional-interpretation, *blush*, of the verb blush will return '1' on input a given sequence

[10] Maybe John simply suffers from high blood pressure which causes his constant facial redness.

[11] E.g. in The trail narrowed at the summit [10, p. 98] and His skin darkens on his right leg near the femoral artery [10, p. 99]. We thank an anonymous reviewer for reminding us of examples of spatial change.

of temporal ‘John’-stages if the values of the measurement (under the contextually presupposed measurement unit) of John’s facial complexion at these stages are increasing, and will return ‘0’ otherwise.

We next discuss the relation of our *associates*-approach to existing work on the temperature puzzle.

6 Relation to Existing Work

Our *associates*-approach distinguishes itself from existing solutions to the temperature puzzle. This is due to the proximity of our approach to Montague’s original solution from [33] (cf. Sect. 3.3) and to its focus on improving the computational properties of this solution (cf. Sect. 2.2):

Firstly, in contrast to the solutions from [3,20,41], and to solutions from event semantics, our solution is not based on an alternative interpretation of (1a) that uses a locative interpretation of the copula (i.e. ‘is *at* ninety’), a measurement-explicit interpretation of the DP ninety (i.e. ‘is ninety *degrees Fahrenheit*’), or an event-based interpretation of the verb rise (s.t. ‘rise’ describes a rising event).

Secondly, in contrast to the solutions from [12,27,29,40], our solution is not directed at a variant of the temperature puzzle (i.e. *Gupta’s problem*; cf. (12)) that arises from the double index-dependence of intensional nouns like temperature; viz. from the dependence of temperature-values on the index-argument of a particular individual concept [i.e. *inner index-dependence*] and the dependence of noun-interpretations on the index of evaluation[12] [i.e. *outer index-dependence*] (cf. [40]). As a result of this double dependence, Montague semantics blocks the intuitively valid inference from (12):

a. Necessarily, the temperature of the air in my refrigerator is the same as the temperature of the air in your refrigerator.
b. The temperature of the air in my refrigerator is rising.
――――――――――――――――――――――――――――――
c. The temperature of the air in your refrigerator is rising. (12)

It should come as no surprise that the different solutions to Gupta’s problem can be integrated into our *associates*-approach to Partee’s temperature puzzle. However, our approach even provides its own solution to the puzzle, which also involves computability considerations. We will detail this solution in a sequel to this paper.

7 Conclusion and Outlook

We have presented a computable, low-type, context-sensitive solution to Partee’s temperature puzzle which uses the countable approximation of continuous functionals via their associates. The success of our solution is challenged by the

[12] As a result of this dependence, rise may denote a different set of individual concepts at different indices.

restriction of associates to *continuous* functionals. This restriction prevents the application of our approach to expressions that are traditionally interpreted as *dis*continuous functionals (e.g. mostly above 90).

Its exclusion of discontinuous intensional verbs hampers the generality of the presented approach. However, in natural language, discontinuous expressions are rather rare: of the 369 intensional intransitive verbs listed in [28] (see Sect. 5 for a selection), *only 5* are discontinuous. Their scarcity notwithstanding, discontinuous verbs can be accommodated in Bezem's model $\mathscr{M}$ of strongly majorizable functionals (cf. [23, Ch. 3, 11]). The weak continuity functional ([4, Sect. 5, p. 171]) in this model serves a similar role to the fan functional in the Kleene-Kreisel model: it produces a lower-type correlate of its input functional. However, whereas the associate of a continuous functional is an accurate representation of the continuous functional (in the sense that no information is lost), the output of the weak continuity functional only *partially* represents the input functional in Bezem's model. The detailed development of this account is a project for future work.

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