Impulse Control of Standard Brownian Motion: Discounted Criterion

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Abstract. This paper examines the impulse control of a standard Brownian motion under a discounted criterion. In contrast with the dynamic programming approach, this paper first imbeds the stochastic control problem into an infinite-dimensional linear program over a space of measures and derives a simpler nonlinear optimization problem that has a familiar interpretation. Optimal solutions are obtained for initial positions in a restricted range. Duality theory in linear programming is then used to establish optimality for arbitrary initial positions.

Keywords: Impulse control · Discounted criterion · Infinite dimensional linear programming · Expected occupation measures

1 Introduction

When one seeks to control a stochastic process and every intervention incurs a strictly positive cost, one must select a sequence of separate intervention times and amounts. The resulting stochastic problem is therefore an impulse control problem in which the decision maker seeks to either maximize a reward or minimize a cost. This paper continues the examination of the impulse control of Brownian motion. It considers a discounted cost criterion while a companion paper [5] studies the long-term average criterion. The aim of the paper is to illustrate a solution approach which first imbeds the stochastic control problem into an infinite-dimensional linear program over a space of measures and then reduces the linear program to a simpler nonlinear optimization. Contrasting with the long-term average paper, the dependence of the value function on the initial position of the process requires the use of duality in linear programming to obtain a complete solution.

Impulse control problems have been extensively studied using a quasi-variational approach; now classical works include [1,3] while the recent paper [2]

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examines a Brownian inventory model. This paper extends a linear programming approach used on optimal stopping problems [4]. See [5] for additional references.

Let W be a standard Brownian motion process with natural filtration $\{\mathcal{F}_t\}$. An impulse control policy consists of a pair of sequences $(\tau, Y) := \{(\tau_k, Y_k) : k \in \mathbb{N}\}$ in which τ_k is the $\{\mathcal{F}_t\}$-stopping time of the kth impulse and the $\mathcal{F}_{\tau_k}$-measurable variable Y_k gives the kth impulse size. The sequence $\{\tau_k : k \in \mathbb{N}\}$ is required to be non-decreasing, a natural assumption in that intervention $k + 1$ must occur no earlier than intervention k. For a policy (τ, Y), the impulse-controlled Brownian motion process is given by

$$X(t) = x_0 + W(t) + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t\}} Y_k.$$

The goal is to control the (discounted) second moment of X subject to (discounted) fixed and proportional costs for interventions. Let (τ, Y) be an impulse control policy. Define $c_0(x) = x^2$. Let $k_1 > 0$ denote the fixed costs incurred for each intervention and let $k_2 \geq 0$ be a cost proportional to the size of the intervention. Define the impulse cost function $c_1(y, z) = k_1 + k_2 |z - y|$, in which y denotes the pre-jump location of X (typically far from 0) and z denotes the post-jump location of X which is thought to be close to 0. Let $\alpha > 0$ denote the discount rate. The objective function is

$$
\begin{aligned}
J(\tau, Y; x_0) = \mathbb{E}_{x_0} \Bigg[& \int_0^{\infty} e^{-\alpha s} c_0(X(s)) \, ds \\
& + \sum_{k=1}^{\infty} I_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} c_1(X(\tau_k -), X(\tau_k)) \Bigg].
\end{aligned}
\tag{1}
$$

The controller must balance the desire to keep the process X near 0 so as to have a small second moment against the desire to limit the number and/or sizes of interventions so as to have a small impulse cost. Since the goal is to minimize the objective function, impulse control policies having $J(\tau, Y; x_0) = \infty$ are undesirable. We therefore restrict attention to the impulse policies for which $J(\tau, Y; x_0)$ is finite. Denote this class of *admissible* controls by $\mathcal{A}$.

We make five important observations about impulse policies. Firstly, "0-impulses" which do not change the state only increase the cost so can be excluded from consideration. Secondly, the symmetry of the dynamics and costs means that any impulse (τ_k, Y_k) which would cause $\mathrm{sgn}(X(\tau_k)) = -\mathrm{sgn}(X(\tau_k -))$ on a set of positive probability will have no greater cost (smaller cost when $k_2 > 0$) by replacing the impulse with one for which $\tilde{X}(\tau_k) = \mathrm{sgn}(X(\tau_k -))|X(\tau_k)|$. Thus we can also restrict analysis to those policies for which all impulses keep the process on the same side of 0. Next, any policy (τ, Y) with $\lim_{k \to \infty} \tau_k =: \tau_{\infty} < \infty$ on a set of positive probability will have infinite cost so for every admissible policy $\tau_k \to \infty$ *a.s.* as $k \to \infty$. Next let (τ, Y) be a policy for which there is some k such that $\tau_k = \tau_{k+1}$ on a set of positive probability. Again due to the presence of the fixed intervention cost k_1, the total cost up to time τ_{k+1} will be at least $k_1 \mathbb{E}[e^{-\alpha \tau_k} I(\tau_k = \tau_{k+1})]$ smaller by combining these interventions into a

single intervention on this set. Hence we may restrict policies to those for which $\tau_k < \tau_{k+1}$ *a.s.* for each k.

The final observation is similar. Suppose (τ, Y) is a policy such that on a set G of positive probability $\tau_k < \infty$ and $|X(\tau_k)| > |X(\tau_k-)|$ for some k. Consider a modification of this impulse policy and resulting process $\tilde{X}$ which simply fails to implement this impulse on G. Define the stopping time $\sigma = \inf\{t > \tau_k : |X(t)| \le |\tilde{X}(t)|\}$. Notice that the running costs accrued by $\tilde{X}$ over $[\tau_k, \sigma)$ are smaller than those accrued by X. Finally, at time σ, introduce an intervention on the set G which moves the $\tilde{X}$ process so that $\tilde{X}(\sigma) = X(\sigma)$. This intervention will incur a cost which is smaller than the cost for the process X at time τ_k. As a result, we may restrict the impulse control policies to those for which every impulse decreases the distance of the process from the origin.

2   Restricted Problem and Measure Formulation

The solution of the impulse control problem is obtained by first considering a subclass of the admissible impulse control pairs.

Condition 1. *Let $\mathcal{A}_1 \subset \mathcal{A}$ be those policies (τ, Y) such that the resulting process X is bounded; that is, for $(\tau, Y) \in \mathcal{A}_1$, there exists some $M < \infty$ such that $|X(t)| \le M$ for all $t \ge 0$.*

Note that for each $M > 0$, any impulse control which has the process jump closer to 0 whenever $|X(t-)| = M$ is in the class $\mathcal{A}_1$ so this collection is non-empty. The bound is not required to be uniform for all $(\tau, Y) \in \mathcal{A}_1$. The restricted impulse control problem is one of minimizing $J(\tau, Y; x_0)$ over all policies $(\tau, Y) \in \mathcal{A}_1$.

We capture the expected behavior of the process and impulses with discounted measures. Let $(\tau, Y) \in \mathcal{A}_1$ be given and consider $f \in C^2(\mathbb{R})$. Then upon letting $t \to \infty$ after taking expectations, the general Dynkin's formula results in

$$
\begin{aligned}
f(x_0) = {} & \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s}[\alpha f(X(s)) - (1/2)f''(X(s))]\, ds\right] \\
& + \mathbb{E}_{x_0}\left[\sum_{k=0}^\infty I_{\{\tau_k < \infty\}} e^{-\alpha \tau_k}[f(X(\tau_k-)) - f(X(\tau_k))]\right],
\end{aligned}
\tag{2}
$$

in which the transversality condition $\lim_{t\to\infty} \mathbb{E}_{x_0}\left[e^{-\alpha t}f(X(t))\right] = 0$ follows from the boundedness of X. Note the generator of the Brownian motion process is $Af(x) = (1/2)f''(x)$. To simplify notation, define $Bf(y, z) = f(y) - f(z)$.

Define the discounted expected occupation measure μ_0 and the discounted impulse measure μ_1 such that for each $G, G_1, G_2 \subset \mathbb{R}$,

$$
\begin{aligned}
\mu_0(G) = {} & \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s} I_G(X(s))\, ds\right] \\
\mu_1(G_1 \times G_2) = {} & \mathbb{E}_{x_0}\left[\sum_{k=0}^\infty I_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} I_{G_1 \times G_2}(X(\tau_k-), X(\tau_k))\right].
\end{aligned}
\tag{3}
$$

Notice that the total mass of μ_0 is $1/\alpha$ while μ_1 is a finite measure since $J(\tau, Y; x_0)$ is finite. Rewriting the objective function and Dynkin's formula in terms of these measures imbeds the impulse control problem in the linear program

$$
\begin{cases}
\text{Min.} & \int c_0 \, d\mu_0 \quad + \int c_1 \, d\mu_1 \\
\text{S.t.} & \int (\alpha f - Af) \, d\mu_0 + \int Bf \, d\mu_1 = f(x_0), \quad \forall f \in C^2.
\end{cases}
\tag{4}
$$

We now wish to introduce an auxiliary linear program derived from (4) which only has the μ_1 measure as its variable and has fewer constraints. Define $\phi(x) = e^{-\sqrt{2\alpha}x}$ and $\psi(x) = e^{\sqrt{2\alpha}x}$. Notice that ϕ is a strictly decreasing solution while ψ is a strictly increasing solution of the homogeneous equation $\alpha f - Af = 0$. For each $(\tau, Y) \in \mathcal{A}_1$, the resulting process X is bounded so we can use both ϕ and ψ in (2). This results in the two constraints

$$
\int B\phi(y, z) \, \mu_1(dy \times dz) = \phi(x_0) \quad \text{and} \quad \int B\psi(y, z) \, \mu_1(dy \times dz) = \psi(x_0) \tag{5}
$$

which only constrain the measure μ_1. Note that the monotonicity and positivity of both ϕ and ψ require the support of μ_1 to be such that the two integrals in (5) are positive. We can also take advantage of the symmetry inherent in the problem. Define $p_0(x) = \cosh(\sqrt{2\alpha}x)$. Then averaging the two constraints (5) yields

$$
\int \frac{Bp_0(y, z)}{p_0(x_0)} \, \mu_1(dy \times dz) = 1.
$$

Using $g_0(x) = (\alpha x^2 + 1)/\alpha^2$ in (2), where again the boundedness of X implies that the transversality condition is satisfied, yields

$$
\begin{aligned}
&\mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s} c_0(X(s)) \, ds\right] \\
&= \frac{\alpha x_0^2 + 1}{\alpha^2} - \mathbb{E}_{x_0}\left[\sum_{k=0}^\infty I_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} Bg_0(X(\tau_k-), X(\tau_k))\right].
\end{aligned}
\tag{6}
$$

Let $[c_1 - Bg_0]$ denote the sum of the two functions c_1 and Bg_0. Using (6) in (1) establishes that

$$
J(\tau, Y; x_0) = \frac{\alpha x_0^2 + 1}{\alpha^2} + \mathbb{E}_{x_0}\left[\sum_{k=0}^\infty I_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} [c_1 - Bg_0](X(\tau_k-), X(\tau_k))\right]
$$

and hence that

$$
J(\tau, Y; x_0) = \frac{\alpha x_0^2 + 1}{\alpha^2} + \int [c_1 - Bg_0](y, z) \, \mu_1(dy \times dz) \tag{7}
$$

so the objective function value only depends on the measure μ_1. Since the objective function for each $(\tau, Y) \in \mathcal{A}_1$ has the affine term $g_0(x_0)$, it may be ignored for the purposes of optimization but it must be included to obtain the correct value for the objective function. Now form the auxiliary linear program

$$
\begin{cases}
\text{Min.} & \int [c_1 - Bg_0](y, z)\, \mu_1(dy \times dz) \\
\text{S.t.} & \int \dfrac{Bp_0(y, z)}{p_0(x_0)}\, \mu_1(dy \times dz) = 1.
\end{cases}
\tag{8}
$$

Let $V_1(x_0)$ denote the value of the impulse control problem over policies in $\mathcal{A}_1$, V_{lp} denote the value of (4) and V_{aux} denote the value of (8). The following proposition is immediate.

Proposition 2. $V_{aux}(x_0) \leq V_{lp}(x_0) \leq V_1(x_0)$.

Remark 3. *Our analysis will also involve other auxiliary linear programs as well. One will replace the single constraint in (8) with the pair of constraints (5) while another will limit the constraints in (4) to a single function. Each auxiliary program will provide a lower bound on $V_{lp}(x_0)$ and hence on $V_1(x_0)$.*

2.1 Nonlinear Optimization and Partial Solution

Recall, the admissible impulse policies can be (and are) limited to those for which impulses move X closer to the origin. As a result, the integrand $Bp_0 > 0$ and the constraint of (8) implies that the feasible measures μ_1 of (8) are those for which $Bp_0/p_0(x_0)$ is a probability density. For a feasible μ_1, let $\tilde{\mu}_1$ be the probability measure $\frac{Bp_0}{p_0(x_0)}\mu_1$. Thus we can write the objective function as

$$
\int [c_1 - Bg_0]\, d\mu_1 = \left(\int \frac{c_1 - Bg_0}{Bp_0}\, d\tilde{\mu}_1 \right) p_0(x_0).
$$

Since the goal is to minimize the cost, a lower bound is given by the minimal value of F scaled by the constant $p_0(x_0)$, where

$$
F(y, z) := \frac{c_1(y, z) - Bg_0(y, z)}{Bp_0(y, z)}.
$$

Moreover, should the infimum be attained at some pair (y_*, z_*), then the probability measure $\tilde{\mu}_1(\cdot)$ putting unit point mass on (y_*, z_*) would achieve the lower bound and identify an optimal μ_1 measure for the auxiliary linear program. To solve the stochastic problem, one would need to connect the measure μ_1 back to an admissible impulse control policy in the class $\mathcal{A}_1$ in such a way that the resulting μ_1 measure would be given by (3).

Remark 4. *The objective function $p_0(x_0)F$ has a natural interpretation. First observe that $Bp_0(y, z) = \cosh(\sqrt{2\alpha}y) - \cosh(\sqrt{2\alpha}z)$ so*

$$
p_0(x_0)F(y, z) = [c_1(y, z) - Bg_0(y, z)] \cdot \frac{\cosh(\sqrt{2\alpha}x_0)}{\cosh(\sqrt{2\alpha}y)} \cdot \sum_{n=0}^{\infty} \left(\frac{\cosh(\sqrt{2\alpha}z)}{\cosh(\sqrt{2\alpha}y)} \right)^n.
$$

It can be shown that the first fraction gives the expected discount for the time it takes X to reach $\{\pm y\}$ when starting at x_0. The ratio $\frac{\cosh(\sqrt{2\alpha}z)}{\cosh(\sqrt{2\alpha}y)}$ then gives the expected discount for the time it takes X to again reach $\{\pm y\}$ but this time starting at $\pm z$ so the sum represents the expected discounting for infinitely many cycles. By symmetry, the initial term gives the cost for impulsing from $\pm y$ to $\pm z$ along with the second moment. The minimization therefore optimizes the expected cost over a particular class of impulse policies. We emphasize that the linear program imbedding is not restricted to these policies.

Proposition 5. *There exists pairs (y_*, z_*) and $(-y_*, -z_*)$ such that*

$$F(y_*, z_*) = F(-y_*, -z_*) = \inf_{(y.z):|z|\leq|y|} F(y, z). \tag{9}$$

Moreover, the minimizing pair (y_*, z_*) having nonnegative components is unique.

Proof. First observe

$$F(y, z) = \frac{k_1 + k_2|y - z| + (z^2 - y^2)/\alpha}{\cosh(\sqrt{2\alpha}y) - \cosh(\sqrt{2\alpha}z)}$$

so there exists some pairs (y, z) for which $F(y, z) < 0$ since the difference of the quadratic terms is negative and will dominate the constant and linear terms in the numerator. A straightforward asymptotic analysis show that $F(y, z)$ is asymptotically nonnegative when $y \to \infty$, $z \to \infty$ or $|y - z| \to 0$. Therefore F achieves its minimum at some point (y_*, z_*).

Notice that F is symmetric about 0 in that $F(-y, -z) = F(y, z)$ so it is sufficient to analyze F on the domain $0 \leq z \leq y$. The first-order optimality conditions on F are

$$0 = \frac{\partial F}{\partial y}(y_*, z_*) = \frac{(k_2 - 2y_*/\alpha)[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]}{[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]^2}$$
$$- \frac{\sqrt{2\alpha}\,[k_1 + k_2(y_* - z_*) + (z_*^2 - y_*^2)/\alpha]\sinh(\sqrt{2\alpha}\,y_*)}{[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]^2},$$

$$0 = \frac{\partial F}{\partial z}(y_*, z_*) = \frac{(-k_2 + 2z_*/\alpha)[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]}{[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]^2}$$
$$+ \frac{\sqrt{2\alpha}\,[k_1 + k_2(y_* - z_*) + (z_*^2 - y_*^2)/\alpha]\sinh(\sqrt{2\alpha}\,z_*)}{[\cosh(\sqrt{2\alpha}\,y_*) - \cosh(\sqrt{2\alpha}\,z_*)]^2}.$$

The minimizing pair (y_*, z_*) will be interior to the region since $\frac{\partial F}{\partial z}(y_*, 0) = \frac{-k_2}{\cosh(\sqrt{2\alpha}\,y_*)-1} < 0$.

Simple algebra now leads to the following systems of nonlinear equations for (y_*, z_*):

$$k_2\alpha - 2y_* = \alpha\sqrt{2\alpha}\,\sinh(\sqrt{2\alpha}\,y_*)\cdot F(y_*, z_*),$$
$$k_2\alpha - 2z_* = \alpha\sqrt{2\alpha}\,\sinh(\sqrt{2\alpha}\,z_*)\cdot F(y_*, z_*). \tag{10}$$

The fact that the minimal value of F is negative implies $y_* > z_* > k_2\alpha/2$. Solving for $-\alpha\sqrt{2\alpha}\,F(y_*, z_*)$ in each equation shows that at an optimal pair (y_*, z_*),

$$-\alpha\sqrt{2\alpha}\,F(y_*, z_*) = \frac{2z_* - k_2\alpha}{\sinh(\sqrt{2\alpha}\,z_*)} = \frac{2y_* - k_2\alpha}{\sinh(\sqrt{2\alpha}\,y_*)}.$$

A straightforward analysis of the function $h(x) = [2x - k_2\alpha]/\sinh(\sqrt{2\alpha}\,x)$ on the domain $[k_2\alpha/2, \infty)$ shows that the level sets of h consist of two-point sets and so on the region $0 \leq z \leq y$, the pair (y_*, z_*) is unique.   □

Now that the lower bound given in (9) is determined, it is important to connect an optimizing μ_1^* with an admissible impulse control policy $(\tau, Y) \in \mathcal{A}_1$. The existence of two minimizing pairs (y_*, z_*) and $(-y_*, -z_*)$ allows many auxiliary-LP-feasible measures μ_1 to place point masses at these two points and still achieve the lower bound. This observation leads to a solution to the restricted stochastic impulse control problem.

Theorem 6. *Let (y_*, z_*) be the pair having positive components that minimizes F as identified in Proposition 5. Consider initial positions $-y_* \leq x_0 \leq y_*$. Define the impulse control policy (τ^*, Y^*) as follows:*

$$\tau_1^* = \inf\{t \geq 0 : X(t-) = \pm y_*\} \quad and \quad Y_1^* = \mathrm{sgn}(X(\tau_1^*-))\cdot z_* - X(\tau_1^*-)$$

and for $k = 2, 3, 4, \ldots$, define

$$\tau_k^* = \inf\{t > \tau_{k-1} : X(t-) = \pm y_*\} \quad and \quad Y_k^* = \mathrm{sgn}(X(\tau_k^*-))\cdot z_* - X(\tau_k^*-).$$

Then (τ^*, Y^*) is an optimal impulse control pair for the restricted stochastic impulse control problem and the corresponding optimal value is

$$V_1(x_0) = \frac{\alpha x_0^2 + 1}{\alpha^2} + F(y_*, z_*)\cdot\cosh(\sqrt{2\alpha}\,x_0). \tag{11}$$

Proof. The measure μ_1^* defined from (τ^*, Y^*) using (3) is concentrated on the two points $(-y_*, -z_*)$ and (y_*, z_*). Since the process resulting from the admissible impulse control pair (τ^*, Y^*) remains bounded, conditions (5) can be used to obtain the masses:

$$\mu_1^*(-y_*, -z_*)$$
$$= \frac{\phi(x_0)[\psi(y_*) - \psi(z_*)] - \psi(x_0)[\phi(y_*) - \phi(z_*)]}{[\phi(-y_*) - \phi(-z_*)][\psi(y_*) - \psi(z_*)] - [\psi(-y_*) - \psi(-z_*)][\phi(y_*) - \phi(z_*)]},$$
$$\mu_1^*(y_*, z_*)$$
$$= \frac{\psi(x_0)[\phi(-y_*) - \phi(-z_*)] - \phi(x_0)[\psi(-y_*) - \psi(-z_*)]}{[\phi(-y_*) - \phi(-z_*)][\psi(y_*) - \psi(z_*)] - [\psi(-y_*) - \psi(-z_*)][\phi(y_*) - \phi(z_*)]}.$$

Recall $\phi(x) = e^{-\sqrt{2\alpha}\,x}$ and $\psi(x) = e^{\sqrt{2\alpha}\,x}$ so $\phi(-x) = \psi(x)$ and $\psi(-x) = \phi(x)$. As a result these expressions simplify to

$$\mu_1^*(-y_*, -z_*) = \frac{\phi(x_0)[\psi(y_*) - \psi(z_*)] - \psi(x_0)[\phi(y_*) - \phi(z_*)]}{[\psi(y_*) - \psi(z_*)]^2 - [\phi(y_*) - \phi(z_*)]^2},$$

$$\mu_1^*(y_*, z_*) = \frac{\psi(x_0)[\psi(y_*) - \psi(z_*)] - \phi(x_0)[\phi(y_*) - \phi(z_*)]}{[\psi(y_*) - \psi(z_*)]^2 - [\phi(y_*) - \phi(z_*)]^2}.$$

It is now straightforward to verify that $J(\tau^*, Y^*; x_0)$ equals the value in (11). $\square$

2.2   Full Solution

Theorem 6 solves the problem for initial positions x_0 with $|x_0| \le y_*$. The issue is now one of determining the optimal value and an optimal impulse control pair when $|x_0| > y_*$. From an intuitive point of view, $|x_0| < y_*$ has an optimal control which waits until the state process first hits $\pm y_*$ before having an impulse so one might expect an impulse to occur immediately when $|x_0| \ge y_*$. Since two impulses at the same instant are no better than one, one would anticipate that the after-jump location might be $z \in (-y_*, y_*)$. The cost of an immediate jump from x_0 to z followed by using an optimal impulse control is

$$g(z) := \frac{\alpha x_0^2 + 1}{\alpha^2} + k_1 + k_2(x_0 - z) + \frac{z^2 - x_0^2}{\alpha} + V_1(z)$$

$$= \frac{\alpha z^2 + 1}{\alpha^2} + k_1 + k_2(x_0 - z) + V_1(z).$$

Solving $g'(z) = 0$ to find a minimizer results in

$$0 = -k_2 + 2z/\alpha + \sqrt{2\alpha}\, F(y_*, z_*) \sinh(\sqrt{2\alpha}\, z),$$

which is the first order condition (10) for which both y_* and z_* are solutions. An impulse to y_* would be followed by an immediate jump to z_* and incur two fixed costs whereas a single jump directly to z_* would cost less. This line of reasoning indicates that a single jump to z_* could be an optimal initial impulse.

The goal is to verify that this intuitive reasoning is correct. Define

$$\widehat{V}(y) = \begin{cases} k_1 + k_2(|y| - z_*) + V_1(-z_*), & y \le -y_*, \\ V_1(y), & -y_* \le y \le y_*, \\ k_1 + k_2(y - z_*) + V_1(z_*), & y \ge y_*. \end{cases}$$

For $|y| > y_*$, the function $\widehat{V}$ is the cost associated with the process starting at initial position y, having an instantaneous jump from y to $\mathrm{sgn}(y)\, z_*$ and then using the optimal impulse control policy of Theorem 6 thereafter. The following lemma is fairly straightforward so its proof is left to the reader.

Lemma 7. $\widehat{V} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \backslash \{\pm y_*\})$.

The function $\widehat{V}$ therefore has sufficient regularity to use in (2). We now consider the new auxiliary linear program

$$
\begin{cases}
\text{Min.} & \displaystyle\int_{\mathbb{R} \backslash \{\pm y_*\}} c_0(x)\,\mu_0(dx) \quad + \quad \int_{\mathbb{R}^2} c_1(y,z)\,\mu_1(dy \times dz) \\
\text{S.t.} & \displaystyle\int_{\mathbb{R} \backslash \{\pm y_*\}} [\alpha\widehat{V}(x) - A\widehat{V}(x)]\,\mu_0(dx) + \int_{\mathbb{R}^2} [\widehat{V}(y) - \widehat{V}(z)]\,\mu_1(dy \times dz) \\
& \hspace{8cm} = \widehat{V}(x_0)
\end{cases}
\tag{12}
$$

and its dual (having sole variable w)

$$
\begin{cases}
\text{Max.} & \widehat{V}(x_0) \cdot w \\
\text{S.t.} & \left(\alpha\widehat{V}(x) - A\widehat{V}(x)\right) \cdot w \le c_0(x), \quad x \ne \pm y_*, \\
& \left(\widehat{V}(y) - \widehat{V}(z)\right) \cdot w \le c_1(y,z), \quad \forall y, z \in \mathbb{R}.
\end{cases}
\tag{13}
$$

Observe that each linear program has feasible points with costs that are finite. A straightforward weak duality argument therefore shows that each value of (13) corresponding to a feasible variable w is no greater than any value of (12) for a feasible pair of measures and hence the value of (13) is a lower bound on the value of the restricted impulse control problem. Since $\widehat{V}(x_0) > 0$, one seeks as large a positive value as possible for w.

Theorem 8. *The optimal value of* (13) *is* $\widehat{V}(x_0)$ *which is achieved when* $w_* = 1$.

Proof. By symmetry, it is sufficient to examine $x, y, z \ge 0$. Notice that for $0 \le x < y_*$, $\alpha\widehat{V}(x) - A\widehat{V}(x) = x^2 = c_0(x) \ge 0$ and hence the dual variable w cannot exceed 1. The question is whether $w = 1$ is feasible for (13) so examine the rest of the constraints with $w = 1$.

For $x > y_*$, $A\widehat{V}(x) = 0$ so the first constraint of (13) requires

$$0 \le x^2 - \alpha\left(k_1 + k_2(x - z_*) + V_1(z_*)\right) = x^2 - \alpha V_1(y_*).$$

Since the right-hand expression is an increasing function for $x \in [k_2\alpha/2, \infty)$, it suffices to verify its nonnegativity with $x = y_*$:

$$
\begin{aligned}
0 \le y_*^2 - \alpha V_1(y_*) &= y_*^2 - \alpha \left(\frac{\alpha y_*^2 + 1}{\alpha^2} + F(y_*, z_*)\cosh(\sqrt{2\alpha}\,y_*)\right) \\
&= -\frac{1}{\alpha} + \frac{[2y_* - k_2\alpha]\cosh(\sqrt{2\alpha}\,y_*)}{\sqrt{2\alpha}\,\sinh(\sqrt{2\alpha}\,y_*)}
\end{aligned}
$$

in which (10) is used to obtain the last expression. This inequality can be rewritten as

$$\frac{\tanh(\sqrt{2\alpha}\,y_*)}{\sqrt{2\alpha}} \le y_* - k_2\alpha/2. \tag{14}$$

Since (y_*, z_*) is a minimizing pair of the function F, (14) holds and the first family of constraints of (13) is satisfied with $w = 1$.

Consider now the second family of constraints with $w = 1$. There are several cases to examine. When $0 \leq y \leq z$, monotonicity of $\widehat{V}$ on this range shows the condition is trivially satisfied. Next, for $0 \leq z \leq y \leq y_*$, the constraint can be rewritten as

$$V_1(y) \leq k_1 + k_2(y - z) + V_1(z).$$

The right-hand expression gives the cost of an immediate jump from y to z followed by an optimal impulse control policy thereafter whereas the left-hand side gives the optimal cost. Hence this inequality is satisfied. Now consider $y_* \leq z < y$ and observe that $\widehat{V}(y) - \widehat{V}(z) = k_2(y - z) < k_1 + k_2(y - z)$. Finally, for $0 \leq z < y_* < y$ and again using the definition of $\widehat{V}$, the second set of constraints in (13) is equivalent to

$$k_1 + k_2(y - z_*) + V_1(z_*) \leq k_1 + k_2(y - z) + V_1(z)$$

or equivalently

$$k_2(y - y_*) + V_1(y_*) = k_2(y - y_*) + [k_1 + k_2(y_* - z_*) + V_1(z_*)]$$
$$\leq k_2(y - y_*) + [k_1 + k_2(y_* - z) + V_1(z)].$$

This last inequality is true by the optimality of both the pair (y_*, z_*) and the function V_1 on $[-y_*, y_*]$ since the bracketed quantity on the right-hand side gives the cost associated with an initial impulse to z from y_* along with optimal impulse control policy starting from z. Thus the second family of constraints in (13) hold when $w = 1$. $\qquad\square$

We now have the following result.

Theorem 9. *Let (y_*, z_*) be the optimizing pair for F having positive components. Define the impulse control policy (τ^*, Y^*) as follows;*

$$\tau_1^* = \inf\{t \geq 0 : |X(t-)| \geq y_*\} \quad and \quad Y_1^* = \text{sgn}(X(\tau_1^*-)) \cdot z_* - X(\tau_1^*-)$$

and for $k = 2, 3, 4, \ldots$, define

$$\tau_k^* = \inf\{t > \tau_{k-1} : X(t-) = \pm y_*\} \quad and \quad Y_k^* = \text{sgn}(X(\tau_k^*-)) \cdot z_* - X(\tau_k^*-).$$

Then (τ^*, Y^*) is an optimal impulse control pair for the restricted stochastic impulse control problem and the corresponding optimal value is $\widehat{V}(x_0)$.

Proof. The particular choice of (τ^*, Y^*) implies $\widehat{V}(x_0) \leq V_{lp}(x_0) \leq V_1(x_0) \leq J(\tau^*, Y^*) = \widehat{V}(x_0)$. $\qquad\square$

2.3   Solution for General Admissible Impulse Controls

The solution of Sect. 2.2 is restricted to those impulse control policies under which the process X remains bounded. It is necessary to show that no lower cost can be obtained by any policy which allows the process to be unbounded.

Theorem 10. *The impulse control policy (τ^*, Y^*) of Theorem 9 is optimal in the class of all admissible policies and $\widehat{V}(x_0)$ is the optimal value.*

Proof. This argument establishes that $\widehat{V}(x_0)$ is a lower bound on $J(\tau, Y; x_0)$ for every admissible impulse control policy. Theorem 9 then gives the existence of an optimal policy whose cost equals the lower bound.

Choose $(\tau, Y) \in \mathcal{A}$ and let X be the resulting controlled process. Suppose there exists some $K > 0$ such that $\liminf_{t \to \infty} \mathbb{E}_{x_0}[e^{-\alpha t}\widehat{V}(X(t))] \geq K$. Note that

$$\liminf_{t \to \infty} \mathbb{E}_{x_0}\left[e^{-\alpha t}\widehat{V}(X(t))\right] = \liminf_{t \to \infty} \mathbb{E}_{x_0}\left[e^{-\alpha t}\widehat{V}(X(t))I_{\{|X(t)| \geq y_*\}}\right]$$

so the linearity of $\widehat{V}$ on $\{x : |x| \geq y_*\}$ implies that $\mathbb{E}_{x_0}[|X(t)|I_{\{|X(t)| \geq y_*\}}]$ is asymptotically bounded below by $Ke^{\alpha t}$ as $t \to \infty$. Hence by Jensen's inequality for $\epsilon > 0$ and t large,

$$\mathbb{E}_{x_0}\left[X^2(t)\right] \geq \left(\mathbb{E}_{x_0}[|X(t)|I_{\{|X(t)| \geq y_*\}}]\right)^2 \geq K^2 e^{2\alpha t} - \epsilon.$$

Using this estimate in (1) shows $J(\tau, Y; x_0) = \infty$.

Now suppose $J(\tau, Y; x_0) < \infty$ so $\liminf_{t \to \infty} \mathbb{E}_{x_0}[e^{-\alpha t}\widehat{V}(X(t))] = 0$. Then there exists a sequence $\{t_j : j \in \mathbb{N}\}$ such that $\lim_{j \to \infty} \mathbb{E}_{x_0}[e^{-\alpha t_j}\widehat{V}(X(t_j))] = 0$. Note that $|\widehat{V}'| \leq k_2$ so $\int_0^t e^{-\alpha s}\widehat{V}'(X(s))\,dW(s)$, $t \geq 0$, is a martingale. Thus the dual constraints, in conjunction with the finiteness of the expected cost, implies that Dynkin's formula holds when $t = t_j$ for each j. Hence

$$\widehat{V}(x_0) = \mathbb{E}_{x_0}\left[\int_0^{t_j} e^{-\alpha s}[\alpha\,\widehat{V}(X(s)) - A\widehat{V}(X(s))]\,ds\right] - \mathbb{E}_{x_0}\left[e^{-\alpha t_j}\widehat{V}(X(t_j))\right]$$

$$+ \mathbb{E}_{x_0}\left[\sum_{k=0}^{\infty} I_{\{\tau_k \leq t_j\}} e^{-\alpha \tau_k} B\widehat{V}(X(\tau_k-), X(\tau_k))\right]$$

$$\leq \mathbb{E}_{x_0}\left[\int_0^{t_j} e^{-\alpha s} c_0(X(s))\,ds + \sum_{k=0}^{\infty} I_{\{\tau_k \leq t_k\}} e^{-\alpha \tau_k} c_1(X(\tau_k-), X(\tau_k))\right]$$

$$- \mathbb{E}_{x_0}\left[e^{-\alpha t_j}\widehat{V}(X(t_j))\right]$$

Letting $j \to \infty$, an application of the monotone convergence theorem on the first expectation and the convergence to 0 of second expectation establishes that $\widehat{V}(x_0)$ is a lower bound on the expected cost $J(\tau, Y; x_0)$. $\square$

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