Dynamic CT Reconstruction by Smoothed Rank Minimization

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Abstract. We address the problem of dynamic CT reconstruction from parsimoniously sampled sinograms. In this paper we propose a novel approach to solve the aforesaid problem by modeling the dynamic CT sequence as a low-rank matrix. This dynamic CT matrix is formed by stacking each frame as a column of the matrix. As these images are temporally correlated, the dynamic CT matrix would therefore be of low-rank as its columns are not independent. We exploit the low-rank information to reconstruct the CT matrix from its parsimoniously sampled sinograms. Mathematically this is a low-rank matrix recovery problem, and we propose a novel algorithm to solve it. Our proposed method reduces the reconstruction error by 50% or more when compared to previous recovery techniques.

1 Introduction

Traditional knowledge dictates that in order to get a good quality high resolution X-Ray CT image, the sinogram should be densely sampled. Dense sampling of a sinogram requires higher CT dosage than parsimonious sampling. CT reconstruction researchers have been looking for ways to parsimoniously sample the sinogram (thereby reducing the radiation dosage) and to reconstruct a good quality image from it.

Recently, Compressed Sensing (CS) based techniques have shown how transform domain sparsity of the underlying CT image can be exploited in order to recover it from parsimoniously sampled sinogram [1-3]. These studies have shown that CS techniques can indeed be used to cut the CT radiation dose by more than 50% for static CT imaging.

In this work we address the problem of reconstructing dynamic CT images. CS based techniques have also been used in the past to reconstruct the dynamic image sequence for parsimoniously sampled sinograms. In this paper we propose a novel formulation to solve this problem where we model the image sequence as a low-rank matrix. The reconstruction problem is thus recast as a low-rank matrix recovery problem from its parsimoniously sampled sinograms. We also propose a new algorithm to solve the low-rank matrix recovery problem.

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The rest of the paper is organized into several sections. The following section briefly reviews the prevalent CS based recovery algorithms in dynamic CT imaging. In section 3, we briefly discuss the similarity between dynamic CT and dynamic magnetic resonance imaging (MRI). We formulate the problem in section 4 and propose a new algorithm to solve it. The experimental results are shown in section 5. Finally the conclusions of this work and future directions of research are discussed in section 6.

2 Compressed Sensing in Dynamic CT

In CT, the data acquisition model can be expressed as follows:

$$y = Ax \tag{1}$$

Here x is the underlying image (to be reconstructed), y is the sampled sinogram and A is the X-ray transform.

For dynamic CT, the sinogram is sampled in an interleaved fashion, so the A matrix changes with time. The data acquisition model for the t^{th} frame is as follows:

$$y_t = A_t x_t \tag{2}$$

This is an inverse problem; one is supposed to reconstruct x_t given A_t and y_t. For non-iterative reconstruction using Filtered Back Projection (FBP), the sinogram needs to be densely sampled; dense sampling translated to higher ionizing radiation for the subject. Researchers in CT reconstruction aim to reconstruct the image from smaller number of sinogram samples. CS based techniques are useful to achieve this goal; CS exploits the sparsity of the image in order to reconstruct it from a smaller number of sinogram samples than was deemed necessary previously [1-3].

Recent papers however have shown how CS techniques can be used for dynamic CT reconstruction [4, 5]. The first step is to generate a static FBP reference image (x_0) from the interleaved projections. Once this reference image is computed, the reconstruction of the t^{th} frame is solved via the following optimization problem,

$$\min_x \alpha \left\|\Psi_1(x_t - x_0)\right\|_p^p + (1-\alpha)\left\|\Psi_2 x_t\right\|_p^p \text{ subject to } y_t = A_t x_t \tag{3}$$

where Ψ_1 and Ψ_2 are sparsifying transforms (wavelet or gradient). The l_p-norm ($0<p\leq 1$) is the sparsity promoting objective function. There are two sparsity promoting terms. The first term assumes that the difference between the current frame and the reference image is sparse in Ψ_1. The second term assumes the t^{th} frame is sparse in Ψ_2. The scalar α controls the relative importance of the two sparsity promoting terms.

This technique (3) is called Prior Image Constrained Compressed Sensing (PICCS). This was originally developed with convex sparsity promoting l_1-norm [4] but was later shown to yield even better results with non-convex l_p-norm (NCPICCS) [5]. It should be noted that even though the frames are reconstructed separately, this is an offline technique because the reference image x_0 can only be generated after the full sequence has been collected.

3 Compressed Sensing in Dynamic MRI

In MRI, the data acquisition model is the same as the CT for both static (1) and dynamic (2) scenarios. For MRI, the matrix A is the Fourier transform and the y is the sampled Fourier coefficients (called K-space in MRI). In MRI the challenge is different from CT. For MRI the challenge is to reduce the data acquisition time. Thus A or A_t is not the full Fourier transform (F), it is an under-sampled Fourier transform ($A = RF$); where R is the sampling mask.

Even though the challenges in CT and MRI are different, the fundamental mathematical problem remains the same. When the K-space is parsimoniously sampled, the inverse problems represented by (1) and (2) become under-determined. CS is used to reconstruct the MR images by exploiting their sparsity in a domain such as the wavelet or gradient.

In dynamic MRI reconstruction the main idea is to maximally exploit the spatio-temporal redundancies of the dynamic MRI sequence. The CS based techniques reconstruct the dynamic MRI sequence by solving an optimization problem of the following form [6, 7]:

$$\min_x \left\| \Psi_1 \otimes \Psi_2 (x') \right\|_1 \text{ subject to } y' = A'x' \tag{4}$$

where y' is the vector formed by concatenating all the acquired vectors y_t's, similarly x' is the vector formed by concatenating all the unknown x_t's and A' is a block diagonal matrix formed by A_t's as the blocks. Ψ_1 is the sparsifying transform along the temporal direction and Ψ_2 is the sparsifying transform along the spatial direction.

CS based sparsity promoting techniques are not the only solution for dynamic MRI reconstruction. In general, one can use a different sampling mask for each frame. But if the sampling mask is the same for all the frames, i.e. if A_t=A for all t's, then (2) can be expressed as

$$Y = AX \tag{5}$$

where Y is a matrix formed by stacking the y_t's as columns, similarly X is formed by stacking the x_t's as columns.

In [8] it is argued that the matrix X is rank deficient; this is because the MRI time-frames are correlated with each other. Thus the columns of X are therefore not independent and thus X can be modeled as a low-rank matrix [8]. Therefore X can be recovered by solving the following problem,

$$\min_X rank(X) \text{ such that } Y = AX \tag{6}$$

In general, minimizing the rank is a combinatorial problem and it is thus not feasible for large scale systems such as (6). Thus a matrix factorization based approach was proposed in [8] in order to recover X.

4 Proposed Solution

In this work, we propose to solve the dynamic CT reconstruction problem (2) by modeling the sequence of CT images as a low-rank matrix. Our work is motivated by the studies in dynamic MRI reconstruction [8]. In the dynamic CT sequence, the frames are correlated temporally. When the frames from the sequence (x_t's) are stacked as columns of a matrix (X), the resulting matrix ($X = [x_1|\ldots|x_t|\ldots|x_N]$, assuming N frames in all) is rank-deficient; this is because the columns are correlated. We propose to recover this matrix by exploiting its rank-deficiency.

For dynamic CT, the sinograms for different time-frames are sampled in an interleaved fashion. The data acquisition model is expressed as follows:

$$y' = A'x' \tag{7}$$

where y' is the vector formed by concatenating all the (acquired) y_t's, similarly x' is the vector formed by concatenating all the (unknown) x_t's and A' is a block diagonal matrix formed by A_t's as the blocks.

It must be understood by now that x' and X are just two different ways to represent the same group of vectors x_t's. In x' they are concatenated one after the other, and in X they are stacked as its columns. To exploit the prior information that X is low-rank, we exploit the rank deficiency of X in order to recover it:

$$\min_X rank(X) \text{ subject to } y' = A'x' \tag{8}$$

This is an NP hard problem. There are two solutions – i) replace the rank by its nearest convex or non-convex surrogate (i.e. nuclear norm or the Schatten-p norm); or, ii) use matrix factorization.

The second approach is computationally faster but does not provide any recovery guarantees. The first approach that recovers the low-rank matrix via nuclear norm minimization [9] provides theoretical recovery guarantees for solving problems like (8). In practice however, it has been found that the non-convex Schatten-p norm minimization yields even better results than the nuclear norm minimization [10].

Our work is motivated by the smoothed l_0-minimization (SL0) [11] algorithm in CS; SL0 is faster and more accurate than most state-of-the-art l_1-minimization algorithms. SL0 approximately solves the l_0-norm minimization, i.e. it does not substitute the NP hard l_0-norm by its convex (l_1-norm) or non-convex (l_p-norm) surrogates. In this work we propose to approximately solve the rank-minimization problem (8) by a similar approach.

The matrix X can be expressed in terms of its singular value decomposition (SVD): $X = U\Sigma V^T$, where U and V are the left and right singular vectors and Σ is the diagonal matrix consisting of the singular values σ_j's. The rank of a matrix is the number of non-zero singular values. We define a function for every singular value,

$$\gamma(\sigma) = \begin{cases} 1 \text{ when } \sigma > 0 \\ 0 \text{ when } \sigma = 0 \end{cases} \tag{9}$$

Based on the above definition, the rank is expressed as $rank(X) = \sum_j \gamma(\sigma_j)$. The function $\gamma(\sigma_j)$ is spiky; i.e. it has the value 0 or 1. Following [11] we replace the spiky function by a smooth zero-mean Gaussian whose spread can be varied by changing its standard deviation (θ) – $f_\theta(\sigma) = e^{(-\sigma^2/2\theta^2)}$

The function is wide when θ is large and becomes narrow when its value reduces. In the limit that the θ is zero, the above function has the following property,

$$\lim_{\theta \to 0} f_\theta(\sigma) = \begin{cases} 1 \text{ when } \sigma = 0 \\ 0 \text{ when } \sigma > 0 \end{cases} \quad (10)$$

Therefore, $\lim_{\theta \to 0} f_\theta(\sigma) = 1 - \gamma(\sigma)$. This allows for approximating the rank by $\lim_{\theta \to 0} F_\theta(x) = \sum_{j=1}^{n} f_\theta(\sigma_j) = \sum_{j=1}^{n} 1 - \gamma(\sigma_j) = n - rank(X)$, (where n is the minimum of the number of rows or the number of columns in X). Therefore the rank minimization problem (8) can be recast as follows,

$$\max_X F_\theta(\mathrm{x}) \text{ subject to } y' = A'x' \quad (11)$$

Since the objective function is smooth, it is easy to solve (11) by gradient based methods. The main idea behind the algorithm proposed below is that at each iteration, (11) is solved for a particular value of θ; then in the following iteration the value of θ is decreased and (11) is solved again. This continues till the solution converges (i.e. when there is no significant change in the solution).

<u>Algorithm for Smoothed Rank Minimization</u>

Initialization – Obtain the initial solution $\hat{x}'^{(0)} = \min_x \|y' - A'x'\|_2^2$. Rearrange $\hat{x}'^{(0)}$ in matrix form $\hat{X}^{(0)}$. Compute the SVD, $\hat{X}^{(0)} = U^{(0)} \Sigma^{(0)} V^{(0)T}$

At iteration k – Continue the following steps till solution is reached (i.e. till θ is greater than a specified value)

1. Choose $\theta = c\sigma_1$, where $\mathrm{c} > 4$.
2. Maximize (11) for the current value of θ. The Steepest Ascent method is used to achieve this.
   a. Initialize, $s = diag(\Sigma^{(k-1)})$. Here *diag()* operator forms a vector from the diagonal elements
   b. Let $\Delta s = [s_1 \cdot e^{-s_1^2/2\theta^2}, ..., s_n \cdot e^{-s_n^2/2\sigma^2}]^T$.
   c. Update: $s \leftarrow s - \mu \Delta s$, where μ is a small constant.

Express $\hat{\Sigma}^{(k)} = diag(s)$. Here *diag()* generates a matrix with diagonal elements.

d. Generate the matrix $\hat{X}^{(k)} = U^{(k-1)}\hat{\Sigma}^{(k)}V^{(k-1)T}$ and $\hat{x}'^{(k)} = vec(\hat{X}^{(k)})$.

e. Project the solution back to the feasible set by $x'^{(k)} \leftarrow \hat{x}'^{(k)} - A'^{T}(A'A'^{T})^{-1}(A'\hat{x}'^{(k)} - y')$.

3. Rearrange $x'^{(k)}$ in matrix form and compute the SVD, $X^{(k)} = U^{(k)}\Sigma^{(k)}V^{(k)T}$ and return to step 1 until convergence.

5 Experimental Evaluation

We compared the above proposed technique with the Non-convex Prior Image Constrained Compressed Sensing (NCPICCS) method [7] since this method [7] yields the best reconstruction for dynamic CT when *α = 0.7* and *p = 0.7* (refer to problem formulated in (3)). The reconstruction accuracy in our study is measured in terms of Relative Mean Squared Error (RMSE) as this metric has been used previously for the same purpose [5].

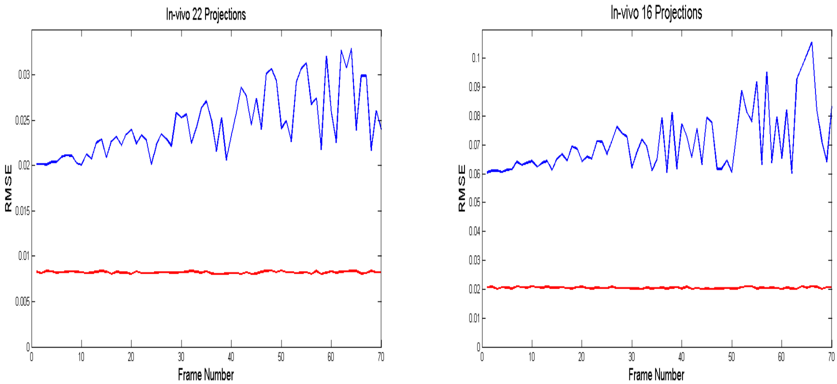

Fig. 1. Variation of RMSE with time. Blue plot represents error from NCPICCS and red plot represents error from proposed method.

We use a portion of the experimental data used in [5]. The reconstructions were carried out on a synthesized Shepp Logan phantom and on an in-vivo animal kidney perfusion CT scans. The Shepp Logan phantom was modified in [5] such that the uppermost ellipses in the simulated original object changed attenuation through time as follows:

$$r = at^{b}\exp(-t/c) \tag{12}$$

where t is the time and the parameters a, b, and c control the amplitude, width, and speed of decay of the gamma-variate function. The values, $a=.05$, $b=7$, and $c=2$ were used to simulate tissue perfusion. A total of 20 time points were simulated, with time steps $t=0.5$ *s*.

The in-vivo study scans were performed at 80 kV, using 160 mA s, with 24X1.2 mm collimation and 0 mm table feed. Using a 0.33 sec. gantry rotation time, 70 exposures (images) were acquired with 0.67 sec interval between consecutive frames.

For the simulated Shepp Logan, the reconstruction was carried using 4 and 6 projections (with parallel beam geometry). For the in-vivo experiment the number of projections were 16 and 22. These values were suggested in [5]. The frame-by-frame RMSE's are plotted in Fig. 1. Owing to limitations in space, we only show the results for the in-vivo data.

We see that our proposed method (red plot) reduces the RMSE by 50% or more. Also the variation in error from our proposed method is less compared to NCPICCS. Both of these observations stem from the same fact. NCPICCS and other PICCS based methods reconstruct the images frame-by-frame, whereas our method reconstructs all the frames simultaneously. During reconstruction, our method makes better use of the spatio-temporal redundancy compared to PICCS. That is why our proposed method yields more stable (less variation in time) and better reconstruction results.

To corroborate the numerical results, we show the ground truth, reconstructed and difference (between ground truth and reconstructed) images. Owing to limitations in space, we only show one frame from each of the datasets. The contrast of the difference images is magnified 10 times for visual clarity. From the difference images, it is clearly seen that our proposed method is better than PICCS; the difference images are darker. The improvement from our proposed method is better evident from the phantom. The PICCS reconstructed phantom image clearly shows reconstruction artifacts; the artifacts are absent in our proposed method.

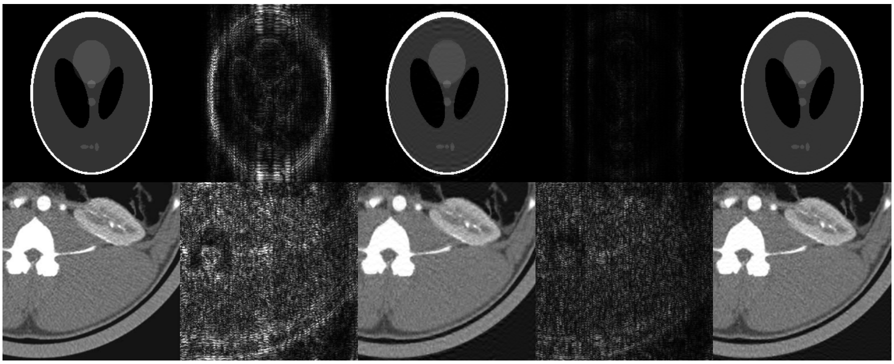

Fig. 2. Left to right: Ground truth, Difference image from NCPICCS, Reconstructed image from NCPICCS, Difference image from proposed method, Reconstructed image from proposed method

6 Conclusion

In this work we propose a novel technique to reconstruct dynamic CT image sequences. The temporal correlation of the CT images allows us to model the entire sequence as a low-rank matrix. We exploit this low-rank structure of the matrix while reconstructing the sequence. The proposed method yields considerably better results than the well known PICCS based technique for reconstructing dynamic CT images from parsimoniously sampled sinograms.

In the future, we want to combine the sparsity promoting reconstruction with the proposed low-rank model to achieve even better reconstruction.

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