A Study of Monodromy in the Computation of Multidimensional Persistence

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Abstract. The computation of multidimensional persistent Betti numbers for a sublevel filtration on a suitable topological space equipped with a $\mathbb{R}^n$-valued continuous filtering function can be reduced to the problem of computing persistent Betti numbers for a parameterized family of one-dimensional filtering functions. A notion of continuity for points in persistence diagrams exists over this parameter space excluding a discrete number of so-called *singular* parameter values. We have identified instances of nontrivial monodromy over loops in nonsingular parameter space. In other words, following cornerpoints of the persistence diagrams along nontrivial loops can result in them switching places. This has an important incidence, e.g., in computer-assisted shape recognition, as we believe that new, improved distances between shape signatures can be defined by considering continuous families of matchings between cornerpoints along paths in nonsingular parameter space. Considering that nonhomotopic paths may yield different matchings will therefore be necessary. In this contribution we will discuss theoretical properties of the monodromy in question and give an example of a filtration in which it can be shown to be nontrivial.

Keywords: Persistence diagram, topological persistence, multifiltration, shape comparison, shape recognition.

Introduction

The last few decades have seen an explosion of the amount of data to be processed in many scientific contexts, due in large part to the availability of powerful computing technology. This has led researchers to consider methods to study qualitative information, such as the topological analysis of data, which has been applied, for example, to computer imaging [14] and shape analysis [3].

In this context, *topological persistence* has revealed itself to be an increasingly interesting approach for data analysis and comparison. Indeed, it enables a deep

R. Gonzalez-Diaz, M.-J. Jimenez, B. Medrano (Eds.): DGCI 2013, LNCS 7749, pp. 192–202, 2013.

reduction in the complexity of data, by confining the analysis only to the relevant parts [12,13]. Moreover, persistence allows for a stable description and comparison of data, and ensures invariance under different groups of transformations [3]. The main idea of this theory consists in studying the k-dimensional holes (components, tunnels, voids ...) of the sublevel sets of a continuous function, called *filtering function*, varying the level, and using the information gleaned for topological denoising and shape comparison. The relevance of components, tunnels and voids is given by their *persistence*, i.e. the duration of the life of these homological structures if the level is interpreted as time. The more persistent a homological property is, the higher its incidence on shape comparison algorithms, since holes of low persistence are assumed to be the result of noise. A key result shows that if the filtering function is real-valued, then persistent homology can be completely described by a countable collection of points in the real plane, called a *persistence diagram* [10]. The stability of this descriptor with respect to noise makes it important in applications of digital topology and discrete geometry to denoising, where spatial data are only known up to approximation error due to digitization (cf. [6]). Other applications in which persistence has been successfully exploited range from 3D image analysis [1], simplification [17] and reconstruction [18] to image segmentation [16], shape comparison [11] and retrieval [9].

In recent years greater attention has been given to *multidimensional persistence*, i.e. persistent homology for $\mathbb{R}^n$-valued filtering functions (e.g. [7]). This extension of the theory is motivated by the fact that data analysis and comparison often involve the examination of properties that are naturally described by vector-valued functions. In computer vision, for example, photometric properties of digital images constitute a standard feature which is taken into account for their segmentation. In point cloud data analysis, the object of study is usually a finite set of samples from some underlying topological space. Each sample is associated with multiple labels, representing several measurements possibly obtained from multiple modalities. Another example is the analysis of 4D time-varying CT scans in medical imaging.

The study of multidimensional persistent homology is proving to be much harder than that of one-dimensional persistent homology. As an example of this difficulty, we recall the lack of a complete and discrete stable descriptor in the case that the filtering function is vector-valued [7]. Fortunately, a method is available to reduce the multidimensional persistent homology of a function $\varphi : X \to \mathbb{R}^n$ to the one-dimensional persistent homology of each function in a suitable parameterized family $\{\varphi_{(\boldsymbol{m},\boldsymbol{b})} : X \to \mathbb{R}\}$ (cf. [2,5]). For each function $\varphi_{(\boldsymbol{m},\boldsymbol{b})}$ a persistence diagram can be obtained, and from the parameterized family of diagrams the multidimensional persistent homology of φ (in the sense of the corresponding persistent Betti numbers) can be recovered.

While this approach has opened a new line of research, it has also brought new problems and questions to the surface. In this work we illustrate that an interesting link exists between the classical concept of monodromy and the persistence diagrams associated with the functions $\varphi_{(\boldsymbol{m},\boldsymbol{b})}$. In plain words, when we move in our parameter space $\{(\boldsymbol{m}, \boldsymbol{b})\}$ along a closed path around a so-called singular point, the points in the persistence diagrams may exchange their position. This switching of places generates a monodromy that, besides its theoretical relevance, could be important for defining new distances between multidimensional persistent Betti numbers.

The paper is organized as follows. In Section 1 we recall the definitions and results needed in our exposition. In Section 2 we demonstrate the continuity of the movement of each point in persistence diagrams over paths in parameter space. In Section 3 we describe an example where this continuous movement creates a nontrivial monodromy over persistence diagrams. A section illustrating our conclusions ends the paper.

1 Preliminaries

In this section we recall the definitions and results we will be needing in this paper. For the treatment, we refer the reader to [8].

We shall use the following notations: Δ^+ will be the open set $\{(u, v) \in \mathbb{R} \times \mathbb{R} : u < v\}$. Δ will represent the diagonal set $\{(u, v) \in \mathbb{R} \times \mathbb{R} : u = v\}$. We can further extend Δ^+ with points at infinity of the kind (u, ∞), where $|u| < \infty$. Denote this set Δ^*. Let us assume that a topological space X and a continuous function $\varphi : X \to \mathbb{R}$ are given. For any $k \in \mathbb{N}$, if $u < v$, the inclusion map of the sublevel set $X_u = \{x \in X : \varphi(x) \leq u\}$ into the sublevel set $X_v = \{x \in X : \varphi(x) \leq v\}$ induces a homomorphism from the kth homology group of X_u into the kth homology group of X_v. The image of this homomorphism is called the *kth persistent homology group of* (X, φ) *at* (u, v), and is denoted by $H_k^{(u,v)}(X, \varphi)$. In other words, the group $H_k^{(u,v)}(X, \varphi)$ contains all and only the homology classes of k-cycles born before or at u and still alive at v.

In what follows, we shall work with coefficients in a field $\mathbb{K}$, so that homology groups are vector spaces. Therefore, they can be completely described by their dimension, leading to the following definition (cf. [13]).

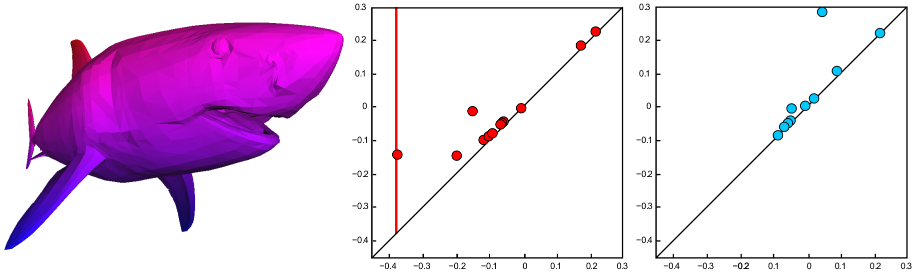
Definition 1 (Persistent Betti Numbers Function). *The* persistent Betti numbers function *of φ, briefly PBN, is the function $\beta_\varphi : \Delta^+ \to \mathbb{N} \cup \{\infty\}$ defined by*

$$\beta_\varphi(u, v) = \dim H_k^{(u,v)}(X, \varphi).$$

Throughout the paper, we shall assume that X is triangulable, implying the finiteness of β_φ for all $(u, v) \in \Delta^+$ [8]. Obviously, for each $k \in \mathbb{Z}$, we have different PBNs of φ (which might be denoted $\beta_{\varphi,k}$), but for the sake of notational simplicity we omit adding any reference to k.

The PBNs of φ can be simply and compactly described by the corresponding *persistence diagrams*. Figure 1(b) and (c) show respectively the 0th and the 1st persistence diagram obtained from the PBNs of the height function defined on a surface model (Fig. 1(a)).

As shown in Figure 1, persistence diagrams can be represented as multisets of points lying in Δ^+. For each point, the u-coordinate denotes the *birth*, in terms



(a) (b) (c)

Fig. 1. (a) A model from the TOSCA dataset [4] filtered by the "height" function (color-coded, left), and the corresponding 0th (b) and 1st (c) persistence diagrams

of the values of the filtering function, of a topological feature (e.g. connected components in the case of Fig. 1(b), holes in the case of Fig. 1(c)), whereas the v-coordinate denotes its *death*. In particular, the red line in Fig. 1(b) can be thought as a point at infinity, i.e. a connected component that *will never die*; indeed, its v-component is equal to $+\infty$. The distance of a point from Δ can be interpreted as the *lifespan* of the associated topological feature, thus reflecting its importance: points far from the diagonal are associated with important or global features, i.e. long-lived ones, while points close to the diagonal correspond to local information such as smaller details and noise.

Formally, a persistence diagram can be defined via the notion of *multiplicity* [12,15]. Following the convention used for PBNs, any reference to k will be dropped in the sequel.

Definition 2 (Multiplicity). *The* multiplicity $\mu_\varphi(u,v)$ *of* $(u,v) \in \Delta^+$ *is the finite, non-negative number given by*

$$\min_{\substack{\varepsilon>0\\ u+\varepsilon<v-\varepsilon}} \beta_\varphi(u+\varepsilon, v-\varepsilon) - \beta_\varphi(u-\varepsilon, v-\varepsilon) - \beta_\varphi(u+\varepsilon, v+\varepsilon) + \beta_\varphi(u-\varepsilon, v+\varepsilon).$$

The multiplicity $\mu_\varphi(u,\infty)$ *of* (u,∞) *is the finite, non-negative number given by*

$$\min_{\varepsilon>0,\, u+\varepsilon<v} \beta_\varphi(u+\varepsilon, v) - \beta_\varphi(u-\varepsilon, v).$$

Definition 3 (Persistence Diagram). *The persistence diagram* $Dgm(\varphi)$ *is the multiset of all points* $p \in \Delta^*$ *such that* $\mu_\varphi(p) > 0$*, counted with their multiplicity, union the singleton* $\{\Delta\}$*, counted with infinite multiplicity.*

Each point $p \in \Delta^*$ with positive multiplicity will be called a *cornerpoint*. A cornerpoint p will be said a *proper cornerpoint* if $p \in \Delta^+$, and a *cornerpoint at infinity* if $p \in \Delta^* \setminus \Delta^+$.

Persistence diagrams show stability properties with respect to the so-called *bottleneck distance* (a.k.a. *matching distance*). Roughly, small changes in the filtering function induce only small changes in the position of the cornerpoints which are far from the diagonal in the associated persistence diagram, and possibly produce variations close to the diagonal [8,10]. An intuition of this fact is given in Figure 2. More precisely, we have the following definition:

Definition 4 (Bottleneck distance). *Let Dgm^1, Dgm^2 be two persistence diagrams. The bottleneck distance $d_B\left(Dgm^1, Dgm^2\right)$ is defined as*

$$d_B(Dgm^1, Dgm^2) = \min_{\sigma} \max_{p \in Dgm^1} d(p, \sigma(p)),$$

where σ varies among all the bijections between Dgm^1 and Dgm^2 and

$$d\left((u,v),(u',v')\right) = \min\left\{\max\left\{|u-u'|,|v-v'|\right\}, \max\left\{\frac{v-u}{2}, \frac{v'-u'}{2}\right\}\right\} \quad (1)$$

for every $(u,v),(u',v') \in \Delta^* \cup \{\Delta\}$.

In practice, the distance d defined in (1) measures the cost of taking a point p to a point p' as the minimum between the cost of moving one point onto the other and the cost of moving both points onto Δ. In particular, the matching of a proper cornerpoint p with Δ can be interpreted as the destruction of p. The stability of persistence diagrams can then be formalized as follows.

Theorem 1 (Stability Theorem). *Let $\varphi, \psi : X \to \mathbb{R}$ be two filtering functions. It holds that*

$$d_B\left(Dgm(\varphi), Dgm(\psi)\right) \leq \|\varphi - \psi\|_\infty.$$

We conclude this subsection by noting that if we use Čech homology, persistence diagrams allow the recovery of all information represented in PBNs [8].

1.1 Multidimensional Setting

The definition of a persistent Betti numbers function can be easily extended to the case of $\mathbb{R}^n$-valued filtering functions [8]. Moreover, it has been proven that the information enclosed in the persistent Betti numbers function of a filtering function $\varphi = (\varphi_1, \ldots, \varphi_n) : X \to \mathbb{R}^n$ is equivalent to the information represented by the set of persistent Betti numbers functions of the parameterized family $\{\varphi_{(\boldsymbol{m},\boldsymbol{b})}\}$ of one-dimensional filtering functions defined by setting

$$\varphi_{(\boldsymbol{m},\boldsymbol{b})}(x) = \min_i\{m_i\} \cdot \max_i \left\{\frac{\varphi_i(x) - b_i}{m_i}\right\}$$

for every $x \in X$, varying $(\boldsymbol{m},\boldsymbol{b})$ in the set of *admissible pairs*

$$Adm_n = \left\{(\boldsymbol{m},\boldsymbol{b}) \in \mathbb{R}^n \times \mathbb{R}^n : \forall i \; m_i > 0, \sum_i m_i = 1, \sum_i b_i = 0\right\}.$$

Intuitively, each admissible pair $(\boldsymbol{m},\boldsymbol{b})$ corresponds to a line of $\mathbb{R}^n$, say $r_{(\boldsymbol{m},\boldsymbol{b})}$, whose generic point is given by $u = \tau\boldsymbol{m} + \boldsymbol{b}$ with $\tau \in \mathbb{R}$. Each such point can be associated with the sublevel set X_u of X defined as $X_u = \{x \in X : \varphi_i(x) \leq u_i,\ i = 1, \ldots, n\}$. The filtration $\{X_u\}_{u \in l_{(\boldsymbol{m},\boldsymbol{b})}}$ corresponds to the one associated with $\varphi_{(\boldsymbol{m},\boldsymbol{b})}$. Details on this approach to multidimensional persistence can be found in [8].

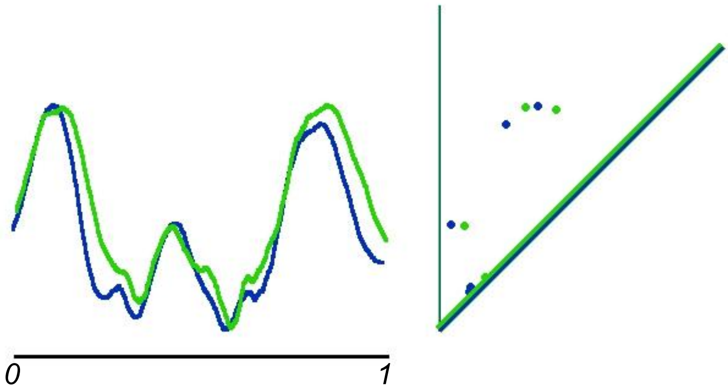


Fig. 2. The change of the filtering function induces a change in the persistence diagram. In this example, the graphs on the left represent two different real-valued filtering functions, defined on the interval $[0, 1]$. The corresponding 0th persistence diagrams are displayed on the right.

Following Proper Cornerpoints. Obviously, when we change the parameter $(\boldsymbol{m}, \boldsymbol{b})$ in Adm_n, the cornerpoints of $\mathrm{Dgm}(\varphi_{(\boldsymbol{m},\boldsymbol{b})})$ move in the topological space $\Delta^* \cup \{\Delta\}$. The main goal of the present work is to describe some properties of such movements, in the case of proper cornerpoints. To this end, we endow the set $\Delta^+ \cup \{\Delta\}$ with the metric d defined in (1).

2 Our Main Theorem

We define the pair $(\boldsymbol{m}, \boldsymbol{b}) \in Adm_n$ as *singular for proper cornerpoints* for φ if at least one proper cornerpoint of $\mathrm{Dgm}(\varphi_{(\boldsymbol{m},\boldsymbol{b})})$ has multiplicity strictly greater than 1. Otherwise, $(\boldsymbol{m}, \boldsymbol{b})$ is *regular for proper cornerpoints*. The concept of singularity and regularity would more generally also include the multiplicities of cornerpoints at infinity, but for our purposes here proper cornerpoints suffice. In the sequel, we will therefore use the terms "regular" and "singular" to refer to regularity and singularity with respect to proper cornerpoints. We denote by $Adm_n^*(\varphi)$ the set of regular pairs of φ. Moreover, φ is said to be *normal* if the set of singular admissible pairs for φ is discrete. We can prove the following result. Let I be the closed interval $[0, 1]$.

Theorem 2. *Let $\varphi : X \to \mathbb{R}^n$ be a normal filtering function. For every continuous path $\gamma : I \to Adm_n^*(\varphi)$ and every proper cornerpoint $p \in Dgm(\varphi_{\gamma(0)})$, there exists a continuous function $c : I \to \Delta^+ \cup \{\Delta\}$ such that $c(0) = p$ and $c(t) \in Dgm(\varphi_{\gamma(t)})$ for all $t \in I$. Furthermore, if there is no $t \in I$ such that $c(t) = \Delta$, c is the only such continuous function.*

Proof. For every $\delta \geq 0$, we set $I^\delta = [0, \delta]$ and consider the following property:

($*$) a continuous function $c^\delta : I^\delta \to \Delta^+ \cup \{\Delta\}$ exists, with $c^\delta(0) = p$ and $c^\delta(t) \in \mathrm{Dgm}(\varphi_{\gamma(t)})$ for all $t \in I^\delta$.

Define the set $A = \{\delta \in [0,1] : \text{ property } (*) \text{ holds}\}$. A is non-empty, since $0 \in A$. Set $\bar{\delta} = \sup A$. We will need to show that $\bar{\delta} \in A$. First let (δ_n) be a non-decreasing sequence of numbers of A converging to $\bar{\delta}$. Since $\delta_n \in A$, for each n there is a continuous function $c^{\delta_n} : I^{\delta_n} \to \Delta^+ \cup \{\Delta\}$ such that $c^{\delta_n}(0) = p$ and $c^{\delta_n}(t) \in \mathrm{Dgm}(\varphi_{\gamma(t)})$ for all $t \in I^{\delta_n}$. Moreover, we can assume that the following holds:

($**$) if $m \leq n$, the restriction of c^{δ_n} to the interval I^{δ_m} coincides with c^{δ_m}.

To clarify this, let us distinguish between two cases. In the first one, we have that $\Delta \notin c^{\delta_n}(I^{\delta_n})$, for any n. Then, if ($**$) failed, by the Stability Theorem 1 a real value $\bar{t} \in I^{\delta_m}$ should exist such that $\mathrm{Dgm}\left(\varphi_{\gamma(\bar{t})}\right)$ would have a proper cornerpoint of multiplicity strictly greater than 1, which is against our assumption that $\gamma(t) \in Adm_n^*(\varphi)$ for all $t \in I$. In order to show this, we can set $\bar{t} = \max\{t \in [0, \delta_m] : c^{\delta_m}(t) = c^{\delta_n}(t)\}$. The second case is when $\Delta \in c^{\delta_n}(I^{\delta_n})$ for some indices n. Let m be the first index at which this happens, and let $t' = \min\{t \in [0, \delta_m] : c^{\delta_m}(t) = \Delta\}$. Then, to ensure that ($**$) holds, for every $n \geq m$ we simply modify all the functions c^{δ_n} by setting $c^{\delta_n}(t) = \Delta$ for all $t \in [t', \delta_n]$.

Therefore, by ($**$) we can define a continuous function $\tilde{c}^{\bar{\delta}} : \left[0, \bar{\delta}\right) \to \Delta^+ \cup \{\Delta\}$ such that $\tilde{c}^{\bar{\delta}}(0) = p$ and $\tilde{c}^{\bar{\delta}}(t) \in \mathrm{Dgm}(\varphi_{\gamma(t)})$ for all $t \in \left[0, \bar{\delta}\right)$. However, to prove that $\bar{\delta} \in A$ we still need to show that $\tilde{c}^{\bar{\delta}}$ can be continuously extended to the point $\bar{\delta}$. The localization of cornerpoints ([8, Prop. 3.8]) implies that, possibly by extracting a convergent subsequence, we can assume that the limit $\lim_n c^{\delta_n}(\delta_n) = \lim_n \tilde{c}^{\bar{\delta}}(\delta_n)$ exists. Once more by the Stability Theorem 1, we have that $\lim_n \tilde{c}^{\bar{\delta}}(\delta_n) \in \mathrm{Dgm}(\varphi_{\gamma(\bar{\delta})})$. Now the function $\tilde{c}^{\bar{\delta}}$ can be extended to $\bar{\delta}$ by setting $\tilde{c}^{\bar{\delta}}(\bar{\delta}) = \lim_n \tilde{c}^{\bar{\delta}}(\delta_n)$, to show that $\bar{\delta} \in A$.

Last, we prove by contradiction that $\bar{\delta} = 1$. Let us suppose that $\bar{\delta} < 1$. If $c^{\bar{\delta}}(\bar{\delta}) = \Delta$, then $c^{\bar{\delta}}$ can be easily extended by setting $c^{\bar{\delta}}(t) = \Delta$ for all $t \in [\bar{\delta}, 1]$. Otherwise, by the Stability Theorem 1 and the fact that $\gamma(\bar{\delta})$ is regular, for any sufficiently small $\varepsilon > 0$ we can pick a real number $\eta > 0$ (small enough with respect to ε) such that there is only one proper cornerpoint $p'(t) \in \mathrm{Dgm}(\varphi_{\gamma(t)})$ with $d\left(p'(t), c^{\bar{\delta}}(\bar{\delta})\right) \leq \varepsilon$ for every t with $\bar{\delta} \leq t \leq \bar{\delta} + \eta$. By setting $c^{\bar{\delta}}(t) = p'(t)$ for every such t, we get a continuous path that extends $c^{\bar{\delta}}$ to the interval $[0, \bar{\delta} + \eta)$. In any case, we get a contradiction of our assumption that $\bar{\delta} = \sup A$.

We now show the uniqueness of c, assuming it does not reach Δ for any $t \in I$. Let $c, c' : I \to \Delta^+$ be two continuous paths such that $c(0) = c'(0) = p$, and $c(t), c'(t) \in \mathrm{Dgm}(\varphi_{\gamma(t)})$ for all $t \in I$. Denote by $\bar{t}$ the greatest value such that $c(t) = c'(t)$ for any t with $0 \leq t \leq \bar{t}$. By the Stability Theorem 1, if $\bar{t} < 1$ then $c(\bar{t})$ would be a proper cornerpoint of $\mathrm{Dgm}(\varphi_{\gamma(\bar{t})})$ having multiplicity strictly greater than 1. This is against our assumption that $\gamma(t) \in Adm_n^*(\varphi)$ for all $t \in I$. Therefore $\bar{t} = 1$, and our statement is proven.

Remark 1. In order to preserve the uniqueness of the function c even in the cases when it reaches Δ for a value $t \in I$, what could be done would be to restrict the domain of c to the subinterval $[0, \tilde{t}]$ such that $\tilde{t} = \sup\{t \in I : \Delta \notin c([0,t])\}$. Indeed, while there is no way to keep track of proper cornerpoints after they've reached Δ, it is also unnecessary to do so, for the simple reason that they then no longer correspond to extant topological properties. This is in contrast to singular admissible pairs, where the definition of c presents a true ambiguity.

3 An Example Illustrating Monodromy in Multidimensional Persistent Homology

Intuition might suggest that a natural correspondence exists between cornerpoints associated to different points in the parameter space Adm_n. Example 3 disproves this belief, showing that this correspondence depends on the path followed. Indeed, when following the persistence diagrams $\mathrm{Dgm}(\varphi_{\gamma(t)})$ along a loop $\gamma : I \to Adm_n^*(\varphi)$, that is, along a continuous path such that $\gamma(0) = \gamma(1)$, nontrivial monodromies may occur if γ is not homotopic to a constant path in $Adm_n^*(\varphi)$. In other words, while $\mathrm{Dgm}(\varphi_{\gamma(0)}) = \mathrm{Dgm}(\varphi_{\gamma(1)})$, there may be a $p \in \mathrm{Dgm}(\varphi_{\gamma(0)})$ such that $c(1) \neq p$.

Example 3 (Nontrivial monodromy). Consider the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined on the plane in the following way: $\varphi_1(x, y) = x$, and

$$\varphi_2(x,y) = \begin{cases} -x & \text{if } y = 0 \\ -x+1 & \text{if } y = 1 \\ -2x & \text{if } y = 2 \\ -2x+\frac{5}{4} & \text{if } y = 3 \end{cases},$$

$\varphi_2(x, y)$ then being extended linearly for every x on the segment joining $(x, 0)$ with $(x, 1)$, $(x, 1)$ with $(x, 2)$, and $(x, 2)$ to $(x, 3)$. On the half-lines $\{(x, y) \in \mathbb{R}^2 : y < 0\}$ and $\{(x, y) \in \mathbb{R}^2 : y > 3\}$, φ_2 is then being taken with constant slope -1 in the variable y. The function φ_2 is shown plotted in Figure 3. As written the domain of φ is not a compact space, but we can easily obtain a compact domain by considering a suitable subset of $\mathbb{R}^2$.

In the case of filtering functions with values in $\mathbb{R}^2$, Adm_2 is the set of pairs $(\boldsymbol{m}, \boldsymbol{b})$ where $\boldsymbol{m} = (a, 1-a)$ with $a \in (0, 1)$, and $\boldsymbol{b} = (\beta, -\beta)$ with $\beta \in \mathbb{R}$. We may therefore represent an admissible pair as $(a, \beta) \in (0, 1) \times \mathbb{R}$. In Figure 4 we can see a "flattened" version of the graph of φ, as if projected on its codomain. It is easily seen that φ admits only one singular admissible pair, that being $(1/4, 0)$. Consider a loop γ in $Adm_2^*(\varphi)$ moving around this singular pair; for example, $\gamma(0) = \gamma(1) = (1/4, -\varepsilon)$, $\gamma(1/4) = (1/4 - \varepsilon, \varepsilon)$, $\gamma(1/2) = (1/4, \varepsilon)$, and $\gamma(3/4) = (1/4+\varepsilon, -\varepsilon)$, with γ linear in $Adm_2^*(\varphi)$ between these points. The 0th order persistence diagram $\mathrm{Dgm}(\varphi_{\gamma(t)})$ is shown for these values of t in Figure 5;

in each case it has two proper cornerpoints, and this property is in fact true for every $t \in I$. If the cornerpoints of $\mathrm{Dgm}(\varphi_{\gamma(0)})$ are denoted p and q, we obtain that $c_p(1) = q$ and $c_q(1) = p$, c_p and c_q being respectively the functions tracking the cornerpoints p and q.

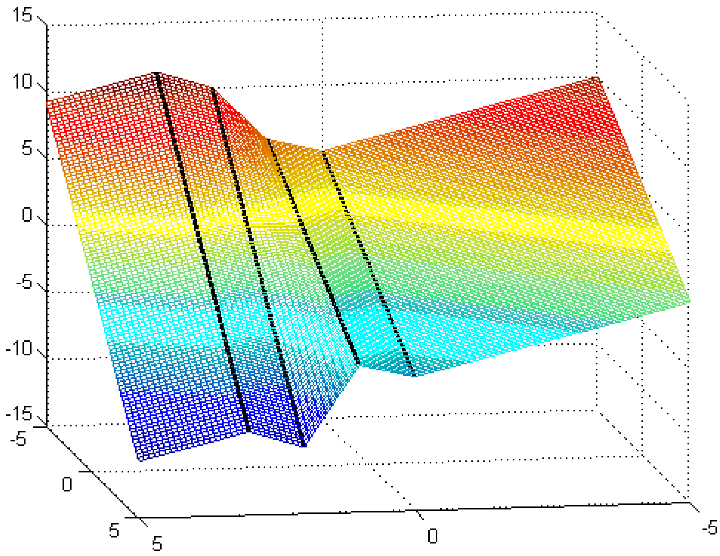


Fig. 3. Function φ_2 of Example 3. Depth is x, width is y

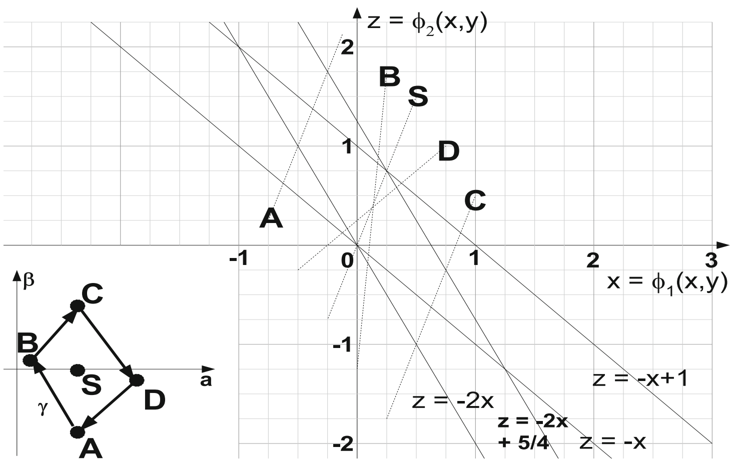


Fig. 4. Codomain of φ for Example 3. Full lines are birth or death points of 0-cycles; dotted lines are $r_{(\boldsymbol{m},\boldsymbol{b})}$ for a few values in Adm_2. The loop $\gamma : I \to Adm_2^*(\varphi)$ is also shown.

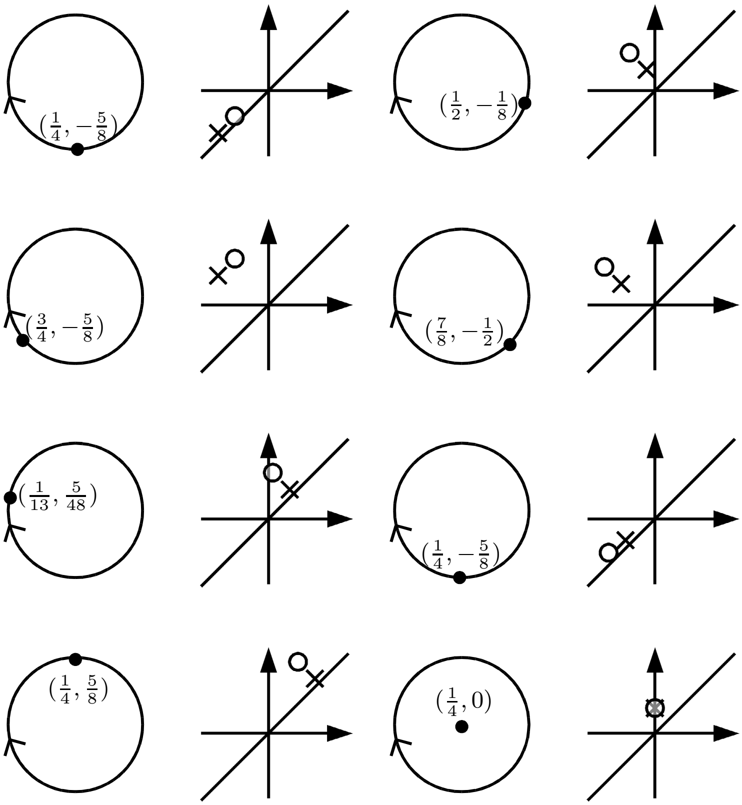


Fig. 5. Schematic of the evolution of the 0th order persistence diagram $\mathrm{Dgm}(\varphi_{\gamma(t)})$ for Example 3 as t goes from 0 to 1. Circle and cross denote the proper cornerpoints. Persistence diagrams are not to scale, but respective positions of cornerpoints are preserved. Similarly, path in (a, β)-space does not actually follow a geometric circle, but is a simple closed curve.

4 Conclusions

In this paper we have shown that monodromy can appear in multidimensional topological persistence, and illustrated how it can be managed by specifying the paths to follow in the parameter space Adm_n of all admissible pairs. This new phenomenon is expected to reveal itself important in shape comparison, opening the way to the definition of new distances between persistence diagrams based on the idea of matchings continuously dependent on the parameters $\boldsymbol{m}$, $\boldsymbol{b}$. The next step in our research will be to study possible methods to construct these new metrics and their application to problems in shape analysis.

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