Planar Lombardi Drawings of Outerpaths

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1 Introduction

A *Lombardi drawing* of a graph is a drawing where edges are represented by circular arcs that meet at each vertex v with perfect angular resolution $360° / \deg(v)$ [3]. It is known that Lombardi drawings do not always exist, and in particular, that planar Lombardi drawings of planar graphs do not always exist [1], even when the embedding is not fixed. Existence of planar Lombardi drawings is known for restricted classes of graphs, such as subcubic planar graphs [4], trees [2], Halin graphs and some very symmetric planar graphs [3]. On the other hand, all 2-degenerate graphs, including all outerplanar graphs, have Lombardi drawings, but not necessarily planar ones [3]. One question that was left open is whether outerplanar graphs always have *planar* Lombardi drawings or not.

In this note, we report that the answer is "yes" for a more restricted subclass: the *outerpaths*, i.e., outerplanar graphs whose weak dual is a path. We sketch an algorithm that produces an outerplanar Lombardi drawing of any outerpath, in linear time.

2 Algorithm Sketch

Let G be a triangulated outerpath; it can be shown that this is no limitation. We define the *spine* of G to be the path connecting all vertices of degree greater than 3. We root the spine at one of its endpoints, v_1, and denote the remaining spine vertices as $v_2, \ldots, v_s$ along the path. We define the *hull* of G to be the cycle bounding the outer face. Finally, we define the *petals* of G to be the remaining edges, grouped into connected components called *flowers*. In Fig. 1, the spine is drawn in red, the hull in blue, and the petals in yellow. We define the (spine/hull/petal) *stubs* of a vertex v as the $\chi_v := \deg(v)$ equally spaced tangent vectors that describe the orientations of all incident edges of v. It is known that for two fixed vertices u and v with a common neighbor w, all positions for w yielding the same angle θ_w between the edges uw and vw in w lie on a so-called *placement circle* through u and v [3].

We sketch the main ideas of our two-step drawing algorithm. In the first step we draw the subgraph induced by the spine vertices in an x-monotone fashion. Assume initially that all spine vertices have degree at least 6. We place v_1 at $(0,0)$ and rotate it so that the vertical line $\ell_1 : x = 0$ bisects the angle between its two hull stubs. Subsequently, we place v_i on the vertical line ℓ_i at distance 1 from v_{i-1} at the unique position where the next spine edge, i.e., a circular arc from v_{i-1} tangent to the outgoing spine stub of v_{i-1}, intersects ℓ_i at an angle of $\pm 1.5 \cdot 360° / \chi_{v_i}$ depending on whether the flower of v_i is to be placed above or below the spine. Interestingly, the resulting spine drawing can always be completed in a planar way, except when there is a vertex v of degree $\chi_v = 5$

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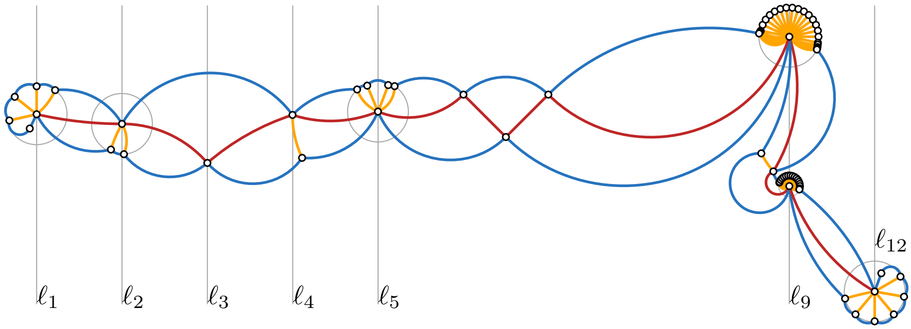

Fig. 1. A Lombardi drawing of an outerpath with spine degrees 7, 6, 4, 5, 8, 4, 4, 4, 30, 5, 30, and 9, which exhibits the different cases considered by our algorithm

whose spine neighbors satisfy $1/\chi_{v_{i-1}} + 1/\chi_{v_{i+1}} \leq 1/15$. When such vertices occur, we switch to a more complicated vertical placement scheme to avoid edge crossings in the second step. Due to the missing petals the procedure for vertices of degree 4 is also slightly different.

In the second step we draw the flowers and the remaining hull edges. Each flower is drawn inside a circle of radius ε centered at its spine vertex, where ε is chosen small enough so that no flower can intersect any non-incident part of the drawing. Assume that a spine vertex v has $k \geq 2$ incident petals and that C is a circle centered at v. We place the leftmost and the rightmost petal vertices of v at the intersection points of their placement circles for an angle $\theta = 120°$ with C that lie on the correct side of the spine. If two or more petals remain, we slightly decrease the radius of C and recurse with the two outermost remaining petals. If a single petal remains, its placement circles define a unique position; this case also applies for $k = 1$. If no petal remains, generally one of the two previous petals must be moved to the unique suitable position that creates three incident angles of 120°. We choose the radius of the first circle C as $\frac{5}{6}\varepsilon$ so that the hull edges do not extend beyond distance ε from v.

Theorem 1. *Every outerpath has an outerplanar Lombardi drawing, and this drawing can be constructed in linear time.*

References

1. Duncan, C.A., Eppstein, D., Goodrich, M.T., Kobourov, S.G., Löffler, M.: Planar and Poly-arc Lombardi Drawings. In: van Kreveld, M., Speckmann, B. (eds.) GD 2011. LNCS, vol. 7034, pp. 308–319. Springer, Heidelberg (2012)
2. Duncan, C.A., Eppstein, D., Goodrich, M.T., Kobourov, S.G., Nöllenburg, M.: Drawing Trees with Perfect Angular Resolution and Polynomial Area. In: Brandes, U., Cornelsen, S. (eds.) GD 2010. LNCS, vol. 6502, pp. 183–194. Springer, Heidelberg (2011)
3. Duncan, C.A., Eppstein, D., Goodrich, M.T., Kobourov, S.G., Nöllenburg, M.: Lombardi drawings of graphs. J. Graph Algorithms and Applications 16(1), 85–108 (2012)
4. Eppstein, D.: Planar Lombardi Drawings for Subcubic Graphs. In: Didimo, W., Patrignani, M. (eds.) GD 2012. LNCS, vol. 7704, pp. 126–137. Springer, Heidelberg (2013) To appear arXiv:1206.6142