

Extension of Barreto-Voloch Root Extraction Method

Zhengjun Cao and Xiao Fan

Department of Mathematics, Shanghai University, Shanghai, China
caozhj@yahoo.cn

Abstract. Root extraction is a classical problem in computers algebra. It plays an essential role in cryptosystems based on elliptic curves. In 2006, Barreto and Voloch proposed an algorithm to compute r th roots in \mathbb{F}_{q^m} for certain choices of m and q . If $r \parallel q-1$ and $(m, r) = 1$, they proved that the complexity of their method is $\tilde{O}(r(\log m + \log \log q)m \log q)$. In this paper, we extend the Barreto-Voloch algorithm to the general case that $r \parallel q^m - 1$, without the restrictions $r \parallel q - 1$ and $(m, r) = 1$. We also specify the conditions that the Barreto-Voloch algorithm can be preferably applied.

Keywords: Barreto-Voloch algorithm, Adleman-Manders-Miller algorithm.

1 Introduction

Consider the problem to find a solution to $X^r = \delta$ in \mathbb{F}_{q^m} , where $q = p^d$ for some prime p and some integer $d > 0$. Clearly, it suffices to consider the following two cases:

$$(1) (r, q^m - 1) = 1, \quad (2) r \mid q^m - 1$$

Root extraction is a classical problem in computational algebra and number theory. It plays an essential role in cryptosystems based on elliptic curves. The typical applications of root extraction are point compression in elliptic curves and operation of hashing onto elliptic curves [3,4,9].

Adleman, Manders and Miller [1] proposed a method to solve the problem, which extends Tonelli-Shanks [7,10] square root algorithm. The basic idea of Adleman-Manders-Miller r th root extraction in \mathbb{F}_q can be described as follows. If $r \mid q - 1$, we write $p - 1$ in the form $r^t \cdot s$, where $(s, r) = 1$. Given a r th residue δ , we have $(\delta^s)^{r^{t-1}} = 1$. Since $(s, r) = 1$, it is easy to find the least nonnegative integer α such that $s \mid r\alpha - 1$. Hence, $(\delta^{r\alpha-1})^{r^{t-1}} = 1$. If $t - 1 = 0$, then δ^α is a r th root of δ . From now on, we assume that $t \geq 2$. Given a r th non-residue $\rho \in \mathbb{F}_q$, we have

$$(\rho^s)^{i \cdot r^{t-1}} \neq (\rho^s)^{j \cdot r^{t-1}} \quad \text{where } i \neq j, \quad i, j \in \{0, 1, \dots, r-1\}$$

Set $K_i = (\rho^s)^{i \cdot r^{t-1}}$ and $\mathbb{K} = \{K_0, K_1, \dots, K_{r-1}\}$. It is easy to find that all K_i satisfy $X^r = 1$. Since $\left((\delta^{r\alpha-1})^{r^{t-2}} \right)^r = 1$, there is a unique $j_1 \in \{0, 1, \dots, r-1\}$ such that $(\delta^{r\alpha-1})^{r^{t-2}} = K_{r-j_1}$ (where $K_r = K_0$). Hence, $(\delta^{r\alpha-1})^{r^{t-2}} K_{j_1} = 1$. That is

$$(\delta^{r\alpha-1})^{r^{t-2}} (\rho^s)^{j_1 \cdot r^{t-1}} = 1$$

Likewise, there is a unique $j_2 \in \{0, 1, \dots, r-1\}$ such that

$$(\delta^{r\alpha-1})^{r^{t-3}} (\rho^s)^{j_1 \cdot r^{t-2}} (\rho^s)^{j_2 \cdot r^{t-1}} = 1$$

Consequently, we obtain j_1, \dots, j_{t-1} such that

$$(\delta^{r\alpha-1}) (\rho^s)^{j_1 \cdot r} (\rho^s)^{j_2 \cdot r^2} \dots (\rho^s)^{j_{t-1} \cdot r^{t-1}} = 1$$

Thus, we have

$$(\delta^\alpha)^r \left((\rho^s)^{j_1 + j_2 \cdot r + \dots + j_{t-1} \cdot r^{t-2}} \right)^r = \delta$$

It means that $\delta^\alpha (\rho^s)^{j_1 + j_2 \cdot r + \dots + j_{t-1} \cdot r^{t-2}}$ is a r th root of δ . The complexity of Adleman-Manders-Miller r th root extraction algorithm is $\mathcal{O}(\log^4 q + r \log^3 q)$. Notice that the algorithm can not run in polynomial time if r is sufficiently large.

In 2006, Barreto and Voloch [2] proposed an algorithm to compute r th roots in \mathbb{F}_{q^m} for certain choices of m and q . If $r \parallel q-1$ and $(m, r) = 1$, where the notation $a^b \parallel c$ means that a^b is the highest power of a dividing c , they proved that the complexity of their method is $\tilde{\mathcal{O}}(r(\log m + \log \log q)m \log q)$.

Our contributions. We extend the Barreto-Voloch root extraction method to the general case that $r \parallel q^m - 1$, without the restrictions $r \parallel q - 1$ and $(m, r) = 1$. We also specify the conditions that the Barreto-Voloch algorithm can be preferably applied.

2 Barreto-Voloch Method

Barreto-Voloch method takes advantage of the periodic structure of v written in base q to compute r th roots in \mathbb{F}_{q^m} , where $v = r^{-1} \pmod{q^m - 1}$ if $(r, q^m - 1) = 1$. This advantage is based on the following fact [2]:

Fact 1. *Let \mathbb{F}_{q^m} be a finite field of characteristic p and let s be a power of p . Define the map*

$$\phi_n : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m}, y \mapsto y^{1+s+\dots+s^n} \text{ for } n \in \mathbb{N}^*$$

We can compute $\phi_n(y)$ with $\mathcal{O}(\log n)$ multiplications and raisings to powers of p .

Notice that raising to powers of p has negligible cost, if we use a normal basis for $\mathbb{F}_{q^m}/\mathbb{F}_q$. Since it only requires $\mathcal{O}(\log n)$ multiplications and raisings to powers of p to compute $y^{1+s+\dots+s^n}$, where p is the characteristic of \mathbb{F}_{q^m} and s is a power

of p , their method becomes more efficient for certain choices of m and q . They obtained the following results [2].

Lemma 1. *Given q and r with $(q(q-1), r) = 1$, let $k > 1$ be the order of q modulo r . For any $m > 0$, $(m, k) = 1$, let $u, 1 \leq u < r$ satisfy $u(q^m - 1) \equiv -1 \pmod{r}$ and $v = \lfloor q^m u / r \rfloor$. Then $rv \equiv 1 \pmod{q^m - 1}$. In addition, $v = a + b \sum_{j=0}^{n-1} q^{jk}$, $a, b < q^{2k}$, $n = \lfloor m/k \rfloor$.*

Theorem 1. *Let q be a prime power, let $r > 1$ be such that $(q(q-1), r) = 1$ and let $k > 1$ be the order of q modulo r . For any $m > 0$, $(m, k) = 1$, the complexity of taking r th roots in \mathbb{F}_{q^m} is $\tilde{O}((\log m + r \log q)m \log q)$.*

Lemma 2. *Given q and r with $r \mid (q-1)$ and $((q-1)/r, r) = 1$, for any $m > 0$, $(m, r) = 1$, let $u, 1 \leq u < r$ satisfy $u(q^m - 1)/r \equiv -1 \pmod{r}$ and $v = \lceil q^m u / r \rceil$. Then $rv \equiv 1 \pmod{(q^m - 1)/r^2}$. In addition, $v = a + b \sum_{j=0}^{n-1} q^{jr}$, $a, b < q^{2r}$, $n = \lceil m/r \rceil$.*

Theorem 2. *Let q be a prime power and let $r > 1$ be such that $r \mid (q-1)$ and $((q-1)/r, r) = 1$. For any $m > 0$, $(m, r) = 1$, given $x \in \mathbb{F}_{q^m}$ one can compute the r th root of x in \mathbb{F}_{q^m} , or show it does not exist, in $\tilde{O}(r(\log m + \log \log q)m \log q)$ steps.*

3 Analysis of Barreto-Voloch Method

3.1 On the Conditions of Barreto-Voloch Method

In Theorem 1, it requires that

$$(q(q-1), r) = 1 \text{ and } (m, k) = 1$$

where $k > 1$ is the order of q modulo r . These conditions imply $(q^m - 1, r) = 1$. But these are not necessary to the general case. Likewise, in Theorem 2, it requires that

$$r \parallel q - 1 \text{ and } (m, r) = 1$$

These imply $r \parallel q^m - 1$. But these are not necessary, too. We will remove the restrictions and investigate the following cases:

$$(1) (r, p^m - 1) = 1; \quad (2) r \parallel p^m - 1.$$

where p is a prime. As for the general case, $p^m - 1 = r^\alpha s$, $\alpha \geq 2$, $(r, s) = 1$, we refer to [1].

3.2 On the Technique of Periodic Structure

As we mentioned before, Barreto-Voloch method takes advantage of the periodic structure of v written in base q . Precisely, in Lemma 1

$$v = a + b \sum_{j=0}^{n-1} q^{jk}, a, b < q^{2k}, n = \lfloor m/k \rfloor \tag{1}$$

where $k > 1$ is the order of q modulo r . From the expression, we know it requires that $n = \lfloor m/k \rfloor \geq 1$. It is easy to find that *the advantage of Barreto-Voloch method*

due to the periodic expansion in base q requires that m is much greater than k . That is, the length of such periodic expansion, n , should be as large as possible.

Since raising to a power of p is a linear bijection in characteristic p , the complexity of such operation is no larger than that of multiplication, namely, $\tilde{O}(m \log p)$ using FFT techniques [5,6,8]. In light of that $q = p^d$ for some prime p , it is better to write v as

$$v = a' + b' \sum_{j=0}^{n'-1} p^{jk'}, a', b' < p^{2k'}, n' = \lfloor md/k' \rfloor \tag{2}$$

where k' is the order of p modulo r . That is, the periodic expansion in base p could produce a large expansion length, instead of the original periodic expansion in base q . This claim is directly based on the following fact

$$n' = \lfloor md/k' \rfloor \geq \lfloor md/kd \rfloor = n \tag{3}$$

(This is because $k' | kd$. See the definitions of k, k' .)

4 Extension of Barreto-Voloch Method

4.1 Taking r th Roots When r is Invertible

We first discuss the problem to take r th roots over \mathbb{F}_{p^m} if $(r, p^m - 1) = 1$, where p is a prime.

Lemma 3. Suppose that $(p^m - 1, r) = 1$. Let k be the order of p modulo r . Let $u, 1 \leq u < r$ satisfy $u(p^m - 1) \equiv -1 \pmod{r}$. Then $rv \equiv 1 \pmod{p^m - 1}$, where $v = \lfloor p^m u/r \rfloor$. In addition, if $m > k$, then $v = a + b \sum_{j=0}^{n-1} p^{jk}, a, b < p^{2k}, n = \lfloor m/k \rfloor$.

Proof. Since $u(p^m - 1) \equiv -1 \pmod{r}$ and $1 \leq u < r$, we have $p^m u/r = \lfloor p^m u/r \rfloor + (u - 1)/r$ and $r \lfloor p^m u/r \rfloor \equiv 1 \pmod{p^m - 1}$. Let $z = u(p^k - 1)/r$. Then z is an integer and $z < p^k - 1$. Hence, $p^m u/r = p^m z / (p^k - 1)$. If $m > k$, then we have the following expansion

$$p^m z / (p^k - 1) = p^{m-k} z \sum_{j=0}^{\infty} p^{-jk} = p^{m-nk} z \sum_{j=0}^{n-1} p^{jk} + p^{m-k} z \sum_n^{\infty} p^{-jk}$$

Take $a = \lfloor p^{m-k} z \sum_n^{\infty} p^{-jk} \rfloor, b = p^{m-nk} z$. This completes the proof. □

Theorem 3. Suppose that $(p^m - 1, r) = 1$. Let k be the order of p modulo r . If $m > k$, then the complexity of taking r th roots of δ in \mathbb{F}_{p^m} is $\tilde{O}((\log m + k \log p)m \log p)$.

Proof. Given $\delta \in \mathbb{F}_{p^m}$, clearly, $\delta^{r^{-1}}$ is a root of $X^r = \delta$ if $(p^m - 1, r) = 1$, where r^{-1} is the inverse of r modulo $p^m - 1$.

By Lemma 3, if $m > k$, then $r^{-1} = a + b \sum_{j=0}^{n-1} p^{jk} \pmod{p^m - 1}, a, b < p^{2k}, n = \lfloor m/k \rfloor$. Raising to the power $\sum_{j=0}^{n-1} p^{jk}$ takes $\mathcal{O}(\log n)$ multiplications

and raisings to powers of p . The raising to the power a takes $\mathcal{O}(k \log p)$ multiplications due to the bound on the exponent. So does the raising to the power b . The total computation cost is therefore $\mathcal{O}(\log m + k \log p)$ operations of complexity $\tilde{\mathcal{O}}(m \log p)$ (if directly using the form $r^{-1} = \frac{u(p^m-1)+1}{r}$, it takes time $\tilde{\mathcal{O}}(m^2 \log^2 p)$). This completes the proof. \square

4.2 Taking r th Roots When r is Not Invertible

We now discuss the problem to take r th roots over \mathbb{F}_{p^m} if $r \parallel p^m - 1$, where p is a prime.

Lemma 4. *Suppose that $r \parallel p^m - 1$. Let k be the order of p modulo r . Let $u, 1 \leq u < r$ satisfy $u(p^m - 1)/r \equiv -1 \pmod{r}$ and $v = \lceil p^m u/r^2 \rceil$. Then $rv \equiv 1 \pmod{(p^m - 1)/r}$. In addition, if $m > kr$, then $v = a + b \sum_{j=0}^{n-1} p^{jkr}$, $a, b < p^{2kr}$, $n = \lfloor m/kr \rfloor$.*

Proof. Since $u(p^m - 1)/r \equiv -1 \pmod{r}$ and $1 \leq u < r$, we have $p^m u/r^2 = \lceil p^m u/r^2 \rceil + (u-r)/r^2$ and $r \lceil p^m u/r^2 \rceil \equiv 1 \pmod{(p^m - 1)/r}$. Let $z = u(p^{kr} - 1)/r^2$. Then z is an integer and $z < p^{kr} - 1$. Hence, $p^m u/r^2 = p^m z / (p^{kr} - 1)$. If $m > kr$, then we have the following expansion

$$p^m z / (p^{kr} - 1) = p^{m-kr} z \sum_{j=0}^{\infty} p^{-jkr} = p^{m-nkr} z \sum_{j=0}^{n-1} p^{jkr} + p^{m-kr} z \sum_n^{\infty} p^{-jkr}$$

Take $a = \lfloor p^{m-kr} z \sum_n^{\infty} p^{-jkr} \rfloor, b = p^{m-nkr} z$. This completes the proof. \square

Theorem 4. *Suppose that $r \parallel p^m - 1$. Let k be the order of p modulo r . If $m > kr$, then one can compute the r th root of δ in \mathbb{F}_{p^m} , or show it does not exist, in $\tilde{\mathcal{O}}((\log m + kr \log p)m \log p)$ steps.*

Proof. Given $\delta \in \mathbb{F}_{p^m}$, we have $\delta^{p^m-1} = 1$. If $r \parallel p^m - 1$ and $\delta^{(p^m-1)/r} = 1$, then there exists an integer v such that $\frac{p^m-1}{r} \mid vr - 1$ and $(\delta^v)^r = \delta$. Hence, it suffices to compute the inverse of r modulo $\frac{p^m-1}{r}$.

By Lemma 4, if $m > kr$, $r^{-1} \equiv v = a + b \sum_{j=0}^{n-1} p^{jkr} \pmod{(p^m - 1)/r}$, $a, b < p^{2kr}, n = \lfloor m/kr \rfloor$. Since raising to the power $\sum_{j=0}^{n-1} p^{jkr}$ takes $\mathcal{O}(\log n)$ multiplications and raisings to powers of p . Raising to the power a takes $\mathcal{O}(kr \log p)$ multiplications due to the bound on the exponent. So does raising to the power b . The cost of raising to v is therefore $\mathcal{O}(\log m + kr \log p)$ operations of complexity $\tilde{\mathcal{O}}(m \log p)$. To check that $\rho = \delta^v$ is a correct root, we compute ρ^r with cost $\tilde{\mathcal{O}}(m \log r \log p)$. If δ is a r th power, then $\rho^r = \delta$, otherwise ρ^r is not equal to δ . The total computation cost is therefore $\tilde{\mathcal{O}}((\log m + kr \log p)m \log p)$ (if directly using the form $r^{-1} = \frac{u(p^m-1)+r}{r^2}$, it takes time $\tilde{\mathcal{O}}(m^2 \log^2 p)$). This completes the proof. \square

5 Conclusion

In this paper, we analyze and extend the Barreto-Voloch method to compute r th roots over finite fields. We specify the conditions that the Barreto-Voloch algorithm can be preferably applied. We also give a formal complexity analysis of the method.

Acknowledgements. We thank the anonymous referees' for their detailed suggestions. This work is supported by the National Natural Science Foundation of China (Project 60873227), and the Key Disciplines of Shanghai Municipality (S30104).

References

1. Adleman, L., Manders, K., Miller, G.: On Taking Roots in Finite Fields. In: Proceedings of the 18th IEEE Symposium on Foundations of Computer Science, pp. 175–177. IEEE Press, New York (1977)
2. Barreto, P., Voloch, J.: Efficient Computation of Roots in Finite Fields. *Designs, Codes and Cryptography* 39, 275–280 (2006)
3. Boneh, D., Boyen, X., Shacham, H.: Short Group Signatures. In: Franklin, M. (ed.) CRYPTO 2004. LNCS, vol. 3152, pp. 41–55. Springer, Heidelberg (2004)
4. Boneh, D., Franklin, M.: Identity-based Encryption from the Weil Pairing. *SIAM J. Computing* 32(3), 586–615 (2003)
5. Gao, S., Gathen, J., Panario, D., Shoup, V.: Algorithms for Exponentiation in Finite Fields. *J. Symbolic Computation* 29, 879–889 (2000)
6. Gathen, J., Gerhard, J.: *Modern Computer Algebra*, 2nd edn. Cambridge University Press (2003)
7. Shanks, D.: Five Number-theoretic Algorithms. In: Proc. 2nd Manitoba Conf., pp. 51–70. *Numer. Math.* (1972)
8. Shoup, V.: *A Computational Introduction to Number Theory and Algebra*. Cambridge University Press (2005)
9. Smart, N.: An Identity Based Authenticated Key Agreement Protocol Based on the Weil Pairing. *Electronics Letters* 38, 630–632 (2002)
10. Tonelli, A.: Bemerkung über die Auflösung quadratischer Congruenzen. *Nachrichten der Akademie der Wissenschaften in Göttingen*, 344–346 (1891)