Convergence of Binomial-Based Derivative Estimation for C^2 Noisy Discretized Curves

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Abstract. We present new convergence results for the integer-only binomial masks method to estimate derivatives of digitized functions. The results work for C^2 functions and as a consequence we obtain a complete uniform convergence result for parametrized C^2 curves.

Introduction

In the framework of image and signal processing, as well as shape analysis, a common problem is to estimate derivatives of functions or tangents and curvatures of curves and surfaces, when only some (possibly noisy) sampling of the function or curve is available from acquisition. This problem has been investigated through finite difference methods, scale-space ([7],[4]), and discrete geometry ([5],[6]).

In [8], a new approach to derivative estimation from discretized data is proposed. As in scale-space approaches, this approach is based on simple computations of "convolutions". However, unlike scale-space methods, this approach is oriented towards integer-only models and algorithms, and is based on a discrete approach to analysis on $\mathbb{Z}$. Unlike existing approaches from discrete geometry, our approach is not based on discrete line reconstruction, which involves uneasy arithmetical calculations and complicated algorithms. Implementation of the binomial approach is straightforward.

As far as the speed of convergence is concerned, in [8] the method for tangents is proved uniform worst-case $O(h^{2/3})$ for C^3 functions, where h is the size of the pixel, while that of [6] is uniform $O(h^{1/3})$ and $O(h^{2/3})$ in average. Moreover, the estimator of [8] allows some (uniformly bounded) noise. Furthermore, the method of [8] allows to have a convergent estimation of higher order derivatives, and in particular a uniform $O(h^{4/9})$ estimation for the curvature of a generic curve.

In this paper, we prove a new convergence result which works for C^2 functions. Moreover, the upper bound for the error is the same as in [8] for small mask size and better for very large mask size.

To deal with parametrized curves in $\mathbb{Z}^2$, we introduced in [8] a new notion of pixel-length parametrization which solves the problem of correspondence between parametrizations of discrete and continuous curves, which arises from the

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non isotropic character of $\mathbb{Z}^2$. However, the reparametrized curve is only C^2 so that [8] didn't contain a full uniform convergence result for parametrized curves. This problem is solved in this paper using our convergence results for C^2 functions, and we provide a complete uniform convergence result for parametrized curves.

The paper is organize as follows. First we find a definition of a uniform noise model and an error model, as well as statements of our main convergence results for real functions. For the sake of comparison, some results from [8] are quoted. Then we find a series of lemmas which outline the proofs. We also give hints as to how to reduce the computational complexity. Finally, we state similar results for parametrized curves. A couple of experiments are presented here. We plan to submit an extended version including all proofs and experiments soon.

1 Estimation of Derivatives for Real Function

Functions for which domain and range are 1−dimensional, without any assumption on the nature of these sets, are called *real functions*. We call *discrete function* a function from $\mathbb{Z}$ to $\mathbb{Z}$.

First we establish a relationship between a continuous function and its discretization. Let $f : R \longrightarrow R$ be a real function and let $\Gamma : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a discrete function. Let h be the discretization step (i.e. the size of a pixel). We introduce a (possibly noisy) discretization of f:

Definition 1. *The function Γ is a discretization of f with error ϵ with discretization step h on the interval $[a; b]$ if for any integer i such that $a \leq ih \leq b$ we have:*

$$h\Gamma(i) = f(ih) + \epsilon_h(i)$$

We consider the following particular cases :

- Rounded case: $|\epsilon_h(i)| \leq \frac{1}{2}h$ which is equivalent to $\Gamma(i) = \left[\frac{f(ih)}{h}\right]$
- Floor case: $0 \leq \epsilon_h(i) \leq h$ which is equivalent to $\Gamma(i) = \lfloor\frac{f(ih)}{h}\rfloor$
- Uniform Noise case: $0 \leq |\epsilon_h(i)| \leq Kh^\alpha$ with $0 < \alpha \leq 1$ and K a positive constant. Note that the round case and the floor case are particular cases of uniform noise with $\alpha = 1$ (see Figure 1).

Definition 2. *The* discrete derivative *of a sequence u with mask size m, denoted by $\Delta_{2m-1} \star u$, is defined by:*

$$(\Delta_{2m-1} \star u)(n) = \frac{1}{2^{2m-1}} \left(\sum_{i=-m+1}^{i=m} \binom{2m-1}{m-1+i} (u(n+i) - u(n-1+i)) \right)$$

In order to show that the discrete derivative of a discretized function provides an estimate for the continuous derivative of the real function, we would like to evaluate the difference between $(\Delta_{2m-1} \star \Gamma)(n)$ and $f'(nh)$.

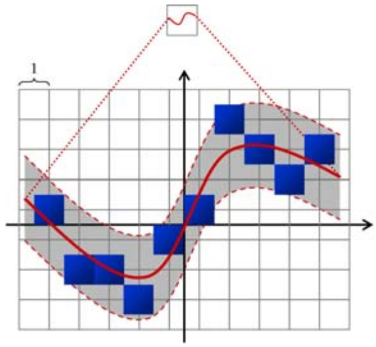

Fig. 1. The uniform noise model

Theorem 1 ([8]). *Suppose that* $f : R \longrightarrow R$ *is a* C^3 *function and* $f^{(3)}$ *is bounded,* $\alpha \in]0, 1]$, $K \in \mathbb{R}_+^*$ *and* $h \in \mathbb{R}_+^*$. *Suppose* $\Gamma : Z \longrightarrow Z$ *is such that* $|h\Gamma(i) - f(hi)| \leq Kh^\alpha$ *(uniform noise case). Then for* $m = \lfloor h^{2(\alpha-3)/3} \rfloor$, *we have* $|(\Delta_{2m-1} \star \Gamma)(n) - f'(nh)| \in O(h^{2\alpha/3})$

The proof of this theorem is based on the following more general upper bound:

Theorem 2. *Under the assumptions of Theorem 1, for some constant* K' *and* m *sufficiently large,*

$$|\Delta_{2m-1} \star \Gamma(n) - f'(nh)| \leq \frac{h^2 m}{4} \|f^{(3)}\|_\infty + \frac{K' h^{\alpha-1}}{\sqrt{m}}$$

In this paper, we provide the following new result:

Theorem 3. *Suppose that* f *is a* C^2 *function and* $f^{(2)}$ *is bounded. Suppose* $\Gamma : Z \longrightarrow Z$ *is such that* $|h\Gamma(i) - f(hi)| \leq Kh^\alpha$ *(uniform noise case). Then if* $m = \lfloor h^{(\alpha-2)/1,01} \rfloor$ *we have*
$|(\Delta_{2m-1} \star \Gamma)(n) - f'(nh)| \in O(h^{(0,51(\alpha-0,01)/1,01})$.

This result is interesting mainly because it works for C^2 functions, which allows us to definitely solve the case of parametrized curves in Section 3. The proof of this theorem is based on the following new more general upper bound:

Theorem 4. *Under the assumptions of Theorem 3, for some constant* K' *and* m *sufficiently large,*

$$|\Delta_{2m-1} \star \Gamma(n) - f'(nh)| \leq hm^{0,51} \|f^{(2)}\|_\infty + \frac{K' h^{\alpha-1}}{\sqrt{m}}$$

To compare our new Theorem 4 with previous Theorem 2, besides the regularity assumption, note that

– For a sufficiently small mask size m relative to h (namely $m \leq \min(h^{2(\alpha-3)/3}, h^{(\alpha-2)/1.01}))$, the upper bounds are the same because they have the same dominant terms.

– For a sufficiently large mask size m relative to h (namely $m \gg h^{-\frac{1}{0.49}}$) we have $hm^{0.51} \ll h^2 m$ and the (dominant term of the) upper bound of Theorem 4 is strictly better than the one given by Theorem 2.

In order to study the error $(\Delta_{2m-1} \star \Gamma)(n) - f'(nh)$, we decompose it into two errors:

– The real approximation error $\left(\Delta_{2m-1} \star \frac{f(ih)}{h}\right)(n) - f'(nh)$
– The input data error $\left(\Delta_{2m-1} \star \frac{\epsilon_h(i)}{h}\right)(n)$

1.1 Upper Bound for the Real Approximation Error

Here are the bounding results of this subsection. The second one is new:

Lemma 1. *Under the assumptions of Theorem 2*

$$\left|\left(\Delta_{2m-1} \star \frac{f(ih)}{h}\right)(n) - f'(nh)\right| \leq \frac{h^2 m}{4}\|f^{(3)}\|_\infty$$

Lemma 2. *Under the assumptions of Theorem 4*

$$\left|\left(\Delta_{2m-1} \star \frac{f(ih)}{h}\right)(n) - f'(nh)\right| \leq h\sqrt{m}\|f^{(2)}\|_\infty \left(2 + \left(\frac{3}{2}ln(m)\right)^{3/2}\right)$$

An important tool for our proofs are Bernstein polynomials defined by:

$$B_n(u_0, u_1, ..., u_n)(x) = \sum_{i=0}^{i=n} \binom{n}{i} u_i x^i (1-x)^{n-i}$$

We define the Bernstein approximation for a function ϕ with domain $[0;1]$ by:

$$B_n(\phi)(x) = B_n\left(\phi(0), ..., \phi\left(\frac{i}{n}\right), ..., \phi(1)\right)(x)$$

We denote in a classical way $p_{n,i}(x) = \binom{n}{i} x^i(1-x)^{n-i}$ the polynomials of the Berntein basis.

We denote $\frac{f(ih)}{h} = f_i$ and when sequences $(u_{i,j,...})$ depend on several integer parameters, we denote it by $(u_{i,j,...})_i$ to specify that sums are made over different values of the parameter i, with other parameters fixed.

We make use of the following convergence theorems for Bernstein approximations:

Theorem 5 ([2]). *If* ϕ *is* C^3 *on* $[0,1]$, *then*

$$\left|(B_n\phi)'\left(\frac{1}{2}\right) - \phi'\left(\frac{1}{2}\right)\right| \leq \frac{\|\phi^{(3)}\|_\infty}{8n}$$

Now we have the new

Theorem 6. *If ϕ is C^2 on $[0,1]$, then*

$$\left|(B_{2m}\phi)'\left(\frac{1}{2}\right) - \phi'\left(\frac{1}{2}\right)\right| \leq \frac{\|\phi^{(2)}\|_\infty}{2\sqrt{m}}\left(2 + \left(\frac{3}{2}ln(m)\right)^{3/2}\right)$$

1.2 Upper Bound for the Input Data Error

Lemma 3 ([8]). *For any bounded sequence u, we have:*

$$|(\Delta_{2m-1} \star u)(n)| \leq \frac{1}{4^{m-1}}\|u\|_\infty \binom{2m-1}{m-1}$$

2 Reducing the Complexity

Considering the mask size suggested in Theorem 3, the mask size is $O(h^{(\alpha-2)/1.01})$. Depending on the implementation, the computational complexity for computing one derivative value at discrete samples is then $O(h^{(\alpha-2)/1.01})$ if we precompute the binomial mask, which requires a space of $O(h^{(\alpha-2)/1.01})$. This complexity is not as good as the one induced by tangent estimators from discrete geometry which is in $O(h)$ in [5]. Indeed, some runtime tests for comparison processed in [3] show an important runtime for the method of [8] (without using the substantial complexity optimization suggested in Section 2.3 of [8]) for small values of h. As already noted there, it is possible to reduce the complexity of our estimator without changing the convergence speed. To do that, we show that values at extremities of the smoothing kernel H_n are negligible. This can be adapted to Theorem 3.

Theorem 7 ([8]). *Let $\beta \in \mathbb{N}^*$ and $m \in \mathbb{N}$. If $k = \frac{m}{2} - \sqrt{\frac{\beta m \ln(m)}{2}}$ then:*

$$\frac{1}{2^m}\sum_{j=0}^{k}\binom{m}{j} \leq \frac{1}{m^\beta} \tag{1}$$

Theorem 7 means that the sum of the k first and the k last coefficients of the smoothing kernel are negligible with respect to the whole kernel. The parameter β enables to define what negligible mean. If we take $\beta = \frac{0.51(\alpha-0.01)}{2-\alpha}$, then we have $\frac{1}{m^\beta} \leq h^{0.51(\alpha-0.01)/1.01}$ Using the result of Theorem 7, it is possible to reduce the size of the derivative kernel D_n recommended by Theorem 3 without affecting the proven convergence speed. Indeed, we compute in the computation of the derivative (Definition 2) only the terms of the sum between $n = -\sqrt{\frac{\beta m \ln(m)}{2}}$ and $n = +\sqrt{\frac{\beta m \ln(m)}{2}}$, which involves an $O(\sqrt{m \ln(m)})$ complexity. As a function of h, the complexity is then $O(h^{0.51(\alpha-0.01)/2.01}\sqrt{-ln(h)})$ (which is less than $O(h^{0.254(\alpha-0.01)})$).

3 Derivatives Estimation for Parametrized Curves

The proofs of this section are omitted for lack of space.

3.1 Tangent Estimation

We assume that a planar simple closed C^1−parametrized curve C (i.e. the parametrization is periodic and one-to-one on a period) is given together with a family of parametrized discrete curves (Σ_h) with Σ_h contained in a tube with radius $H(h)$ around C.

Here we estimate the tangent at a point of C by a binomial digital tangent at a point of Σ_h which is not too far. The goal of this section is to bound the error of this estimation and in particular to show that this error uniformly converges to 0 with h. Note that in [8], the results are valid for C^3 curves and don't work at points with horizontal of vertical tangents. We overcome these limitations, and also use weaker hypothesis on the discretization of C.

Definition 3. *The binomial discrete tangent at M_i is the real line going through M_i directed by the vector $(\Delta_{m+1} \star (x_i), \Delta_{m+1} \star (y_i))$, when this vector is nonzero.*

Theorem 8. *Let g be a C^2 parametrization of a simple closed curve C. Suppose that for all i we have $\|g(ih) - h\Sigma_h(i)\|_\infty \leq Kh^{\alpha-1}$.*
Then for some constant K' and m sufficiently large,

$$\|\Delta_{2m-1} \star \Sigma_h(n) - g'(nh)\| \leq hm^{0,51}\|g^{(2)}\|_\infty + \frac{K'h^{\alpha-1}}{\sqrt{m}}$$

The proof of Theorem 8 is similar to that of Theorem 4, and we can derive a theorem similar to Theorem 3 for parametrized curves under the hypothesis of Theorem 8.

The hypothesis of Theorem 8 is stronger than the discretization being simply contained in a tube, because not only the discrete curve must be close to the continuous curve, but the parametrization of the discretization must be close to the parametrization of the continuous curve. This is the reason for introducing Pixel-Length parametrizations in the next section.

3.2 Pixel-Length Parametrization of a Curve

Definition 4. *A parametrization of a real curve γ is pixel-length if for all u et u' such that γ_x and γ_y are monotonic between u and u', we have*

$$\|\gamma(u) - \gamma(u')\|_1 = |u - u'|.$$

The idea is that for a curve with a pixel-length parametrization, the speed on the curve is the same as the speed of a discretization of the curve with each edgel taking the same time (see Figure 2).

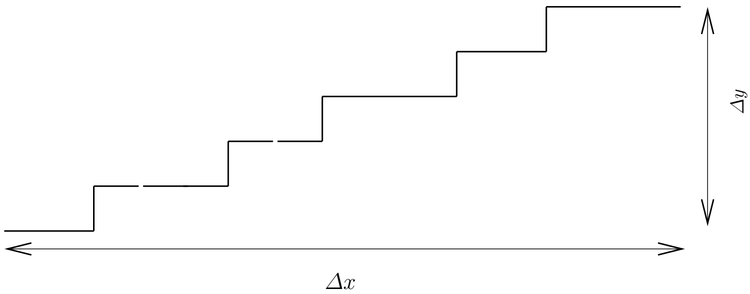

Fig. 2. On a monotonic parametrized discrete curve, the number of edgels between two points is equal to the norm $||.||_1$ of the difference between these two points

Lemma 4. *Let $g : R \longrightarrow R^2$ be a C^1 simple closed regular curve. Let us define σ_a by $\sigma_a(u) = \int_a^u \|g'(t)\|_1 \, dt$ and γ by $\gamma(u) = g[\sigma_a^{-1}(u)]$. Then γ is a C^1 pixel-length parametrization of g and $\|\gamma'(u)\|_1 = 1$.*
Moreover, suppose g is C^2 and generic in the sense that the set of values of u in a period of g for which either $g'_x(u) = 0$ or $g'_y(u) = 0$ is finite, and $g'_x(u) = 0$ implies $g_x"(u) = 0$ and $g'_y(u) = 0$ implies $g_y"(u) = 0$. Then γ is C^2.

Thanks to this lemma, we may now reduce the C^2 regular generic case to the C^2 pixel-length regular case.

3.3 Tangent Estimation for C^2 Pixel-Length Parametrization

Definition 5. *Two monotonic $\mathbb{R}-$ or $\mathbb{Z}-$valued functions are said to have* similar variations *if either they are both increasing or they are both decreasing.*

Lemma 5. *Let $\gamma : R \longrightarrow R^2$ be a pixel-length parametrization of a curve C, and let $\Sigma : Z \longrightarrow Z^2$ be a 4-connected discrete parametrized curve, lying in a tube of C with radius H. Suppose γ_x and Σ_x (resp. γ_y and Σ_y) are monotonous with similar variations. If $\|\gamma(0) - h\Sigma(0)\|_2 \leq D$, then for all i, we have $\|\gamma(ih) - h\Sigma(i)\|_2 \leq (H + D)\sqrt{2}$*

Lemma 6. *Let $\gamma : R \longrightarrow R^2$ be a pixel-length parametrization of a curve C, and let $\Sigma : Z \longrightarrow Z^2$ be a 4-connected discrete parametrized curve, lying in a tube of C of width H. Suppose moreover that for all $i < j$ such that γ_x and γ_y are monotonic in $[ih; jh]$, then Σ_x and Σ_y are monotonic and Σ_x and γ_x have similar variations and Σ_y and γ_y have similar variations.*

If $\|\gamma(0) - h\Sigma(0)\|_2 \leq D$, then for all i, we have

$$\|\gamma(ih) - h\Sigma(i)\|_2 \leq D2^{l/2} + (2h + H\sqrt{2})\frac{2^{l/2} - 1}{\sqrt{2} - 1} - 2h$$

where $l \geq 1$ is the number of points with horizontal or vertical tangents on the real curve between parameters 0 and ih.

Theorem 9. *Let γ be a C^2 pixel-length parametrization of a simple closed curve C. Let $\Sigma_h : Z \longrightarrow Z^2$ be a 4-connected discrete parametrized curve, lying in a tube of C of width Kh^α. Suppose moreover that for all $i < j$ such that γ_x and γ_y are monotonic in $[ih; jh]$, then $(\Sigma_h)_x$ and $(\Sigma_h)_y$ are monotonic and $(\Sigma_h)_x$ and γ_x have similar variations and $(\Sigma_h)_y$ and γ_y have similar variations. Consider a fixed point on the curve C. We suppose wlog that it is $\gamma(0)$. Consider any point M_0 of Σ_h such that $\|M_0 - \gamma(0)\|_2 \leq Kh^\alpha$. We suppose wlog that M_0 is $\Sigma_h(0)$. Then there is a constant K' such that for m sufficiently large,*

$$\|\Delta_{2m-1} \star \Sigma_h(0) - \gamma'(0)\| \leq hm^{0,51} \|\gamma^{(2)}\|_\infty + \frac{K'h^{\alpha-1}}{\sqrt{m}}$$

3.4 Tangent Estimation for a General C^2 Curve

Let C be a simple closed real curve C. If we assume that g is a C^2 generic parametrization of C, then Lemma 4 provides a C^2 pixel-length parametrization of C and Theorem 9 provides a convergent estimation of the derivative of the pixel-length parametrization, hence of the tangent of the curve. The method for solving the problem in a non generic case consists in a uniform C^2 approximation of C by a family of curves with a generic parametrization.

Lemma 7. *Let $\boldsymbol{g}$ be a C^2 parametrization of a real curve C. There exists a family $\boldsymbol{g}_n$ of C^2 generic parametrizations of real curves C_n such that $\boldsymbol{g}_n$ uniformly converges to $\boldsymbol{g}$ and $\boldsymbol{g}'_n$ converges uniformly to $\boldsymbol{g}'$.*

Theorem 10. *Let C be a simple closed curve C and $M_0 \in C$. Suppose that $\boldsymbol{g}$ is a regular C^2−parametrization of C and wlog $M_0 = \boldsymbol{g}(0)$. Let $\boldsymbol{\Sigma}_h : Z \longrightarrow Z^2$ be a 4-connected discrete parametrized curve, lying in a tube of C of width Kh^α. Suppose moreover that for all $i < j$ such that g_x and g_y are monotonic in $[ih; jh]$, then $(\Sigma_h)_x$ and $(\Sigma_h)_y$ are monotonic and $(\Sigma_h)_x$ and g_x have similar variations and $(\Sigma_h)_y$ and g_y have similar variations. Let $\boldsymbol{T}_0$ be a tangent vector to C in M_0 such that $\|\boldsymbol{T}_0\|_1 = 1$.*
Suppose that $\|\boldsymbol{\Sigma}_h(0) - M_0\|_2 \leq Kh^\alpha$ (up to a translation on the parameters of Σ, this is alway possible). Then there is constants K' and K" such that for m sufficiently large,

$$\|\Delta_{2m-1} \star \boldsymbol{\Sigma}_h(0) - \boldsymbol{T}_0\| \leq K'hm^{0,51} + \frac{K'h^{\alpha-1}}{\sqrt{m}}$$

Proof. (Sketch)
Following Lemma 7, let us introduce a family g_n. For a convenient choice of n, $\boldsymbol{\Sigma}_h$ lies in a tube of C with width $\frac{3K}{2}h^\alpha$. Following Lemma 4, we introduce γ_n as a C^2 pixel-length parametrization for each one. We use Theorem 9 to approximate $\Delta_{2m-1} \star \boldsymbol{\Sigma}_h(0)$ with $\boldsymbol{\gamma}'_n(0)$ and a suitable n to approximate $\boldsymbol{\gamma}'_n(0)$ with $\boldsymbol{T}_0$. □

4 Experiments

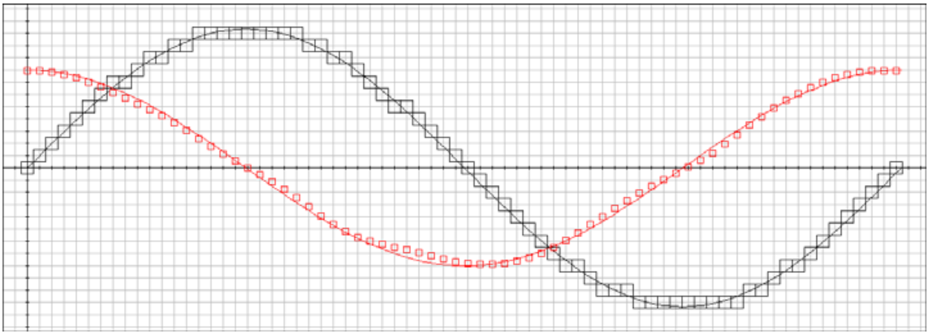

Fig. 3. Estimation of the derivative of the function $x \longmapsto \sin(2\pi x)/(2\pi)$ with $h = 0.014$ and $m = 31$

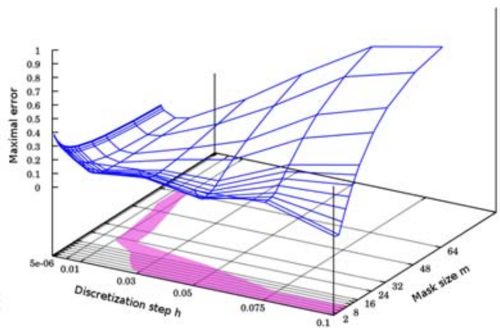

Fig. 4. Maximal error as a function of the discretization step and the mask size. The practical optimum remains to be theoretically determined as Theorem 1 and Theorem 3 make require different mask sizes.

5 Conclusion

We have provided convergence results for the binomial derivative estimator for C^2 real curves and parametrized curves. Note that the proofs in this papers are made for masks with odd size, but similar results can be obtained for masks with even size.

Note that the restriction to closed curve has been used here for convenience, but the proofs can be extended to non closed curves.

Note that when h tends to 0, the number of pixels of the boundary discretization of a convex shape is of the order of $h^{-2/3}$ (adapted from [1]). Hence the size of the required effective mask to ensure convergence (Section 2), which is $O(\sqrt{m \ln(m)})$. Hence under application of Theorem 3 and its generalization to parametrized curves, the mask size, is $O(\sqrt(h^{(\alpha-2)/1.01\ln(1/h)}))$, and so its real size tends to 0 as a proportion of the number of pixels involved. For this reason, the binomial estimator can be called a **local estimator**.

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