Distances on Lozenge Tilings

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Abstract. In this paper, a structural property of the set of lozenge tilings of a $2n$-gon is highlighted. We introduce a simple combinatorial value called *Hamming-distance*, which is a lower bound for the the number of *flips* – a local transformation on tilings – necessary to link two tilings. We prove that the flip-distance between two tilings is equal to the Hamming-distance for $n \leq 4$. We also show, by providing a pair of so-called *deficient* tilings, that this does not hold for $n \geq 6$. We finally discuss the $n = 5$ case, which remains open.

1 Introduction

Lozenge tilings are widely used by physicists as a model for quasicrystals [9], following the celebrated Penrose tilings and its pentagonal symmetry. A basic operation acting over such tilings is the *flip*: whenever three lozenge tile a hexagon, rotating this hexagon by 180 yields a new lozenge tiling. This leads to define the *tiling space* of a fixed domain as the graph whose vertices are all the possible tilings of this domain, with two tilings being connected by an edge if and only if one can transform one into the other by a performing a flip. Combinatorial properties of tiling spaces are not trivial, especially for $n \geq 4$. The connectivity of these spaces has been proved in [10] in the case of finite and simply connected domains. Nevertheless, connectivity turns out to not hold for infinite tiling (*e.g.*, the Penrose tiling), even by allowing infinite sequences of flips [4].

In this paper, we consider the case of finite domains, more precisely $2n$-gons. Tilings are thus flip-connected. It is however generally unclear what is the flip-distance between two given tilings, *i.e.*, the minimal number of flips necessary to link both tilings. We give a natural lower bound, called the *Hamming-distance*, which relies on elementary geometrical considerations (namely de Bruijn lines), and we adress the question, whether this bound is tight or not.

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The first result of this paper (section 4) is that this bound is tight for octogonal tilings ($n = 4$ case). This extends a previously known similar result for hexagonal tilings [11] ($n = 3$ case). However, the lack of a distributive lattice structure on the octogonal tiling space makes the proof more difficult and the result more surprising.

The second result of this paper (section 5) is that this bound is not tight for $n \geq 6$. Indeed, there exist pairs of tilings (not that much) such that their Hamming-distance is strictly lower than their flip-distance (we explicitly provide such a pair in the $n = 6$ case).

The $n = 5$ case remains open, because of a huge case-study that we did not manage to complete. We only here sketch (section 6) the way the proof should be carried out.

Let us stress that our results, although they looks like the ones obtained in [8], are different. Indeed, in [8], it is proven that, in any unitary $2n$-gon, there exists a fixed special tiling $\mathcal{T}_0$ such that the flip-distance and the Hamming-distance between any other tiling $\mathcal{T}$ and $\mathcal{T}_0$ are equal. But this situation cannot be extended to any arbitrary pairs of tilings of general (non-unitary) $2n$-gons.

2 $2n$-Gons, Tilings and de Bruijn Lines

For $n > 1$, let $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be pairwise non-colinear vectors of $\mathbb{R}^2$ and $(m_1, \ldots, m_n)$ be positive integers. We call $(m_1, \ldots, m_n)$*-gon* the subset $Z(V, M)$ of $\mathbb{R}^2$ defined by:

$$Z(V, M) = \left\{ \sum_{k=1}^{n} \lambda_k \ \boldsymbol{v}_k \mid \lambda_k \in [-m_k, m_k] \right\}.$$

This is called, for short, a $2n$-gon. It is said to be *regular* if $\boldsymbol{v}_k = (\cos \frac{k\pi}{n}, \sin \frac{k\pi}{n})$ for $k = 1, \ldots, n$. We here consider only regular $2n$-gons. It is said to be *unitary* if $m_k = 1$ for $k = 1, \ldots, n$.

For $1 \leq i < j \leq n$, we denote by T_{ij} the *lozenge prototile* $T_{ij} = \{\lambda \boldsymbol{v}_i + \mu \boldsymbol{v}_j, -1 \leq \lambda, \mu \leq 1\}$. A *lozenge tiling* of a $2n$-gon is a set of translated copies of lozenge prototiles with pairwise disjoint interiors and whose union is the whole $2n$-gon. Let $\mathcal{T}$ be a lozenge tiling, the *vertices* (resp. *edges*) of $\mathcal{T}$ are the vertices (resp. edges) of the tiles which belong to $\mathcal{T}$.

The combinatorial structure of tilings of a $(m_1, \ldots, m_n)$-gon depends only on $(m_1, \ldots, m_n)$, and not on the v_i's. This important property, which does not hold in dimension 3 or higher [7], ensures that we can w.l.o.g. restrict our study to regular $2n$-gons.

Height functions for tilings have been introduced by Thurston [12]. Here, for each integer $k \in \{1, ..., n\}$ and each tiling $\mathcal{T}$ of $Z(V, M)$, we define the *k-located height function* $h_{T,k}$ as the function from vertices to $\mathbb{Z}$ such that, for any edge $[x, x + \boldsymbol{v}_i]$ of $\mathcal{T}$:

$$h_{T,k}(x + \boldsymbol{v}_i) = \begin{cases} h_{T,k}(x) + 1 & \text{if } i = k, \\ h_{T,k}(x) & \text{otherwise.} \end{cases}$$

We use the normalized k-located height function such that, for each vertex x, $h_{T,k}(x) \geq 0$ and there exists a vertex x_0 with $h_{T,k}(x_0) = 0$. The existence of height function and uniqueness of normalized height functions is well known [3].

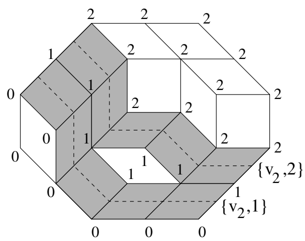

Fig. 1. The 2-located height function and two de Bruijn lines

Following [6], we define the *de Bruijn line* $\mathcal{S}_{i,j}$ of T as the set of tiles whose normalized i-located function is $j-1$ on one edge, and j on the opposite one. See Figure 1.

A de Bruijn line $\mathcal{S}_{i,j}$ is said of *type* i. It is interesting to note that two distinct de Bruijn lines of the same type do not intersect, while two de Bruijn lines of different types share a single tile. Conversely, each tile is the intersection of exactly two de Bruijn lines of different types.

Each de Bruijn line $\mathcal{S}_{i,j}$ disconnects $\mathcal{T}$

- $\triangle(\mathcal{S}_{i,j})$ is the set of tiles for which the i-located function is at least j on any vertex;
- $\bigtriangledown(\mathcal{S}_{i,j})$ is the set of tiles for which the i-located function is at most $j-1$ on any vertex.

Three de Bruijn lines of pairwise different types i, j, k define a sub-tiling of the zonotope, called *pseudo-triangle of type* ijk. A *minimal* (for inclusion) pseudo-triangle is reduced to three tiles.

3 Hamming-Distance and Flip-Distance

We introduce in this section two distance over tilings.

3.1 Flip-Distance

Two tiles are *adjacent* if they share an edge. Assume that three tiles of a tiling $\mathcal{T}$ are pairwise adjacent (*i.e.*, form a minimal pseudo-triangle). In this case, one can replace in a unique way these three tiles by three other tiles of the same type, to obtain another tiling $\mathcal{T}'$ of the same domain as $\mathcal{T}$. This operation is called a *flip* (see. Fig.3). The *tiling space* of $Z(V, M)$ is the undirected graph whose vertices are tilings of $Z(V, M)$, with two of them being connected by an edge if and only if they differ by a flip.

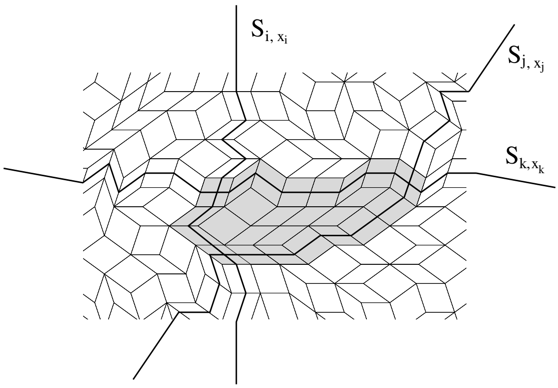

Fig. 2. In gray, a pseudo-triangle

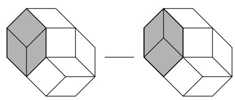

Fig. 3. Two neighbor tilings. One can pass from one to the other one by a single flip

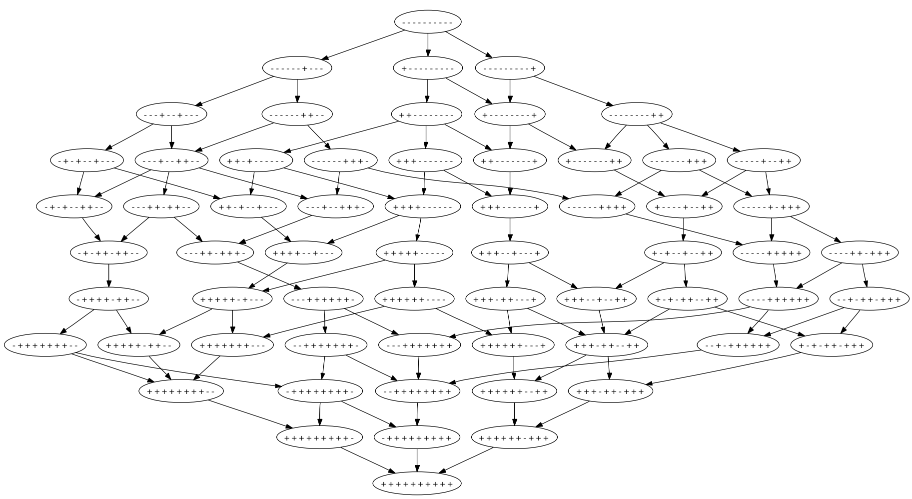

Fig. 4. The tiling space of the unitary decagon

Definition 1. *The* flip-distance *between two tilings* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *of* $Z(V, M)$*, denoted by* $d_F(\mathcal{T}_1, \mathcal{T}_2)$*, is the length of the shortest path connecting* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *in the tiling space. It is a finite value because the tiling space is connected [10].*

The figure 4 illustrates the topology of such a graph.

3.2 Hamming-Distance

Let $\mathbb{T}$ be the sign function defined on any triple $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$, $i < j < k$, of de Bruijn lines by:

$$\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = \begin{cases} + \text{ if the tile } \mathcal{S}_{i,\alpha_i} \cap \mathcal{S}_{j,\alpha_j} \text{ belongs to } \triangle(\mathcal{S}_{k,\alpha_k}), \\ - \text{ otherwise.} \end{cases}$$

Thus, to each tiling corresponds a one dimensionnal array $\mathbb{T}$ of $+$ or $-$, indexed on the set $\mathcal{L}_{\mathbf{m}}$ of all possible triples. It can be proven that $\mathbb{T}$ totally characterizes the tiling.

Nevertheless, there exist some $\mathbf{m}$-uples of $\{-,+\}^{\mathbf{m}}$ that do not correspond to any tiling. A characterization of the $\mathbf{m}$-uples induced by tilings has been given by Chavanon-Rémila [5]. It uses a set of "local" conditions, in a sense that each of them involve finitely many entries of the array.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two tilings. Let respectively $\mathbb{T}$ and $\mathbb{T}'$ be the corresponding arrays. A triple $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$ (or, by extension, the pseudo-triangle defined by the corresponding de Bruijn lines) is said to be *inverted* when

$$\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) \neq \mathbb{T}'(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}).$$

This leads to define:

Definition 2. *The* Hamming-distance *between $\mathcal{T}$ and $\mathcal{T}'$, denoted by $d_H(\mathcal{T}, \mathcal{T}')$, is the number of inverted triples.*

Note that this is exactly the classical Hamming-distance between $\mathbb{T}$ and $\mathbb{T}'$. Recall that the goal of this paper is to compare flip-distance and Hamming-distance. On the one hand, one easily shows:

Proposition 1. *For any two tilings $\mathcal{T}$ and $\mathcal{T}'$ of a $2n$-gon, one has:*

$$d_H(\mathcal{T}, \mathcal{T}') \leq d_F(\mathcal{T}, \mathcal{T}').$$

Indeed, since a flip modifies only one triangle, there is at least as much flips as inverted triangles (which must be reverted). On the other hand, it is far less obvious whether the converse inequality holds, that is whether a given pair of tilings is *deficient* or not:

Definition 3. *A pair $(\mathcal{T}_1, \mathcal{T}_2)$ of tilings is said to be* deficient *if its flip-distance is strictly greater than its Hamming-distance.*

4 The Octogonal Case

In this section, we show that there is no deficient pair of tilings of a 8-gon:

Proposition 2 ([11]). *For hexagons (i. e. $n = 3$), the Hamming-distance between two lozenge tilings is equal to the flip-distance between them.*

This result, which also holds for any polygon, is strongly related to the structure of distributive lattice of the space of lozenge tilings, (for $n = 3$). It also can be interpreted in terms of "stepped surface" [1]: the Hamming-distance is exactly the volume of the solid inbetween the stepped surfaces defined by the tilings. It is possible to crush this gap by deforming the stepped surfaces decreasing the current volume. But for the general $2n$-gons one cannot use the same type of arguments, since the lattice structure and the volume interpretation are both lost.

Distances between the tilings of the unitary 8-gon. It is well-known that there exists only 8 tilings of the unitary 8-gon centered in Ω. These 8 tilings are isometrically equivalent. They can be obtain of acting the dihedral group of order 16 (which is isomorph to the group of isometry that preserves the octagon) from one of them. More precisely, the flip which is in general a local move can be seen here as the global map $s \circ \rho$ or $s \circ \rho^{-1}$ where ρ is the rotation of angle $2\pi/2n$ centered in Ω and s is the central symmetry of center Ω. The tiling space of these tilings is a cycle (of length 8) which is also the orbit of a tiling under $s \circ \rho$. We can easily remark that the Hamming-distance between two unitary 8-gon tilings is equal to their flip-distance. Up to isometry, there only exists 4 types of pair of tilings which corresponds to important configurations related to the next lemma 1.

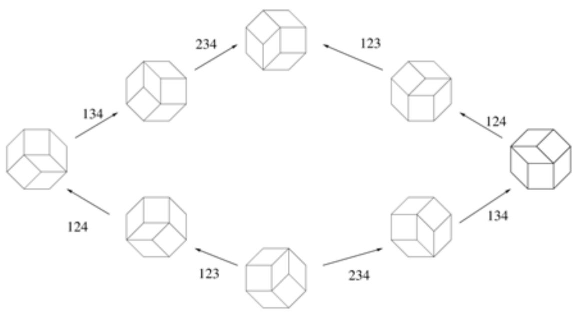

Fig. 5. The tiling space of the unitary octagon

Distances between general 8-gon tilings. The previous result, for unitary 8-gons, turns out to be true for general 8-gons. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings. Let us introduce another array $\mathbb{B}$ as follows:

– $\mathbb{B}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = 1$ if the pseudo-triangle $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$ is inverted (*i.e.*, $\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) \neq \mathbb{T}'(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$.),
– $\mathbb{B}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = 0$ otherwise.

Some consistence conditions (lemmas 1 and 2) in $\mathbb{B}$ are similar (and local) to those appearing in the characterization of $\mathbb{T}$.

Lemma 1. *Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings of a $(m_1, \ldots, m_n)$-gon. Let us consider four de Bruijn lines, say S_1, S_2, S_3, S_4. We assume that their relative positions in $\mathcal{T}_1$ are such that (see respectively Fig.6a and 6b):*

1. *S_3 and S_4 have the same type and S_4 cuts the pseudo-triangle (S_1, S_2, S_3);*
2. *the pseudo-triangle (S_2, S_3, S_4) is included in the pseudo-triangle (S_1, S_2, S_3).*

Then:

1. *$(\mathbb{B}(S_1, S_2, S_3),\ \mathbb{B}(S_1, S_2, S_4))$ belongs to $\{(0,0), (0,1), (1,1)\}$;*
2. *$(\mathbb{B}(S_1, S_3, S_4),\ \mathbb{B}(S_1, S_2, S_4),\ \mathbb{B}(S_1, S_2, S_3),\ \mathbb{B}(S_2, S_3, S_4))$ belongs either to $\{(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,1,1,1)\}$, or to the set obtained by exchanging 0's and 1's (in other words, it is a monotonic sequence).*

Proof. First, note that we have to keep the order between the de Bruijn lines of a same type in both tilings. Thus, it suffices to consider only four lines of different types, what is equivalent to consider the case of the unitary 8-gon. Since there is only eight tilings of the unitary 8-gon, the claim can be check by an exhaustive case-study.

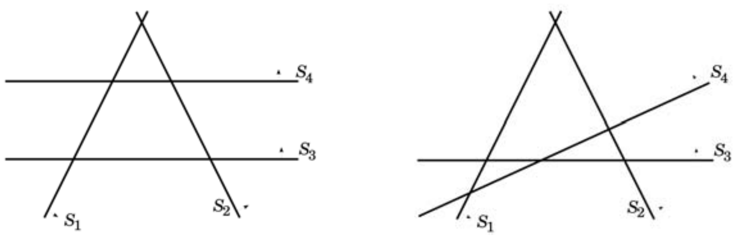

Fig. 6. The configurations for the lemma 1

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings of a $(m_1, \ldots, m_n)$-gon. If a pseudo-triangle (S_i, S_j, S_k) of $\mathcal{T}_1$ is inverted and if it is only cutted by de Bruijn lines of type i, j or k, then every pseudo-triangle (S'_i, S'_j, S'_k) included in (S_i, S_j, S_k) is inverted. In particular, (S_i, S_j, S_k) contains an inverted minimal sub-pseudo-triangle.*

Proof. The fact that every pseudo-triangle (S'_i, S'_j, S'_k) included in (S_i, S_j, S_k) is inverted follows from lemma 1, by induction on the number of de Bruijn lines.

Theorem 1. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings of a $(m_1, \ldots, m_4)$-gon. Then, their Hamming-distance is equal to their flip-distance.*

Proof. We first prove that for every pair of distinct tilings $(\mathcal{T}_1, \mathcal{T}_2)$ of the 8-gon, $\mathcal{T}_1$ contains an inverted minimal pseudo-triangle (a closer flip is feasible on it).

Considering all these previous lemmas, an induction on the number m_4 of de Bruijn lines of type 4 can be done.

For initialization, if $m_4 = 0$, the 8-gon is actually a hexagon for which the result is previously known. Suppose that $m_4 = 1$ and when we remove the de Bruijn line S of type 4, $\mathcal{T}_1 \backslash S$ are identical to $\mathcal{T}_2 \backslash S$. In this case, the positions of the de Bruijn line S in $\mathcal{T}_1$ and $\mathcal{T}_2$ mark the boundary of a stepped surface U in $\mathcal{T}_1 \backslash S$ which is a hexagonal tiling (see figure 7).

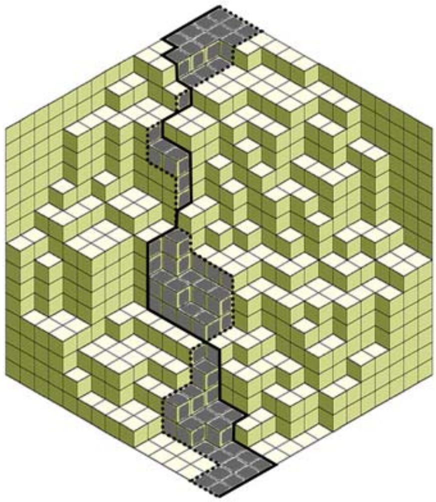

Fig. 7. The bold (resp. dotted) line indicates the position of the de Bruijn line S in $\mathcal{T}_1$ (resp. in $\mathcal{T}_2$). The stepped surface U is in dark gray inbetween the two de Bruijn lines.

The flip-distance and the Hamming-distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ are exactly the number of tiles in U. Indeed, one easily proves that, in any case, a flip can be done, corresponding to a local transformation on the position of the de Bruijn line S in $\mathcal{T}_2$ (i.e. surrounding a unique tile by the other side).

Now, for the other cases, we can remove a de Bruijn line $S = \mathcal{S}_{4,\alpha}$, of type 4, in both tilings $\mathcal{T}_1$, $\mathcal{T}_2$ in such a way that $\mathcal{T}_1 \backslash S$ are distinct to $\mathcal{T}_2 \backslash S$ (this is always possible when $m_4 > 1$). Henceforward, we are going to only work on the tiling $\mathcal{T}_1$. By hypothesis of induction, there exists an inverted minimal pseudo-triangle (S_a, S_b, S_c), in the tiling $\mathcal{T}_1$ obtained by removing of S and sticking the two remaining parts of the initial tiling.

After this, let us replace the removed de Bruijn line S. The only tricky case arises when S cuts the pseudo-triangle (S_a, S_b, S_c) and the types of de Bruijn lines S_a, S_b and S_c are respectively 1, 2 and 3. Moreover, the minimal sub-pseudo-triangle of (S_a, S_b, S_c) (which is (S_a, S_b, S) or (S_b, S_c, S)) is not inverted (in any other configuration, the existence of an inverted minimal pseudo-triangle is trivial).

We can consider without loss of generality that $\mathbb{T}(S_a, S_b, S_c) = +$ and that (S_a, S_b, S) is the non-inverted minimal sub-pseudo-triangle of (S_a, S_b, S_c). The 3 other cases are in fact isometrically equivalent.

Since (S_a, S_b, S) is not inverted, we have by lemma 1.2 applying on the pseudo-lines $\{S_a, S_b, S_c, S\}$ that the pseudo-triangle (S_a, S_c, S) is inverted. If this pseudo-triangle is minimal, then the result ensues. Otherwise, (S_a, S_c, S) can only be cut by de Bruijn lines of type 1 or 2, because of the minimality of the pseudo-triangle (S_a, S_b, S_c) in $\mathcal{T}_1 \backslash S$. Let $\mathcal{S}_{1,j_1}$ (resp. $\mathcal{S}_{2,j_2}$) be (if there exists) the de Bruijn line of type 1 (resp. type 2), with j_1 minimal such that $\mathcal{S}_{1,j_1}$ cuts (S_a, S_c, S) (resp. with j_2 minimal such that $\mathcal{S}_{2,j_2}$ cuts (S_a, S_c, S)) (see fig.8). The pseudo-triangle $(\mathcal{S}_{1,j_1}, S_c, S)$ (resp. $(\mathcal{S}_{2,j_2}, S_c, S)$) is inverted. Indeed this follows of applying lemma 1.1 to the pseudo-triangles (S_a, S_c, S) (resp. (S_b, S_c, S)) which are both inverted. If one of them is minimal, we can conclude.

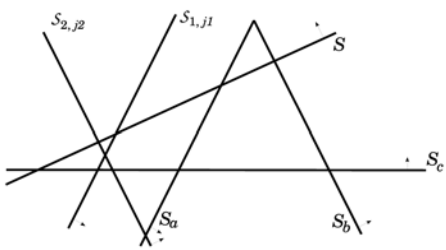

Fig. 8. A configuration involving the de Bruijn lines $\mathcal{S}_{1,j_1}$ and $\mathcal{S}_{2,j_2}$ such that their intersection is in (S_a, S_c, S)

Otherwise, the tile $\mathcal{S}_{1,j_1} \cap \mathcal{S}_{2,j_2}$ belongs to $\triangle(S_c) \cap \bigtriangledown(S)$ and necessarily at least one of the pseudo-triangles $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S_c)$ and $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S)$ is inverted (by applying lemma 1.2 on $\{S_c, S, \mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}\}$ with $(\mathcal{S}_{1,j_1}, S_c, S)$ inverted). But $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S_c)$ (resp. $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S)$) can only be cut by de Bruijn lines of type 1 (resp. type 2). Because of the minimality of j_1 and j_2, it includes by lemma 2 an inverted minimal sub-pseudo-triangle of type 123 (resp. type 124). Thus, we always have a inverted minimal pseudo-triangle in $\mathcal{T}_1$.

Now, we can prove the theorem. Assume that $(\mathcal{T}_1, \mathcal{T}_2)$ is a deficient pair of tiling of a $(m_1, \ldots, m_4)$-gon, with their flip-distance being minimal among such pairs. In particular, $\mathcal{T}_1$ could not contain any inverted minimal pseudo-triangle, because such a pseudo-triangle would correspond be flippable, thus yielding a tiling $\mathcal{T}_1'$ which would contradict the minimality of the flip-distance between $\mathcal{T}_1$ and $\mathcal{T}_2$. This is however impossible, according to what we proved above.

Corollary 1. *A $(m_1, \ldots, m_4)$-gon tiling (resp. $4 \rightarrow 2$ tiling of the plane $\mathbb{R}^2$) is uniquely determined by the value of $\mathbb{T}$ for its minimal pseudo-triangles.*

Proof. Consider $\mathcal{T}_1$ and $\mathcal{T}_2$ be a pair of tiling of a $(m_1, \ldots, m_4)$-gon with the same set M of minimal pseudo-triangle. If $\mathbb{T}_1(v) = \mathbb{T}_2(v)$ for every $v \in M$, then $\mathcal{T}_1$ has no inverted minimal pseudo-triangle. So, by theorem 1, $\mathcal{T}_1 = \mathcal{T}_2$. For the $4 \rightarrow 2$ tilings of the plane, it suffices to take the limit when (m_1, m_2, m_3, m_4) tends to infinity (no matter how).

5 The Dodecagonal Case (and beyond)

It turns out that there exist deficient pairs of tilings of the unitary 12-gon (and thus of any $2n$-gon, for $n \geq 6$). To shows this, we achieved (by using a computer) an exhaustive search of all the deficient pair of tilings of the unitary 12-gon. More precisely, this shows that there exist, up to isometry, only two deficient pairs of tilings. For these two pairs the Hamming-distance is equal to 16, while the flip-distance is equal to 18. The first pair (Fig. 9) yields, by symmetry, 12 different deficient pairs of tilings. The second pair (Fig.10) yields, by symmetry, 4 different deficient pairs of tilings. These 16 pairs of tilings are the only deficient ones, among the $(908)^2$ possible pairs of tilings.

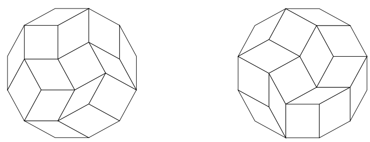

Fig. 9. The first deficient pair of tilings of the unitary 12-gon

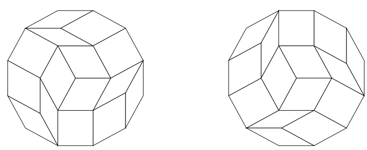

Fig. 10. The second deficient pair of tilings of the unitary 12-gon

6 Towards the Decagonal Case

The two previous sections solved the $n \leq 4$ and the $n \geq 6$ cases. We here discuss the $n = 5$ case, where we conjecture:

Conjecture 1. *The Hamming-distance between any two tilings of a* 10*-gon is equal to their flip-distance.*

The problem for proving (or disproving!) this conjecture is mainly technical. Indeed, a huge and tedious case-study need to be achieved. Let us just here state two proven lemmas, which should be the cornerstones of a complete proof.

Lemma 3 (Harp lemma). *Consider an inverted pseudo-triangle P of type ijk. Assume that it is cutted only by a set $\mathcal{S}$ of de Bruijn lines whose types are not in $\{i, j, k\}$. Then, the configuration formed by $\mathcal{S}$ and the three de Bruijn lines defining P contains an inverted minimal pseudo-triangle.*

Lemma 4 (10-cycle lemma). *Assume that there exist tuples* $(m_1, \ldots, m_5)$ *such that the* $(m_1, \ldots, m_5)$*-gon has deficient pairs of tilings. Let us consider a minimal such tuple. Consider a pair of tilings with minimal flip-distance among the deficient pairs of the* $(m_1, \ldots, m_5)$*-gon. In particular, the inverted pseudo-triangles of this pair cannot be minimal,* i.e., *flippable, because flipping such a pseudo-triangle would yield a deficient pair with a smaller flip-distance. Then, there is an infinite sequence* $(P_n)_n$ *of inverted pseudo-triangles such that:*

- P_n *is cutted by a de Bruijn line, say* S_n *;*
- P_{n+1} *is defined by* S_n *and two of the three lines defining* P_n*, with the types of these two lines being characterized by the types of* S_n *and* P_n *(Fig. 11).*

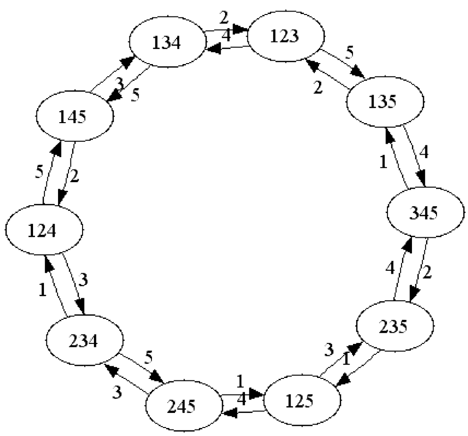

Fig. 11. The vertices of this 10-cycle are the possible types of the P_n's. Each edge links a P_n to P_{n+1}, with its label being the type of the de Bruijn line S_n which cuts P_n.

To fill the gap between the previous lemma and the above conjecture, it would remain to show that a pseudo-triangle cannot appear twice in the sequence (P_n), and thus that this sequence cannot be infinite. This would prove that decagons cannot have deficient pairs of tiling. However, proving (or disproving) that a pseudo-triangle cannot appear twice requires a huge case-study that has not yet been achieved (except for the unitary decagon, where an exhaustive computer-assisted search has shown that there is no deficient pair). This is a work in progress.

7 Conclusion

To conclude this paper, let us mention that the problem here adressed is only a part of the general study of combinatorics of lozenge tilings spaces, where a lot of questions remain so far open (*e.g.*, the size or the diameter of the set of tilings of a given domain, as well as the way to perform random sampling on this set *etc.*). More specifically, without going into details which are beyond the scope of this paper, our question is motivated by the problem of the growth of quasicrystals (seen as lozenge tilings), which can be roughly stated as follows: "given a random tiling, can we transform it into a certain specific tiling (quasicrystal) by performing only some restricted types of flips?"

References

1. Arnoux, P., Berthé, V., Fernique, T., Jamet, D.: Functional stepped surfaces, flips and generalized substitutions. Theor. Comput. Sci. 380, 251–265 (2007)
2. Björner, A., Las Vergnas, M., Sturmfels, B., White, N., Ziegler, G.M.: Oriented matroids. Encyclopedia of Mathematics, vol. 46. Cambridge University Press, Cambridge (1993)
3. Bodini, O., Latapy, M.: Generalized Tilings with height functions. Morfismos 7, 47–68 (2003)
4. Bodini, O., Fernique, T., Rémila, É.: Characterizations of flip-accessibility for rhombus tilings of the whole plane. Info. & Comput. 206, 1065–1073 (2008)
5. Chavanon, F., Rémila, É.: Rhombus tilings: decomposition and space structure. Disc. & Comput. Geom. 35, 329–358 (2006)
6. De Bruijn, N.G.: Dualization of multigrids. J. Phys. Fr. C3 47, 9–18 (1986)
7. Desoutter, V., Destainville, N.: Flip dynamics in three-dimensional random tilings. J. Phys. A: Math. Gen. 38, 17–45 (2005)
8. Felsner, S., Weil, H.: A theorem on higher Bruhat orders. Disc. & Comput. Geom. 23, 121–127 (2003)
9. Grnbaum, B., Shephard, G.C.: Tilings and patterns. W. H. Freeman, New York (1986)
10. Kenyon, R.: Tiling a polygon with parallelograms. Algorithmica 9, 382–397 (1993)
11. Rémila, É.: On the lattice structure of the set of tilings of a simply connected figure with dominoes. Theor. Comput. Sci. 322, 409–422 (2004)
12. Thurston, W.P.: Conway's tiling group. American Math. Month. 97, 757–773 (1990)