Maximum Expected Utility of Markovian Predicted Wealth

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Abstract. This paper proposes an ex-post comparison of portfolio selection strategies based on the assumption that the portfolio returns evolve as Markov processes. Thus we propose the comparison of the ex-post final wealth obtained with the maximization of the expected negative exponential utility and expected power utility for different risk aversion parameters. In particular, we consider strategies where the investors recalibrate their portfolios at a fixed temporal horizon and we compare the wealth obtained either under the assumption that returns follow a Markov chain or under the assumption we have independent identically distributed data. Thus, we implement an heuristic algorithm for the global optimum in order to overcome the intrinsic computational complexity of the proposed Markovian models.

Keywords: Markov chains, expected utility, portfolio strategies, heuristic, computational complexity.

1 Introduction

In this paper, we model the return portfolios with a Markov chain. Under this distributional hypothesis we compare expected utility portfolio strategies with the assumption that returns are independent identically distributed.

The Markovian hypothesis have been widely used in financial modeling. In particular, in option theory, portfolio theory and risk management theory most of the parametric processes used are Markov processes (for portfolio models see, among others, Staino et al. (2007), Rachev et al. (2007), for option pricing models see, among others, Cox et al. (1979), De Giovanni et al. (2008), Iaquinta and Ortobelli (2008), for risk management models see, among others, Longerstaey and Zangari (1996), Lamantia et al. (2006b). In addition, using the methodology proposed by Christoffersen (1998) we can easily show that the Markovian hypothesis of asset returns cannot be rejected (see Lamantia et al. (2006a)). However, even if most of the parametric processes used in financial applications are Markov processes, only recently it has been shown that we can easily maximize inter-temporal performance measures assuming return portfolios following a Markov chain (see Angelelli and Ortobelli (2008)). In this paper we first propose some algorithms that reduce the complexity of the portfolio selection problems based on the Markovianity of the gross

G. Allen et al. (Eds.): ICCS 2009, Part II, LNCS 5545, pp. 588–597, 2009.

returns. In particular, we use the method discussed by Iaquinta and Ortobelli (2006) for non parametric Markovian processes where the transition matrix depends directly on the portfolio weights. This algorithm permits to predict future asset returns and their distributions in polynomial computational times. However, the dependence on the portfolio weights of the transition matrix implies that the computational complexity of these portfolio problems is much higher than assuming that historical observations of returns are independent identically distributed. As a matter of fact, if we use classic methods for global optimum (such as simulated annealing type algorithms see Leccadito *et al.* (2007)) we cannot solve these problems in reasonable computational times. In order to reduce the computational complexity of these portfolio selection strategies, we use the optimization heuristic proposed by Angelelli and Ortobelli (2008). That algorithm permits to check the n-dimensional simplex to approximate the global optimum. Secondly, we propose an empirical comparison among portfolio selection strategies based on the optimization of expected utility of future wealth. We use the negative exponential utility and the power utility with different degrees of risk aversion. We propose an ex-post analysis where we compare the sample path of wealth obtained assuming that the investors recalibrate their portfolios at a fixed temporal horizon. Since any of these portfolio strategies is based on the estimation of the distribution of the returns at future times, we get a substantial difference when portfolio selection strategies are developed using the Markovian assumption respect to those based on the assumption that returns are independent identically distributed. So, when we apply Markovian strategies on twenty components of the Dow Jones Industrials, we show that we always get higher returns with respect to returns obtained by means of classic strategies.

The paper is organized as follows. In Section 2 we show how to model non parametric Markov chains and we formalize the maximum expected utility problem with Markov chains. In Section 3 we discuss the ex-post empirical comparison. In the last Section, we briefly summarize the paper.

2 Maximum Expected Utility with Non Parametric Markov Processes

In this section we deal the portfolio selection problem among n risky assets with gross returns $z_{t+1}=[z_{1,t+1},\ldots,z_{n,t+1}]'$ assuming that the portfolio process is described by a homogeneous Markov chain with N states. In particular, we assume that investors want to maximize their utility of wealth at a given future date T. We denote by $x=[x_1,\ldots,x_n]'$ the vector of the positions taken in the n risky assets, then the portfolio return during the period $[t, t+1]$ is given by $z_{(x),t+1}=x'z_{t+1}=\sum_{i=1}^{n}x_i z_{i,t+1}$.

2.1 The Markovian Evolution Process

Next, we consider the range $(\min_k z_{(x),k};\max_k z_{(x),k})$ of the portfolio gross returns, where $z_{(x),k}$ is the k-th past observation of the portfolio $z_{(x)}$. Without loss of generality we assume that the N states $z_{(x)}^{(i)}$ of portfolio gross return are ordered as follows $z_{(x)}^{(i)} > z_{(x)}^{(i+1)}$ for $i = 1,...,N-1$. Since we want to have a recombining tree of the Markov chain, we first divide the portfolio support $(\min_k z_{(x),k}; \max_k z_{(x),k})$ in N intervals $(a_{(x),i}; a_{(x),i-1})$ where $a_{(x),i} = \left(\frac{\min_k z_{(x),k}}{\max_k z_{(x),k}}\right)^{i/N} \cdot \max_k z_{(x),k}$, $i = 0,1,\ldots,N$ is decreasing with index i. Then, we compute the return associated to each state as the geometric average of the extremes of the interval $(a_{(x),i}; a_{(x),i-1})$, that is

$$z_{(x)}^{(i)} := \sqrt{a_{(x),i} a_{(x),i-1}} = \max_k z_{(x),k} \left(\frac{\max z_{(x),k}}{\min z_{(x),k}}\right)^{\left(\frac{1-2i}{2N}\right)}, \; i = 1,2,...,N. \tag{1}$$

Consequently $z_{(x)}^{(i)} = z_{(x)}^{(1)} u^{1-i}$, where $u = \left(\frac{\max z_{(x),k}}{\min z_{(x),k}}\right)^{1/N} > 1$. Let us assume that the initial wealth W_0 at time 0 is equal to 1, while for each possible wealth W_t at time t we have N possible different values $W_{t+1} = W_t z_{(x)}^{(i)}$ (i=1,…,N) at time t+1. Thanks to the recombining effect of the Markov chain we have $1 + k(N-1)$ possible values after k steps of wealth $W_k(x)$ that are given by the formula $w_{(x)}^{(i,k)} = (z_{(x)}^{(1)})^k u^{(1-i)}$ $i = 1,...,(N-1)k+1$, where the i-th node at time k of the Markovian tree corresponds to wealth $w_{(x)}^{(i,k)}$. Moreover, all possible values of the random wealth $W_k(x)$ can be stored in a matrix with k columns and $1 + k(N-1)$ rows resulting in $O(Nk^2)$ memory space requirement. Since we assume homogeneous Markov chain the transition matrix $P = [p_{i,j}]$ does not depend on time and the entries $p_{i,j}$ are estimated using the maximum likelihood estimates $\hat{p}_{i,j} = \frac{\pi_{ij}(K)}{\pi_i(K)}$, where $\pi_{ij}(K)$ is the number of observations (out of K observations) that transit from the i-th state to the j-th state and $\pi_i(K)$ is the number of observations (out of K observations) in the i-th state (see D'Amico (2003) for the statistical properties of these estimators). Following the idea of Iaquinta and Ortobelli (2006) we can compute the distribution function of the future gross returns. In particular, as shown by Angelelli and Ortobelli (2008), the $(N-1)k+1$ dimensional vector $p^{(k)}$ (representing the unconditional distribution at a given time k = 0,1,2,...,T of wealth $W_k(x)$) can be computed by means of a sequence of matrixes $\left\{Q^{(k)}\right\}_{k=0,1,...,T}$, where $Q^{(k)} = [q_{i,j}^{(k)}]_{\substack{1\le i \le (N-1)k+1 \\ 1 \le j \le N}}$ and $q_{i,j}^{(k)}$ is the unconditional probability at time k to obtain the wealth $w_{(x)}^{(i,k)}$ and to be in the state $z_{(x)}^{(j)}$. In particular, $Q^{(0)} = [p_1,..., p_N]$, where p_i is the unconditional probability to be in the i-th state at time *0*. Thus, $p^{(0)} = 1 = Q^{(0)} \cdot \mathbf{1}_N$, where $\mathbf{1}_N$ is the unity vector column. In general, for $k = 1,...,T$, the vector $p^{(k)}$ is given by $p^{(k)} = Q^{(k)} \cdot \mathbf{1}_N$, where $Q^{(k)}$ is recursively defined as $Q^{(k)} = \mathtt{diagM}(Q^{(k-1)} \cdot P)$ being **diagM** a linear operator defined for any $m, n \in N$ as **diagM:** $R^{mn} \to R^{(m+n-1)n}$ that at any $m \times n$ matrix $A = [a_{ij}]$ associates the $(m+n-1)\times n$ matrix obtained by simply shifting down the *j*-th column by $(j-1)$ rows (see Iaquinta and Ortobelli (2006), Angelelli and Ortobelli (2008) for further details). The matrix $Q^{(k)}$ is the so called *unconditional evolution matrix* of the Markov chain or simply *evolution matrix.* Moreover, the algorithm to compute the probabilities has a computational complexity of $O(N^3k^2)$.

2.2 The Portfolio Selection Problem

The static portfolio selection problem when no short sales are allowed, can be represented as the maximization of the expected utility applied to the random portfolio of gross returns $z_{(x),t+1}$ subject to the portfolio weights belonging to the *n*-dimensional simplex $S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1; x_i \geq 0 \right\}$, i.e., $\max_{x \in S} E(u(z_{(x),t+1}))$, for a given utility function *u*. This represents the classic *myopic utility functional* that does not use the time evolution of the wealth process. In a dynamic context we consider an initial wealth $W_0 = 1$ and all admissible wealth processes $W(x) = \{W_t(x)\}_{t \geq 0}$ depending by an initial portfolio $x \in S$ are defined on a filtered probability space $\left(\Omega, \Im, (\Im_t)_{0 \leq t \leq \infty}, P\right)$. In this case we can distinguish two cases: the case where the investors recalibrate the portfolio at some given date *T* (*European portfolio selection strategies*) and the case where the investors recalibrate the portfolio at some given date $t \leq T$ if some particular events $A_t \in \Im_t$ happen (*American portfolio selection strategies*). In this paper we deal only European portfolio selection strategies where investors recalibrate their portfolio every *T* periods solving the problem:

$$\max_{x \in S} E\left(u(W_T(x))\right). \tag{2}$$

According to Angelelli and Ortobelli (2008) definition we call *OA expected utility* the above functional $E(u(W_T(x)))$ when it is computed under the assumption that the gross return of each portfolio follows a Markov chain with *N* states. The European OA expected utility is given by

$$E(u(W_T(x))) = u\left(\hat{W}_T(x)\right) \cdot Q^{(T)} \cdot \mathbf{1}_N = u\left(\hat{W}(x)\right) \cdot p^{(T)}, \tag{3}$$

where $\hat{W}_T(x) = \left[w_{(x)}^{(1,T)}, \ldots, w_{(x)}^{(N-1)T+1,T)} \right]$ is the $(N-1)T+1$ dimensional vector of the final wealth and $u\left(\hat{W}_T(x)\right) = \left[u(w_{(x)}^{(1,T)}), \ldots, u(w_{(x)}^{(N-1)T+1,T)}) \right]$. Since Angelelli and Ortobelli (2008) have shown that standard optimization algorithms are not adequately suited to solve the global optimization problem (2) of OA expected utility, we use the same optimization heuristic proposed by Angelelli and Ortobelli (2008) to solve portfolio optimization problems. So, starting by an initial feasible portfolio solution $\overline{x}$, the heuristic algorithm tries to iteratively update the current solution by a better one. Improving solutions, if any, are searched on a predefined grid of points fixed on the directions $x-e_i$ for $i=1,2,...,n$, where x is the current portfolio and e_i is the portfolio where the share of asset i is equal to 1 and all other assets have share equal to 0. If a better solution is found on a search direction the current solution is updated and the search is continued from the new one. If no direction provides an improved solution the search ends. Next, we recall some empirical results provided by Angelelli and Ortobelli (2008), who tested the performance of the optimization heuristic algorithm versus function **fmincon** provided with the optimization toolbox of MATLAB. The results are synthesized in Table 1 that reports the percentage in average of variations :

- of the estimated function $\Delta f = \frac{f_{\text{heuristic}} - f_{\text{fmincon}}}{f_{\text{fmincon}}}$, where $f_{[]}$ represents the optimal objective function obtained using the [ ] algorithm ([ ] can be either fmincon or the heuristic);
- of the time $\Delta t = \frac{Timef_{\text{heuristic}} - Timef_{\text{fmincon}}}{Timef_{\text{fmincon}}}$ where $Timef_{[]}$ represents the computational time necessary to optimize the objective function using the [ ] algorithm;
- of the portfolio weights $\Delta x = \sum_{i=1}^{n} \left| x_{\text{heuristic}}^{(i)} - x_{\text{fmincon}}^{(i)} \right|$ where $x_{[]}^{(i)}$ represents the i-th optimal weight obtained using the [ ] algorithm.

Table 1. Performance comparison between fmincon and the optimization heuristic (see Angelelli and Ortobelli (2008) for a definition of these strategies)

Functional	Δf	Δt	Δx
Myopic Sharpe	-0.002%	502.963%	0.010
OA-Sharpe	163.084%	328.202%	1.447
Myopic Rachev	2.696%	213.160%	0.774
OA-Rachev	15465.330%	240.005%	1.681

Table 1 underlines the limit of **fmincon Matlab** procedure to approximate a global optimum when the functionals admit many local maxima. These results tell us that the heuristic algorithm generally needs more computational time of **fmincon Matlab** procedure. However, we have a significant improvement in terms of objective function and portfolio weights when we use the heuristic. Moreover, the heuristic well approximates the optimum when this is unique; indeed there is just a little difference with the myopic Sharpe functional in terms of values and portfolios. From the results we deduce that the **fmincon** procedure can be used only for myopic strategies that admit an unique optimum (such as the myopic Sharpe strategy). Thus, as suggested by Angelelli and Ortobelli (2008), the main advantages of this algorithm are:

1) The algorithm permits to approximate the global optimum with a given error when the objective function is a non-constant concave function (the optimum is unique) and some particular lines are not contour lines of the objective function.
2) The algorithm permits to explore the whole simplex.
3) The computational complexity is much less than that of classic algorithms for global optimum such as Simulated Annealing type algorithms.

3 An Ex-post Comparison among OA Portfolio Strategies Based on the Maximum Expected Utility

In this section, we propose an ex post comparison among European OA expected utility strategies and the myopic ones. In the empirical comparisons, we consider the optimal allocation among 20 assets components of the Dow Jones Industrials[1] on the period from 1/3/1985 till 5/1/2008 for a total of 5884 daily observations. The work of Kondor et al. (2007) on the sensitivity to estimation error of portfolios optimized under various risk measures suggests that we need a large number of observations when we want to propose portfolio models considering rare events. As a matter of fact, Papp et al. (2005), Kondor et al. (2007) have shown that we could loose robustness of the approximations if the number of observations is not adequate to the number of assets. In addition, some empirical experiments show that, if we increase the number of the states, we need an increasing number of observations. For this reason we forecasted the future wealth using a non parametric Markov chain with only few states N=3, N=5 states and $K = 2000$ historical observations. We assume investors recalibrate the portfolio every $T = 60$ days starting from 1/3/1985. The comparison consists in the ex post evaluation of the wealth produced by the strategies. We compare the performance of myopic and OA expected utility strategies based on the following HARA utility functions:

1) negative exponential utility function:

$$u(W) = -\exp(-aW)\text{; with } a=1, 5, 10, 15, 20.$$

2) power utility function:

$$u(W) = \frac{W^g}{g}\text{; with } g=-1, -0.6, -0.2, 0.2, 0.6, 1, 1.4, 1.8, 2.2, 2.6, 3.$$

With myopic strategies the expected utility of each portfolio is approximated considering the last $K = 2000$ observations and computing $E(u(x'z_{t+1})) \approx \frac{1}{K}\sum_{t=1}^{K} u\left(x'z_{t+1}\right)$. For each strategy, we consider an initial wealth $W_0 = 1$ at the date 1/3/1985, and at the k-th recalibration ($k = 0,1,2,...$), the investor should solve:

$$\begin{aligned} &\max_{x^{(k)}} E(u(\hat{W}_{t_k+60}(x^{(k)}))) \\ &\text{s.t.} \\ &\left(x^{(k)}\right)' e = 1, \\ &x_i^{(k)} \geq 0; \quad i = 1,\dots,n, \end{aligned} \tag{4}$$

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[1] We used the following components: 3M Company, Alcoa Inc, American Express, AT&T, Boeing Co, Caterpillar Inc, Coca Cola, Du Pont, Exxon Mobil, General Electric, General Motors, Hewlett Packard, IBM, Johnson and Johnson, McDonalds, Merck, Procter Gamble, United technologies, Wal Mart Stores, Walt Disney.

where $\hat{W}_{t_k+60}$ is the forecasted wealth at time t_{k+1}. So, the ex-post final wealth is given by $W_{t_{k+1}} = W_{t_k}\left(\left(x_M^{(k)}\right)' z^{(ex\ post)}\right)$, where $z^{(ex\ post)}$ is the vector of observed gross returns between t_k and $t_{k+1} = t_k + 60$.

Table 2. Final wealth obtained at date 5/1/2008 using myopic and Markovian strategies and maximizing the expected power utility

Parameter	HARA Power Utility	OA-HARA-power utility Markovian strategies	
g	Myopic strategy	states=3	states=5
-1	3.8135	10.8913	9.1947
-0.6	3.9096	11.102	9.466
-0.2	3.973	11.2265	9.5626
0.2	3.5307	11.3135	9.104
0.6	2.0762	11.2862	9.142
1	1.5423	11.3808	7.9091
1.4	2.2367	11.5531	7.7919
1.8	2.0602	11.324	7.7235
2.2	1.9872	11.2224	8.131
2.6	6.0933	10.9617	7.623
3	3.4551	11.1149	7.9061

Table 3. Final wealth obtained at date 5/1/2008 using myopic and Markovian strategies and maximizing the expected negative exponential utility

	HARA negative exponential utility	OA-HARA negative exponential utility Markovian strategies	
a	Myopic strategies	states=3	states=5
1	3.927	11.1103	9.6179
5	4.187	12.2377	9.6295
10	5.087	10.9565	10.9592
15	5.494	8.8739	12.414
20	5.950	9.4837	7.7379

The output of this analysis is represented in Tables 2, 3, Fig. 1, and Fig. 2. Tables 2 and 3 show the ex-post final wealth at date 5/1/2008 obtained with myopic and Markovian strategies and maximizing expected power utility and expected negative exponential utility. We observe that always the Markovian strategies perform better than myopic strategies. Moreover we also observe that we get better results using three states. We believe that this fact can be justified by a more robust approximation of the forecasted final wealth (see Papp et al. (2005), Kondor et al. (2007)). These results are further confirmed by Fig. 1, and Fig. 2 that describe the ex post sample paths of final wealth. Figures 1 and 2 show better performance of OA expected utility strategies respectively for power utility with risk aversion parameter g= -0.2 and with negative exponential utility with parameter a= 10.

This empirical comparison suggests the use of OA type strategies since with these strategies we get in some cases even three times the final wealth we get with the analogous myopic strategies.

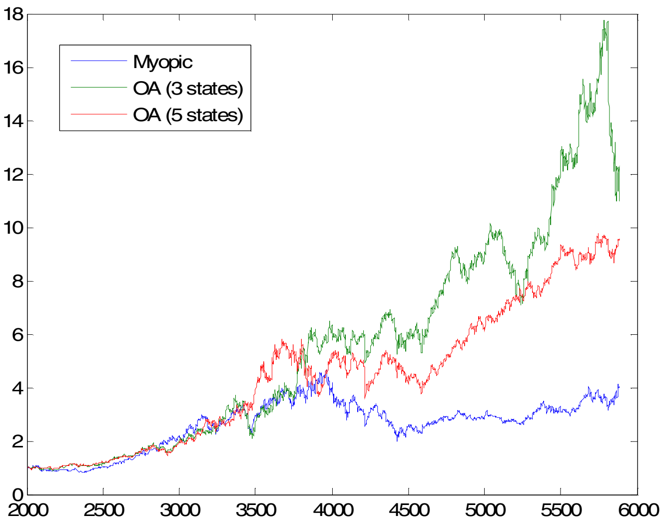

Fig. 1. Performances obtained with HARA power utility (g= -0.2)

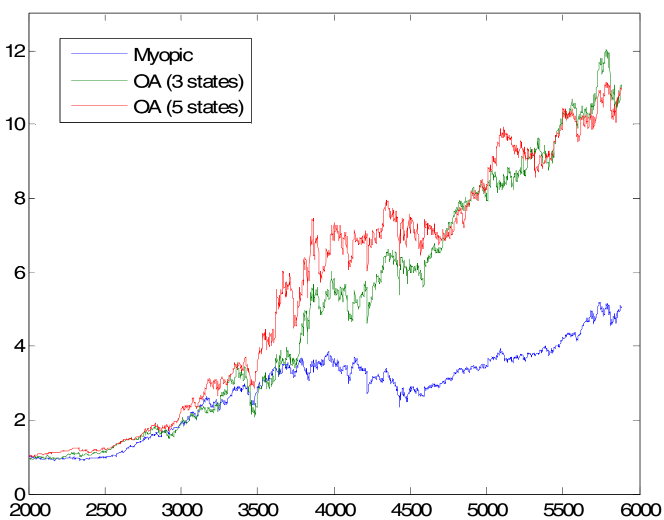

Fig. 2. Performances obtained with HARA negative exponential utility (a=10)

4 Concluding Remarks

This paper analyzes the impact of Markovianity in optimal portfolio choices. We examine how to approximate non parametric Markov processes and we deal the computational complexity of these portfolio selection problems. Thus we propose algorithms that permit to solve computationally complex problems in acceptable computational times. Secondly, we propose an empirical comparison among the myopic portfolio selection models and the Markovian ones. The ex-post empirical comparison among classic approaches and those based on Markovian trees shows the greater predictable capacity of the latter.

The contribution of this paper consists in the computational accessible methodology to solve dynamic expected utility portfolio problems.

Acknowledgments

The authors thank for grants COFIN 60% 2008 and seminar audiences at AMASES 2008 (Trento, Italy), and at X Workshop in Quantitative Finance 2009 (Milan, Italy).

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