Approximating the Crossing Number of Apex Graphs

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Abstract. We show that the crossing number of an apex graph, i.e. a graph G from which only one vertex v has to be removed to make it planar, can be approximated up to a factor of $\Delta(G - v) \cdot d(v)/2$ by solving the *vertex inserting* problem, i.e. inserting a vertex plus incident edges into an optimally chosen planar embedding of a planar graph. Due to a recently developed polynomial algorithm for the latter problem, this establishes the first polynomial fixed-constant approximation algorithm for the crossing number problem of apex graphs with bounded degree.

Keywords: Crossing number, apex graph, vertex insertion.

1 Edge and Vertex Insertion Problems

We assume that the reader is familiar with the standard notation of terminology of graph theory, and especially with topological graphs, see [5]. A graph G is called an *apex graph* if there is a vertex v such that $G - v$ is planar. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise edge crossings in a drawing of G in the plane. Determining the crossing number of a given graph is an NP-complete problem, and exact crossing numbers are in general extremely difficult to compute.

A common heuristic way of finding a drawing of a graph G with few crossings starts with a planar subgraph of G, and then re-inserts the remaining edges one by one in a locally optimal way. The edge insertion problem can be solved to optimality by a linear-time algorithm [3]. A subsequent result [4] uses that algorithm to give an approximation of the crossing number of almost planar graphs (i.e. those made planar by removing one edge) up to a factor of $\Delta(G)$ (recently improved to the best possible $\Delta(G)/2$ in [1]).

A natural generalization of the previous results is to consider the problem of *inserting a vertex* with a specified neighbourhood into a planar embedding of a graph G, with the least number of crossings. Although this shows to be a much harder question than that of edge insertion, a very recent result of [2] reads:

Theorem 1 (Chimani, Gutwenger, Mutzel, and Wolf). *The vertex insertion problem for a planar graph can be optimally solved in polynomial time.*

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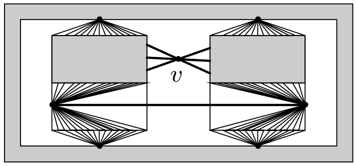

Fig. 1. An example of a vertex v insertion instance requiring many crossings, eventhough the crossing number of the graph is small. The gray regions denote dense subgraphs.

2 Crossing Number Approximation

We can apply Theorem 1 to approximate the crossing number of apex graphs.

Theorem 2. *Let G be a graph and v its vertex such that $G - v$ is planar, the maximum degree in $G - v$ is Δ, and v has degree d in G. Then the vertex insertion problem of v back into a planar embedding of $G - v$ has a solution with at most $d \cdot \lfloor \Delta/2 \rfloor \cdot \mathrm{cr}(G)$ crossings.*

This new result immediately gives us a polynomial approximation algorithm for the crossing number of an apex graph G up to factor $d \cdot \lfloor \Delta/2 \rfloor$. On the other hand, it is possible to construct examples for which optimal solutions to the vertex insertion problem require up to $d \cdot \Delta \cdot \mathrm{cr}(G)/4$ crossings, cf. Fig. 1.

The idea of the proof is as follows (compare to [4]): Assume Γ is a plane embedding of the graph $G - v$ achieving optimality in the vertex v insertion problem, Γ' is a crossing-optimal drawing of the graph G, and let F be a minimal edge set such that $\Gamma' - v - F$ is a plane embedding. Then $|F| \leq \mathrm{cr}(G)$ and the embedding $\Gamma' - v - F$ can be turned into $\Gamma - F$ by a sequence of 1- and 2-flips (Whitney flips), which allows to re-embed the edges F without crossings in $G - v$. The central argument is that the number of new crossings introduced on the edges of v is limited by an iteration of the following claim over all $f \in F$:

Lemma 3. *Let H be an apex graph with a vertex v, having a drawing with ℓ crossings in which $H - v$ is connected and plane embedded. Let an edge f connect vertices of $H - v$. If $(H - v) + f$ is planar, then there is a drawing of $H + f$ with plane embedded $(H - v) + f$ having at most $\ell + d(v) \cdot \lfloor \Delta(H - v)/2 \rfloor$ crossings.*

In contrast to [4], establishing Theorem 2 using this lemma requires a careful consideration of non-biconnected graphs and the fact that the position of the newly introduced vertex v is unknown and probably different between Γ and Γ'.

References

1. Cabello, S., Mohar, B.: Crossing and weighted crossing number of near planar graphs. In: GD 2008. LNCS, vol. 5417. Springer, Heidelberg (to appear, 2009)
2. Chimani, M., Gutwenger, C., Mutzel, P., Wolf, C.: Inserting a vertex into a planar graph. In: ACM-SIAM Symposium on Discrete Algorithms (to appear, 2009)

3. Gutwenger, C., Mutzel, P., Weiskircher, R.: Inserting an edge into a planar graph. Algorithmica 41, 289–308 (2005)
4. Hliněný, P., Salazar, G.: On the Crossing Number of Almost Planar Graphs. In: Kaufmann, M., Wagner, D. (eds.) GD 2006. LNCS, vol. 4372, pp. 162–173. Springer, Heidelberg (2007)
5. Mohar, B., Thomassen, C.: Graphs on surfaces. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore MD, USA (2001)