Minimal Obstructions for 1-Immersions and Hardness of 1-Planarity Testing

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Abstract. A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by no more than one other edge. A non-1-planar graph G is *minimal* if the graph $G - e$ is 1-planar for every edge e of G. We construct two infinite families of minimal non-1-planar graphs and show that for every integer $n \geq 63$, there are at least $2^{\frac{n}{4}-\frac{54}{4}}$ non-isomorphic minimal non-1-planar graphs of order n. It is also proved that testing 1-planarity is NP-complete. As an interesting consequence we obtain a new, geometric proof of NP-completeness of the crossing number problem, even when restricted to cubic graphs. This resolves a question of Hliněný.

1 Introduction

A graph is 1-*immersed* in the plane if it can be drawn in the plane so that each edge is crossed by no more than one other edge. A graph is 1-*planar* if it can be 1-immersed into the plane. It is easy to see that if a graph has 1-immersion in which two edges e, f with a common endvertex cross, then the drawing of e and f can be changed so that these two edges no longer cross. Consequently, we may assume that adjacent edges are never crossing each other and that no edge is crossing itself. We take this assumption as a part of the definition of 1-immersions since this limits the number of possible cases when discussing 1-immersions.

The notion of 1-immersion of a graph was introduced by Ringel [11] when trying to color the vertices and faces of a plane graph so that adjacent or incident elements receive distinct colors.

Little is known about 1-planar graphs. Borodin [1,2] proved that every 1-planar graph is 6-colorable. Some properties of maximal 1-planar graphs are considered in [12]. It was shown in [3] that every 1-planar graph is acyclically 20-colorable. The existence of subgraphs of bounded vertex degrees in 1-planar graphs is investigated in [7]. It was shown in [4,5] that a 1-planar graph with n vertices has at most $4n - 8$ edges and that this upper bound is tight. In the paper [6] it was observed that the class of 1-planar graphs is not closed under the operation of edge contraction.

⋆ This work was done while the first author visited Simon Fraser University.

⋆⋆ Supported in part by ARRS (Slovenia), P1-0297, and by NSERC (Canada).

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I.G. Tollis and M. Patrignani (Eds.): GD 2008, LNCS 5417, pp. 302–312, 2009.

Much less is known about non-1-planar graphs. The basic question is how to recognize 1-planar graphs. This problem is clearly in NP, but it is not clear at all if there is a polynomial time recognition algorithm. We shall answer this question by proving that 1-planarity testing problem is NP-complete.

The recognition problem is closely related to the study of minimal obstructions for 1-planarity. A graph G is said to be a *minimal* non-1-planar graph (MN-*graph*, for short) if G is not 1-planar, but $G - e$ is 1-planar for every edge e of G. An obvious question is:

How many MN-graphs are there? Is their number finite? If not, can they be characterized?

The answer to the first question is not hard: there are infinitely many. This was first proved in [10]. Here we present two additional simple arguments implying the same conclusion.

Example 1. Let G be a graph such that and $t = \lceil \mathrm{cr}(G)/|E(G)| \rceil - 1 \geq 1$, where $\mathrm{cr}(G)$ denotes the crossing number of G. Let G_t be the graph obtained from G by replacing each edge of G by a path of length t. Then $|E(G_t)| = t|E(G)| < \mathrm{cr}(G) = \mathrm{cr}(G_t)$. This implies that G_t is not 1-planar. However, G_t contains an MN-subgraph H. Clearly, H contains at least one subdivided edge of G in its entirety, so $|V(H)| > t$. Since t can be arbitrarily large, this shows that there are infinitely many MN-graphs.

Example 2. Let $K \in \{K_5, K_{3,3}\}$ be one of Kuratowski graphs. For each edge $xy \in E(K)$, let L_{xy} be a 5-connected triangulation of the plane and u, v be adjacent vertices of L_{xy} whose degree is at least 6. Let $L'_{xy} = L_{xy} - uv$. Now replace each edge xy of K with L'_{xy} by identifying x with u and y with v. It is not hard to see that the resulting graph G is not 1-planar (since two of graphs L'_{xy} must "cross each other", but that is not possible since they come from 5-connected triangulations). Again, one can argue that they contain large MN-graphs.

The paper [10] and the above examples prove the existence of infinitely many MN-graphs but do not give any concrete examples. In [10], two specific MN-graphs of order 7 and 8, respectively, are given. One of them, the graph $K_7 - E(K_3)$, is the unique 7-vertex MN-graph and since all 6-vertex graphs are 1-planar, the graph $K_7 - E(K_3)$ is the MN-graph with the minimum number of vertices. Surprisingly enough, the two MN-graphs in [10] are the only explicit MN-graphs known in the literature.

The main problem when trying to construct 1-planar graphs is that we have no characterization of 1-planar graphs. The set of 1-planar graphs is not closed under taking minors, so 1-planarity can not be characterized by forbidding some minors.

In the present paper we construct two explicit infinite families of MN-graphs and, correspondingly, we give two different approaches how to prove that a graph has no plane 1-immersion.

In Sect. 2 we construct MN-graphs based on the Kuratowski graph $K_{3,3}$. To obtain the MN-graphs, we replace six edges of $K_{3,3}$ by some special subgraphs. The non-1-planarity of the obtained MN-graphs follows from the nonplanarity of $K_{3,3}$. Using these MN-graphs, we show that for every integer $n \geq 63$, there are at least $2^{\frac{n}{4} - \frac{54}{4}}$ non-isomorphic minimal non-1-planar graphs of order n. In Sect. 3 we describe a class of 3-connected planar graphs that have no plane 1-immersions with at least one crossing point (PN-*graphs*, for short). Every 3-connected PN-graph has a unique plane

1-immersion, namely, the unique plane embedding of the graph. Hence, if a 1-planar graph G contains as a subgraph a PN-graph H, then in every plane 1-immersion of G the subgraph H is 1-immersed in the plane in the same way. Having constructions of PN-graphs, we can construct 1-planar and non-1-planar graphs with some desired properties: 1-planar graphs that have exactly $k > 0$ different plane 1-immersions; MN-graphs, etc.

In Sect. 4 we construct MN-graphs based on PN-graphs. Each of these MN-graphs G has as a subgraph a PN-graph H and the unique plane 1-immersion of H prevents to draw the remaining part of G on the plane when trying to obtain a plane 1-immersion of G.

Despite the fact that minimal obstructions for 1-planarity (i.e., the MN-graphs) have diverse structure, and despite the fact that discovering 1-immersions of specific graphs can be very tricky, it turned out to be a hard problem to establish hardness of 1-planarity testing. A solution is outlined in Sect. 5, where we show that 1-planarity testing is NP-complete, see Theorem 4. The proof is geometric in the sense that the reduction is from 3-colorability of planar graphs (or similarly, from planar 3-satisfiability). As an interesting consequence we obtain a new, geometric proof of NP-completeness of the crossing number problem, even when restricted to cubic graphs. Hardness of the crossing number problem for cubic graphs was established recently by Hliněný [9], who asked if one can prove this result by a reduction from an NP-complete geometric problem instead of the Optimal Linear Arrangement problem used in his proof.

2 MN-Graphs Based on the Graph $K_{3,3}$

Two cycles of a graph are *adjacent* if they share a common edge. If a graph G is drawn in the plane, then we say that a vertex x lies *inside* (resp. *outside*) a non-self-intersecting embedded cycle C, if x lies in the interior (resp. exterior) of C, and does not lie on C. Having two embedded adjacent cycles C and C', we say that C lies inside (resp. outside) C' if every point of C either lies inside (resp. outside) C' or lies on C'. We assume that in 1-immersions, adjacent edges do not cross each other and no edge crosses itself. Thus, every 3-cycle of a 1-immersed graph is embedded in the plane. Hence, given a 3-cycle of a 1-immersed graph, we can speak about its interior and exterior.

In what follows, throughout the paper, given a 1-immersion of a graph, when we speak about vertices, paths and cycles of the graph, we usually mean (the exact meaning will be always clear from the context) immersed vertices, paths and cycles of the 1-immersed graph.

Now we begin describing a family of MN-graphs based on the graph $K_{3,3}$.

By a *link* $L(x, y)$ connecting two vertices x and y we mean any of the graphs shown in Fig. 1 where $\{z, \overline{z}\} = \{x, y\}$.

By an A-*chain* of length $n \geq 2$ we mean the graph shown in Fig. 2(a). By a B-*chain* of length $n \geq 2$ we mean the graph shown in Fig. 2(c) and every graph obtained from this graph in the following way: for some integers $h_1, h_2, \ldots, h_t$, where $t \geq 1$ and $1 \leq h_1 < h_2 < \cdots < h_t \leq n-2$, for every $i = 1, 2, \ldots, t$, we replace the subgraph at the left of Fig. 2(e) by the subgraph shown at the right of the figure. Note that, by definition, A- and B-chains have length at least 2. We say that the chains in

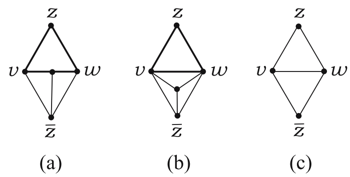


Fig. 1.

Figs. 2(a) and (c) connect the vertices $v(0)$ and $v(n)$ which are called the *end vertices* of the chain. Two chains are *adjacent* if they share a common end vertex. The A-chain in Fig. 2(a) and the B-chain in Fig. 2(c) will be designated in later figures by a single directed (broken) edge, as shown in Figs. 2(b) and (d), respectively, where the arrow points to the end vertex incident with the base link. The vertices $v(0), v(1), \dots, v(n)$ are the *core vertices* of the chains.

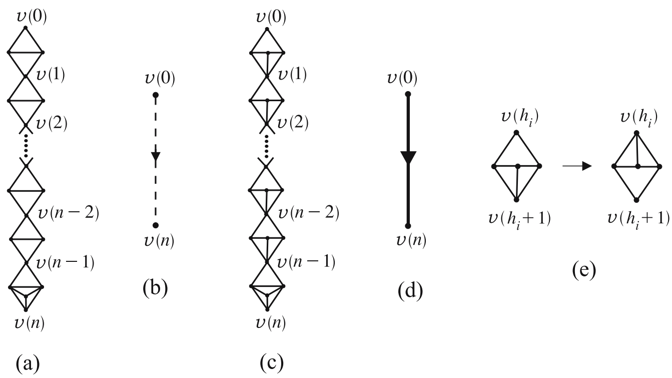


Fig. 2.

By a *chain graph* we mean the graph obtained from $K_{3,3}$ as shown in Fig. 3(a), where the three A-chains and three B-chains can have arbitrary lenghts ≥ 2. The vertices $\Omega(1)$, $\Omega(2)$, and $\Omega(3)$ are the *base vertices* of the chain graph. The edges joining the vertex Ω to the base vertices are called the *Ω-edges*.

We will show that every chain graph is an MN-graph.

Lemma 1. *If G is a chain graph and $e \in E(G)$, then the graph $G - e$ is 1-planar.*

Proof. It is easy to see that $G - e$ is 1-planar for every Ω-edge e. Let us now consider a plane embedding f of $G - \Omega$ of Fig. 3(a) after we delete the vertex Ω. If e is not an Ω-edge, then, because of the symmetry, it suffices to prove that $G - e$ is 1-planar for every edge e belonging to the A- or B-chain incident to $\Omega(2)$. Figs. 3(b) and (c) show how f can be modified to obtain a 1-immersion of $G - e$ for every edge e belonging to the chains incident to $\Omega(2)$ (the edge e is represented by the dotted line). ∎

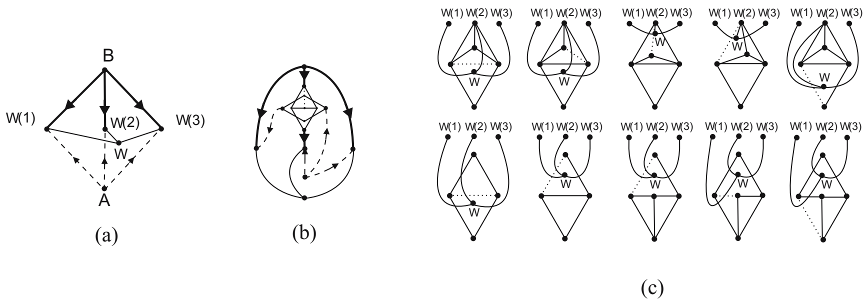


Fig. 3.

We are not aware of a simple argument showing that a chain graph G is not 1-planar. We prove it by *reductio ad absurdum* – assuming that G has a 1-immersion φ, we show that φ has the following properties that eventually yield a contradiction. If Π and Π' are nonadjacent A- and B-chain, respectively, then for every 3-cycle C of Π the following holds: The core vertices of Π' either all lie inside or all lie outside C. If all core vertices of Π' lie inside (resp. outside) C, then at most one vertex of Π' lies outside (resp. inside) C. If Π and Π' are nonadjacent A- and B-chain, respectively, then Π does not cross Π' in φ. The Ω-edges do not cross all three edges of a link incident to the same core vertex of the link. The proof of these properties is deferred for the full paper.

The following theorem shows how chain graphs can be used to construct exponentially many nonisomorphic MN-graphs of order n.

Theorem 1. *For every integer $n \geq 63$, there are at least $2^{\frac{n}{4}-\frac{54}{4}}$ non-isomorphic MN-graphs of order n.*

Proof. The A-chain of length t has $3t+2$ vertices and a B-chain of length t has $4t+1$ vertices. Consider a chain graph whose three A-chains have length 2, 2, and $\ell \geq 2$, respectively, and whose B-chains have length 2, 3, and $t \geq 4$, respectively. The graph has $35+3\ell+4t$ vertices. One can apply the modification shown in Fig. 2(e) to an arbitrary subset of the links of the B-chains of the graph, and thus obtain 2^{t-1} nonisomorphic chain graphs of order $35+3\ell+4t$, where $\ell \geq 2$ and $t \geq 4$. We claim that for every integer $n \geq 63$, there are integers $2 \leq \ell \leq 5$ and $t \geq 4$ such that $n = 35+3\ell+4t$. Indeed, if $m \equiv 0,1,2,3 \pmod 4$, put $\ell = 3,2,5,4$, respectively. If $n = 35+3\ell+4t$, where $2 \leq \ell \leq 5$, then $t \geq n/4 - 50/4$. Hence, there are at least $2^{\frac{n}{4}-\frac{54}{4}}$ non-isomorphic chain graphs of order $n \geq 63$. Since every chain graph is an MN-graph, the theorem follows.

3 PH-Graphs

By a *proper* 1-immersion of a graph we mean a 1-immersion with at least one crossing point. Let us recall that a PN-*graph* is a planar graph that does not have proper 1-immersions. In this section we describe a class of PN-graphs and construct some graphs of the class. They will be used in Sect. 4 to construct MN-graphs.

Two disjoint edges vw and $v'w'$ of a graph are *paired* if the four vertices v, w, v', w' are all four vertices of two adjacent 3-cycles. For every cycle C of a graph denote by $N(C)$ the set of all vertices of the graph not belonging to C but adjacent to vertices of C.

Consider a 3-connected plane graph. By a *basic* k-cycle of the graph we mean the boundary cycle of a k-gonal face of the embedding. By a *nontriangular* basic cycle we mean every basic k-cycle, $k \geq 4$.

Theorem 2. *Suppose that a 3-connected plane graph G satisfies the following conditions:*

(C1) *Every vertex has degree at least 4 and at most 6.*
(C2) *Every edge belongs to at least one 3-cycle.*
(C3) *Every 3-cycle is basic.*
(C4) *Every 3-cycle is adjacent to at most one other 3-cycle.*
(C5) *No vertex belongs to three mutually edge-disjoint 3-cycles.*
(C6) *Every 4-cycle is either basic or is the boundary of two adjacent triangular faces.*
(C7) *For every 3-cycle C, any two vertices of $V(G) \setminus (V(C) \cup N(C))$ are connected by 4 edge-disjoint paths not passing through the vertices of C.*
(C8) *If an edge vw of a nontriangular basic cycle C is paired with an edge $v'w'$ of a nontriangular basic cycle C', then C and C' have no vertices in common and any two vertices a and a' of C and C', respectively, such that $\{a, a'\} \not\subseteq \{v, w, v', w'\}$ are non-adjacent and are not connected by a path a, b, a' of length 2, where b does not belong to C and C'.*
(C9) *G does not contain the subgraphs shown in Fig. 4 (in this figure, 4-valent (resp. 5-valent) vertices of G are encircled (resp. encircled twice)).*

Then G has no proper 1*-immersion.*

The proof of Theorem 2 is long and will be given in the full paper.

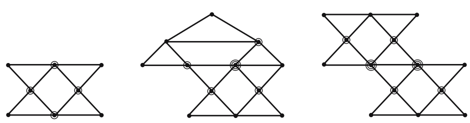
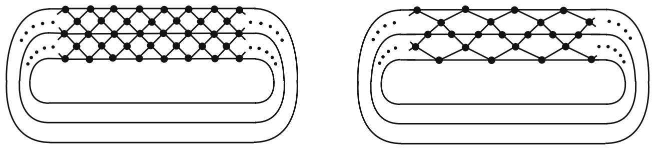


Fig. 4.

Denote by $\mathcal{A}$ the class of all 3-connected plane graphs G satisfying the conditions (C1)–(C9) of Theorem 2. In the full paper we show how to construct graphs of the class $\mathcal{A}$. Figure 5 shows two graphs of $\mathcal{A}$, one of which (in Fig. 5(a)) will be used in Sect. 4 to construct MN-graphs. To simplify checking conditions (C1)–(C9) we construct the graphs to be symmetrical so that, for example, to check the condition (C7) we need to consider only two 3-cycles of a graph.



(a) (b)

Fig. 5.

4 MN-Graphs Based on PN-Graphs

In this section we construct MN-graphs based on the PN-graphs G_n described in Sect. 3.

Denote by S_m, $m \geq 2$, the graph shown in Fig. 6. The graph has $m+1$ cycles of length $12m-2$ labelled by $B_0, B_1, \ldots, B_m$ as shown in the figure. The vertices of B_0 are called the *central vertices* of S_m and are labelled by $1, 2, \ldots, 12m-2$ (see Fig. 6). For every central vertex $x \in \{1, 2, \ldots, 12m-2\}$, denote by x^* the vertex $6m-1+x$ if $x \in \{1, 2, \ldots, 6m-1\}$ and the vertex $x-(6m-1)$ if $x \in \{6m, 6m+1, \ldots, 12m-2\}$. In S_m any pair $\{x, x^*\}$ of central vertices is connected by a *central path* $P(x, x^*)$ of length $6m-3$ with $6m-4$ two-valent vertices.

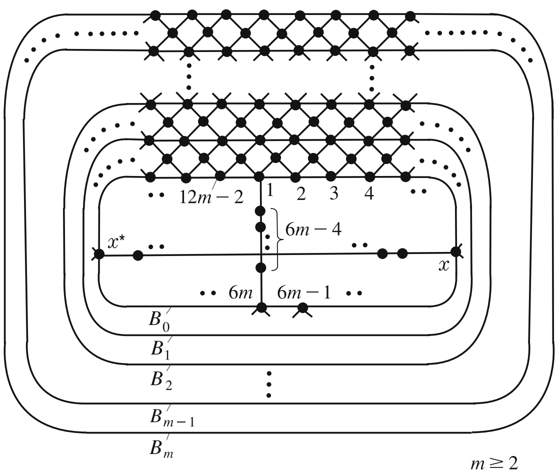


Fig. 6.

For any integers $m \geq 4$ and $n \geq 0$, denote by $\Phi_m(n)$ the set of all $(12m-2)$-tuples $n_1, n_2, \ldots, n_{12m-2}$ of nonnegative integers such that $n_1 + n_2 + \cdots + n_{12m-2} = n$. For every $\lambda \in \Phi_m(n)$, denote by $S_m(\lambda)$ the graph obtained from S_m if for every central vertex $x \in \{1, 2, \ldots, 12m-2\}$, we replace the 8 edges marked by transverse stroke in Fig. 7(a) by $8(1+n_x)$ new edges marked by transverse stroke in Fig. 7(b) (here $x+1 = 1$

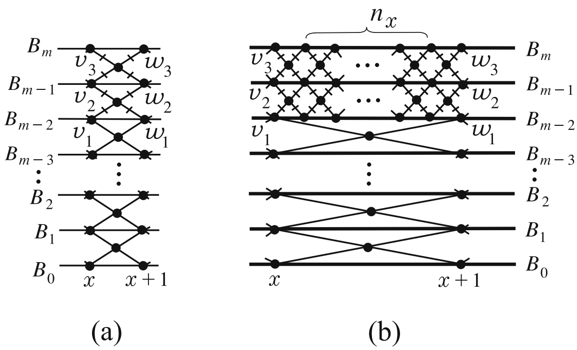


Fig. 7.

for $x = 12m-2$). The graph $S_m(\lambda)$ has $m-2 \quad (12m-2)$-cycles $B_0, B_1, \ldots, B_{m-3}$ and three $(12m-2+n)$-cycles B_{m-2}, B_{m-1}, B_m; all the $m+1$ cycles are depicted in Fig. 7(b) in thick line.

We want to show that for every $m \geq 4$ and for every $\lambda \in \Phi_m(n)$, $n \geq 0$, the graph $S_m(\lambda)$ is an MN-graph.

Lemma 2. *The graph $S_m(\lambda) - e$, where $m \geq 4$, $\lambda \in \Phi_m(n)$, is 1-planar for every edge e.*

Proof. If we delete an edge of a central path, then the remaining $6m-2$ central paths, each with $6m-3$ edges, can be 1-immersed inside B_0 in Fig. 6. If we delete one of the edges depicted in Fig. 8(a) in thick line, then the central path $P(x, x^*)$ can be drawn outside B_0 with $6m-3$ crossing points as shown in the figure (where the path is depicted in thin line) and then the remaining $6m-2$ central paths can be 1-immersed inside B_0. If we delete one of the two edges depicted in Fig. 8(a) in dotted line, then Fig. 8(b) shows how to place the central vertex x so that the path $P(x, x^*)$ can be drawn outside B_0 with $6m-3$ crossing points and then the remaining $6m-2$ central paths can be 1-immersed inside B_0. ∎

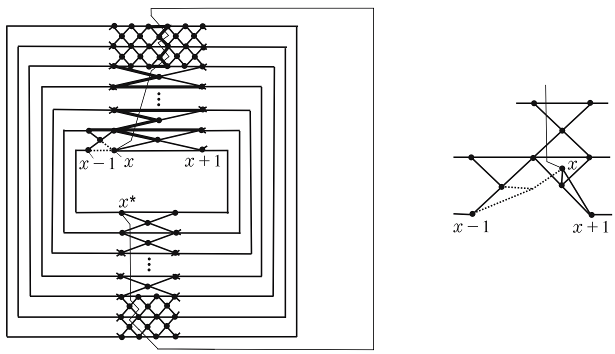
Lemma 3. *The graph obtained from the graph $S_m(\lambda)$, where $m \geq 4$ and $\lambda \in \Phi_m(n)$, by deleting the two-valent vertices of all central paths is a PN-graph.*

Theorem 3. *The graph $S_m(\lambda)$, where $m \geq 4$ and $\lambda \in \Phi_m(n)$, is not 1-planar.*

The proofs of Lemma 3 and Theorem 3 are deferred for the full paper.

We have shown that every graph $S_m(\lambda)$, where $m \geq 4$ and $\lambda \in \Phi_m(n)$, is an MN-graph (the graph has order $(5m-1)(12m-2)+5n$). Clearly, graphs $S_{m_1}(\lambda_1)$ and $S_{m_2}(\lambda_2)$, where $\lambda_1 \in \Phi_{m_1}(n_1)$ and $\lambda_2 \in \Phi_{m_2}(n_2)$, are nonisomorphic for $m_1 \neq m_2$ and for $m_1 = m_2$ and $n_1 \neq n_2$.

Corollary 1. *For any integers $m \geq 4$ and $n \geq 0$, there are at least $\frac{1}{2(12m-2)}\binom{n+12m-3}{12m-3}$ non-isomorphic MN-graphs $S_m(\lambda)$, where $\lambda \in \Phi_m(n)$.*



(a) (b)

Fig. 8.

Proof. It is well known that $|\Phi_m(n)| = \binom{n+12m-3}{12m-3}$. The automorphism group of the graph S_m is the automorphism group of a regular $(12m-2)$-gonal, that is, the dihedral group D_{12m-2} of order $2(12m-2)$. Now the claim follows. ■

5 Testing 1-Immersibility Is Hard

In this section we prove that it is NP-complete to decide if a given input graph is 1-immersible. This shows that it is extremely unlikely that there exists a nice classification of MN-graphs.

The reduction showing completeness for the class NP is from 3-colorability of planar graphs. It is worth mentioning that our method also yields a similar reduction of planar 3-colorability to the problem of computing the crossing number of cubic graphs. NP-completeness of the crossing number problem on cubic graphs was proved recently by Hliněný [9]. The author has observed in [9] that his proof is non-geometric and asked for an accessible proof based on geometric reduction. Our construction, correspondingly adapted, in particular answers the question of Hliněný.

Theorem 4. *It is NP-complete to decide if a given graph is* 1*-immersible in the plane.*

Proof (sketch). Since 1-immersions can be represented combinatorially, it is clear that 1-immersability is in NP. To prove its completeness, we shall make a reduction from a known NP-complete problem, that of 3-colorability of planar graphs of maximum degree at most four [8].

Let G_0 be a given planar graph of maximum degree 4 whose 3-colorability is to be tested. We shall show how to construct, in polynomial time, a related graph $\hat{G}$ such that $\hat{G}$ is 1-immersible if and only if G_0 is 3-colorable. We may assume that G_0 has no vertices of degree less than three.

The construction of $\hat{G}$ involves replacement of each vertex v of G_0 by a *vertex-block* L_v, and replacement of each edge $uv \in E(G_0)$ by an *edge-block* F_{uv} which is

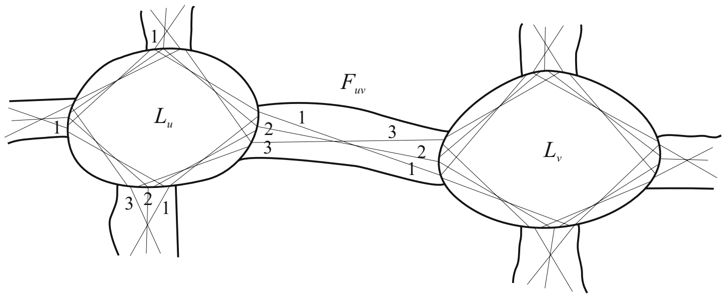


Fig. 9.

henceforth attached to L_u and L_v. Each building block has constant size, so the whole construction can be carried over in linear time. The building blocks L_v and F_{uv} are 1-planar but there is very little flexibility among their 1-immersions. They are pasted together so that their 1-immersions influence each other in such a way that globally consistent choices exist if and only if G_0 has a 3-coloring.

The vertex block essentially consists of a PN-graph L together with several subdivided edges, called *legs*. The legs can "pass through" L in a unique way since the number of degree-two vertices on the legs (the *lengths* of the legs) allow crossing it through a part of L that is not too dense. The legs are connecting vertices of L in a way as shown in Fig. 9, where they are represented by thin lines. Where the legs attach to the "boundary", there is an additional crossing edge, which can be turned outside to cross the leg in the edge-block instead. Each edge-block contains three legs that correspond to three colors 1,2,3, and we say that the leg i is active if it is crossed by the additional edge at the boundary part of the edge-block. A leg i that is active at the connection of L_u and F_{uv} corresponds to the choice of color i for the vertex u of G_0. The construction is made in such a way that an active leg i cannot be active at the other end of the edge-block (so we have proper coloring), that around u at least one leg is active, and that being active in the edge-block $F_u v$, the ith leg is also active in other edge-blocks F_{uw}, for other edges uw of G_0 incident with u. The details are cumbersome and are left for the full version of the paper. ∎

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