Generation and Empirical Investigation of *hv*-Convex Discrete Sets

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Abstract. One of the basic problems in discrete tomography is the reconstruction of discrete sets from few projections. Assuming that the set to be reconstructed fulfils some geometrical properties is a commonly used technique to reduce the number of possibly many different solutions of the same reconstruction problem. Since the reconstruction from two projections in the class of so-called *hv*-convex sets is NP-hard this class is suitable to test the efficiency of newly developed reconstruction algorithms. However, until now no method was known to generate sets of this class from uniform random distribution and thus only ad hoc comparison of several reconstruction techniques was possible. In this paper we first describe a method to generate some special *hv*-convex discrete sets from uniform random distribution. Moreover, we show that the developed generation technique can easily be adapted to other classes of discrete sets, even for the whole class of *hv*-convexes. Several statistics are also presented which are of great importance in the analysis of algorithms for reconstructing *hv*-convex sets.

Keywords: discrete tomography; *hv*-convex discrete set; decomposable configuration; random generation; analysis of algorithms.

1 Introduction

The reconstruction of two-dimensional discrete sets from their projections plays a central role in discrete tomography and it has several applications in pattern recognition, image processing, electron microscopy, angiography, radiology, non-destructive testing, and so on [11,12]. Since taking projections of an object can be expensive or time-consuming the number of projections used in the reconstruction is usually small (in most cases two or four). This can yield extremely many solutions with the same projections or/and NP-hard reconstruction causing the developed reconstruction algorithm hardly applicable in practice. One way to get rid of these problems is to suppose that the discrete set to be reconstructed belongs to a certain class described by some geometrical properties

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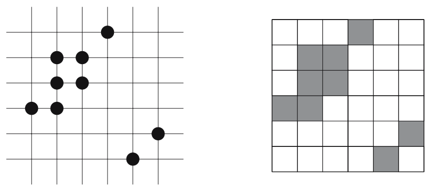
such as convexity, connectedness, etc. One of the first such approaches was presented in [14] where the author gave a reconstruction heuristic for the class of horizontally and vertically convex (shortly, hv-convex) discrete sets using only two projections. Later, it was shown that this reconstruction task is NP-hard [18]. However, by this time it was known that assuming that the set to be reconstructed is also connected makes polynomial-time reconstruction possible [4,7]. Thus, researchers began to study what makes the reconstruction in the general class of hv-convexes so difficult. For certain subclasses it was found that the reconstruction can be done in polynomial time [1,6]. Surprisingly, it was also shown that the reconstruction is no longer intractable if absorption in the projections is present (at least for certain absorption coefficients) [15]. Therefore, during the last few years the class of hv-convex discrete sets became one of the indicators of newly developed exact or heuristic reconstruction algorithms from the viewpoint of effectiveness [5,8]. Unfortunately, all the developed techniques for solving the reconstruction problem in the class of hv-convexes had to face the problem that no method was known to generate sets of this class from uniform random distribution and thus no exact comparison of the techniques was possible. In this paper we outline algorithms for generating certain hv-convex discrete sets from uniform random distributions and study properties of randomly generated hv-convex sets from several point of view. The structure of the contribution is the following. First, the necessary definitions are introduced in Section 2. In Section 3 we describe the generation method for a subclass of hv-convexes. Then, in Section 4 we investigate some properties of randomly generated hv-convex discrete sets of the above class that can affect the complexity of several reconstruction algorithms. In Section 5 we discuss our results and show how the presented generation technique can be adapted to other classes of discrete sets, in particular, even for the whole class of hv-convexes.

2 Preliminaries

Discrete tomography aims to reconstruct a discrete set (a finite subset of the two-dimensional integer lattice defined up to translation) from its line integrals along several (usually horizontal, vertical, diagonal, and antidiagonal) directions. Discrete sets can be represented by binary pictures or binary matrices (see Fig. 1) and thus the above problem is equivalent to the task of reconstructing a binary matrix from its row, column (and sometimes also diagonal and antidiagonal) sums. In the following we will use both terms discrete set and binary matrix depending on technical convenience. To stay consistent, without loss of generality we will assume that the vertical axis of the 2D integer lattice is directed top-down and the upper left corner of the smallest containing rectangle of a discrete set is the position $(1, 1)$. Clearly, definitions given for discrete sets always have natural counterparts in matrix theory. Vice versa, a definition described in matrix theoretical form can be expressed in the language of discrete sets, too.

A discrete set F is *4-connected* (with an other term *polyomino*), if for any two positions $P \in F$ and $Q \in F$ of the set there exist a sequence of distinct positions

$(i_0, j_0) = P, \ldots, (i_k, j_k) = Q$ such that $(i_l, j_l) \in F$ and $|i_l - i_{l+1}| + |j_l - j_{l+1}| = 1$ for each $l = 0, \ldots, k-1$. A discrete set is called *hv-convex* if all the rows and columns of the set are 4-connected, i.e., the 1s of the corresponding representing matrix are consecutive in each row and column. For example, the discrete set in Fig. 1 is hv-convex.



$$\begin{bmatrix} 0&0&0&1&0&0\\ 0&1&1&0&0&0\\ 0&1&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{bmatrix}$$

Fig. 1. A discrete set represented by its elements (*left*), a binary picture (*center*), and a binary matrix (*right*)

The maximal 4-connected subsets of a discrete set F are called the *components of F*. For, example the discrete set in Fig. 1 has four components: $\{(1,4)\}$, $\{(2,2),(2,3),(3,2),(3,3),(4,1),(4,2)\}$, $\{(5,6)\}$, and $\{(6,5)\}$.

3 Generation of Special *hv*-Convex Binary Matrices

The first result towards the uniform random generation of hv-convex binary matrices was given by Delest and Viennot [9] who proved that the number P_n of hv-convex polyominoes with perimeter $2n+8$ is

$$P_n = (2n+11)4^n - 4(2n+1)\binom{2n}{n} . \tag{1}$$

Later, based on the above result in [10] it was shown that the number $P_{m+1,n+1}$ of hv-convex polyominoes with a minimal bounding rectangle of size $(m+1)\times(n+1)$ is

$$P_{m+1,n+1} = \frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - \frac{2mn}{m+n}\binom{m+n}{m}^2 . \tag{2}$$

We first will consider a special class of hv-convex matrices (denoted by $\mathcal{S}$), namely where the components' bounding rectangles are connected to each other with their bottom right and upper left corners and there are no rows or columns with zero-sums. Especially, every hv-convex polyomino belongs to this class, too, as they have only one component. Then, a binary matrix $F \in \mathcal{S}$ of size $m \times n$ is either an hv-convex polyomino or it contains a polyomino of size $k \times l$ (where $k < m$ and $l < n$) as a submatrix in the upper left corner and the remaining part of F is a binary matrix of size $(m-k)\times(n-l)$ which also belongs to the class $\mathcal{S}$. Denoting the number of binary matrices of $\mathcal{S}$ of size $m \times n$ with $Q_{m,n}$ this observation can be expressed by the following recursive formula

$$Q_{m,n} = P_{m,n} + \sum_{k<m,\ l<n} P_{k,l} \cdot Q_{m-k,n-l} . \tag{3}$$

Using Equation (2) and the initial values $Q_{1,j} = P_{1,j} = 1$ ($j = 1, \ldots, n$) and $Q_{i,1} = P_{i,1} = 1$ ($i = 1, \ldots, m$) $Q_{m,n}$ can be calculated by a dynamic programming approach in $O(m^2n^2)$ time with $O(mn)$ memory requirement. Based on this we now can describe the algorithm for generating *hv*-convex binary matrices of $\mathcal{S}$ from uniform random distribution.

Algorithm 1. for generating matrices of $\mathcal{S}$ from uniform random distribution
Input: The integers m and n.
Output: A binary matrix $F \in \mathcal{S}$ of size $m \times n$.

Step 1 calculate $Q_{m,n}$;
Step 2 generate a number $r \in [1, Q_{m,n}]$ from uniform random distribution;
Step 3 **if** ($r > P_{m,n}$)
  { $r = r - P_{m,n}$;
   for $k = 1$ **to** $m - 1$
   for $l = 1$ **to** $n - 1$
   { **if** ($r > P_{k,l} \cdot Q_{m-k,n-l}$) $r = r - P_{k,l} \cdot Q_{m-k,n-l}$;
    else call Algorithm 1 with parameters $m - k$ and $n - l$; }
  }
Step 4 generate the components from uniform random distribution;

This algorithm works as follows. First, in Step 1 it calculates $Q_{m,n} = P_{m,n} + P_{1,1} \cdot Q_{m-1,n-1} + P_{1,2} \cdot Q_{m-1,n-2} + P_{2,1} \cdot Q_{m-2,n-1} + \ldots + P_{m-1,n-1} \cdot Q_{1,1}$. Choosing a number randomly in the interval $[1, Q_{m,n}]$ (Step 2) it can be decided whether it is in the interval $[1, P_{m,n}]$, $[P_{m,n} + 1, P_{m,n} + P_{1,1} \cdot Q_{m-1,n-1}]$, $[P_{m,n} + P_{1,1} \cdot Q_{m-1,n-1} + 1, P_{m,n} + P_{1,1} \cdot Q_{m-1,n-1} + P_{1,2} \cdot Q_{m-1,n-2}]$, etc. Thus, the size of the upper left component can be identified, and this method can be repeated for the remaining set, too (Step 3). Now, we only have to generate the components themselves from uniform random distribution knowing their bounding rectangles which is possible with the algorithm given in [13] (Step 4).

The above method can be extended to *hv*-convex binary matrices possibly having zero row or/and column sums, too (but still having the same configuration of the components as in the class $\mathcal{S}$). This class will be denoted by $\mathcal{S}'$. Clearly, $\mathcal{S} \subset \mathcal{S}'$. In fact, a binary matrix $F \in \mathcal{S}'$ of size $m \times n$ is either an *hv*-convex polyomino or it contains a polyomino of size $k \times l$ (where $k < m$ and $l < n$) as a submatrix in the upper left corner and the remaining part of F is a binary matrix of size $(m - k) \times (n - l)$ such that it possibly has some zero rows, or/and columns in the upper left corner and the remaining part belongs to the class $\mathcal{S}'$. Denoting the number of binary matrices of $\mathcal{S}'$ of size $m \times n$ with $Q'_{m,n}$ we get a formula similar to Equation (3)

$$Q'_{m,n} = P_{m,n} + \sum_{k<m,\ l<n} P_{k,l} \cdot \Big(\sum_{i \le m-k,\ j \le n-l} Q'_{i,j} \Big) \ . \tag{4}$$

Again, on the basis of Equation (2) and the initial values $Q'_{1,j} = P_{1,j} = 1$ ($j = 1, \ldots, n$) and $Q'_{i,1} = P_{i,1} = 1$ ($i = 1, \ldots, m$) the above formula can be evaluated by a dynamic programming approach in $O(m^3n^3)$ time with $O(mn)$ memory requirement. Then, an algorithm similar to Algorithm 1 can be given to generate hv-convex binary matrices of $\mathcal{S}'$ from uniform random distribution.

4 Statistics on *hv*-Convex Matrices

In order to test some important properties of hv-convex binary matrices we have generated test data sets with Algorithm 1 (and its modified version in the case of matrices with possible zero rows or/and columns). Each set of test data consisted of 1000 hv-convex matrix with the same size generated from uniform random distribution from the classes $\mathcal{S}$ and $\mathcal{S}'$. The algorithms were implemented in C++ and the long integer functions of library NTL-5.4 [16] were used. The test run on a PC with Intel Pentium 4 processor of 3.2 GHz and 1 GB RAM under Debian GNU/Linux 3.1, Kernel 2.6.17.13.

4.1 The Number of Special *hv*-Convex Discrete Sets

Our first simple investigation focuses on the number of special hv-convex discrete sets. Table 1 shows the number of hv-convex polyominoes and hv-convex sets from the classes $\mathcal{S}$ and $\mathcal{S}'$ with bounding rectangles of semi-perimeter n for the first 15 values of n denoted by $P(n)$, $Q(n)$, and $Q'(n)$, respectively. The first column can also be calculated by formula (1) and this is Sequence A005436 in [17]. For $n = 5$ the corresponding binary pictures of all three classes are shown in Fig. 2.

Table 1. The number $P(n)$, $Q(n)$, and $Q'(n)$ of hv-convex polyominoes, and hv-convex sets from the classes $\mathcal{S}$ and $\mathcal{S}'$, respectively, depending on the semi-perimeter n of the bounding rectangle

n	$P(n)$	$Q(n)$	$Q'(n)$
2	1	1	1
3	2	2	2
4	7	8	8
5	28	32	34
6	120	139	150
7	528	618	674
8	2344	2779	3056
9	10416	12528	13898
10	46160	56404	63178
11	203680	253152	286570
12	894312	1131849	1296008
13	3907056	5040412	5842442
14	16986352	22359981	26255254
15	73512288	98837102	117642282

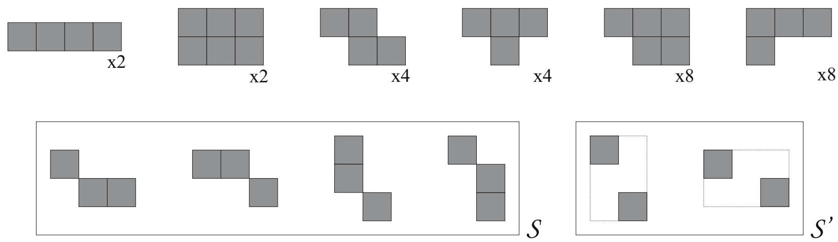


Fig. 2. All the *hv*-convex polyominoes (*first row*) and *hv*-convex sets from the classes $\mathcal{S}$ and $\mathcal{S}'$ (*second row*) with bounding rectangles of semi-perimeter 5. Small numbers in the first row indicate other solutions that can be get by mirroring or/and rotating the given polyomino.

4.2 The Number of Components

The second experiment treats the number of components of special *hv*-convex sets. It is important information when reconstructing such kind of sets. For example, as it was mentioned in Section 1 if the set consists of a single component then the reconstruction can be executed in polynomial-time. Table 2 shows the number of components of the generated sets depending on size when the sets do not have empty rows and columns (top half of the table) and when empty rows and columns are also permitted (bottom half of the table). Note that the sum of the elements in the last two rows are less than 1000. Due to space considerations we omitted 2 sets of size 80×80 and 22 sets of size 100×100 that have more than 15 components. The numerical investigation shows that generating *hv*-convex sets from the classes $\mathcal{S}$ and $\mathcal{S}'$ from uniform random distribution there is a great possibility that the set consists of a single component if the size of the set is small (namely, less than or equal to 20×20) but there is almost no chance to apply the well-known polynomial-time algorithms for reconstructing *hv*-convex polyominoes for sets of greater sizes.

It is interesting and could be quite useful in the reconstruction that the number of components can be estimated in advance knowing only the size of the set. Let $E(n)$ and $D^2(n)$ denote the expected number of components and its variance, respectively, for a set of size $n \times n$ generated from uniform random distribution from the class $\mathcal{S}$ or $\mathcal{S}'$. If $n \leq 100$ then the estimated values of $E(n)$ and $D^2(n)$ can be calculated directly from Table 2. For larger sets a good estimation can be given using the following equations

$$E(n) \approx 0.075n \quad \text{and} \quad D^2(n) \approx 0.04n \tag{5}$$

in the class $\mathcal{S}$, and

$$E(n) \approx 0.100n \quad \text{and} \quad D^2(n) \approx 0.06n \tag{6}$$

in the class $\mathcal{S}'$.

Moreover, for each size of sets the number of components follows a normal-like distribution with expected value $E(n)$ and with variance $D^2(n)$. In order

Table 2. The number of components of 1000 *hv*-convex discrete sets with bounding rectangle of size $n \times n$ generated from uniform random distribution from the classes $\mathcal{S}$ (*top half of the table*) and $\mathcal{S}'$ (*bottom half of the table*)

Size	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5×5	785	191	23	1											
7×7	746	225	27	2											
10×10	659	272	60	9											
20×20	314	403	196	73	12	2									
40×40	29	189	318	257	135	54	8	6	3	1					
60×60	2	36	118	240	260	171	106	40	19	6	1	1			
80×80		3	34	100	183	216	186	138	85	36	12	6	1		
100×100		1	9	30	69	160	189	190	149	92	55	32	17	6	1
5×5	725	241	32	2											
7×7	656	280	54	9	1										
10×10	518	335	120	22	5										
20×20	175	326	315	123	51	10									
40×40	9	72	206	271	205	132	57	33	11	1	3				
60×60		10	33	102	169	224	192	126	87	28	12	5	1	1	
80×80			5	24	55	106	148	196	178	122	75	49	21	14	5
100×100				2	11	30	98	123	147	169	146	94	77	52	29

to check this we have generated two more test sets consisting of 1000 uniformly chosen discrete sets of sizes 200×200 and 500×500 with nonempty rows and columns (the generation of this latter set took about half a day). Figure 3 shows the differences between the test results and the normal distributions with the estimated parameters.

4.3 The Number of Decomposable Configurations

Given an ordered pair of binary matrices (C, D) we say that we get the binary matrix F by *NorthWest gluing* (or shortly, NW-gluing) C to D if

$$F = \begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix} .$$

NE-, SE- and SW-gluings are defined similarly. Then, given a binary matrix F consisting of $k \geq 2$ components we say that the components are in a *decomposable configuration* if the following properties hold

- the sets of the row and column indices of the components are disjoint, and
- if $k > 2$ then we get F by gluing a single component to a binary matrix having a decomposable configuration consisting of $k - 1$ components using one of the four gluing operators.

For example, the four components of the matrix depicted in Fig. 1 are in a non-decomposable configuration. Decomposability was introduced in [1] as a new

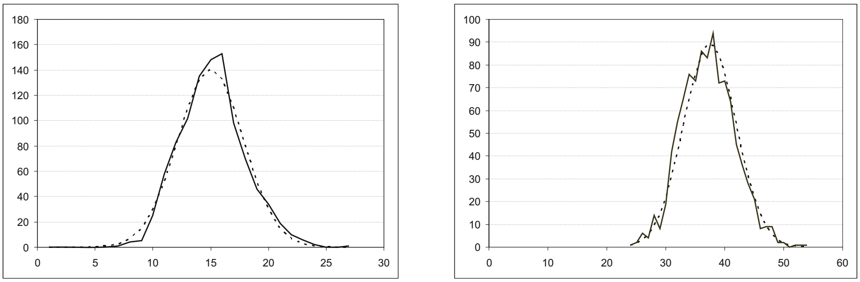


Fig. 3. The number of sets (*vertical axis*) having given number of components (*horizontal axis*) in the test data (*solid lines*) and the corresponding normal distribution (*dashed lines*) for sets of sizes 200×200 (*left*) and 500×500 (*right*)

property that can guarantee polynomial-time reconstruction from four projections. Later, in [2] it was also shown how the knowledge that the components of an *hv*-convex discrete set form a decomposable configuration can facilitate the reconstruction in the case of four projections. In our third experiment we tested whether decomposable configurations often occur in the class of *hv*-convex sets. In order to test this we also generated a uniformly chosen random permutation π of order k for each previously generated test data if the set consisted of $k \geq 2$ components. Then, we permuted the set of the column indices of the components in that test data according to the generated permutation π. Although in this way we did not get a uniform distribution in the whole class of *hv*-convexes a useful estimation could be made about the number of decomposable configurations. Table 3 represents the number of decomposable configurations for each set of test data having size up to 100×100 if empty rows and columns are not present ($N(dec)$) and if empty rows and columns are permitted ($N'(dec)$).

Table 3. The number of decomposable configurations from 1000 *hv*-convex sets of the given size $n \times n$

Size $n \times n$	2	3	4	5	7	10	20	40	60	80	100
$N(dec)$	163	166	160	214	254	340	669	826	599	344	160
$N'(dec)$	176	208	230	275	343	477	777	673	354	124	45

From this study we can see that decomposition reconstruction algorithms (see [1,2]) have the best chance to succeed on the generated test sets if the size of the set is between about 10×10 and 80×80 if the set has no empty rows and columns, and 7×7 and 60×60 if empty rows and columns are possibly present. From the definition it follows that every discrete set consisting of a single component is non-decomposable. Moreover, if a discrete set has two or three components then they necessarily form a decomposable configuration. Finally, the more component the discrete set has the less likely that the components are in a decomposable configuration. Thus, our statistic corresponds to the previous one about the expected number of components.

5 Conclusions and Discussion

We have developed a technique for generating special hv-convex binary matrices from uniform random distribution. Then, several statistics are given that can be useful in the complexity analysis of reconstruction algorithms. In a forthcoming contribution we plan to do an empirical investigation of the effectiveness of several reconstruction algorithms for hv-convex discrete sets that could be valuable in designing further more efficient reconstruction algorithms.

The main advantage of the developed method is that it can be applied for any class of matrices having disjoint components if the components themselves can be generated from uniform random distribution knowing their bounding rectangles and if it is possible to enumerate them. For a simple example let us assume that the discrete set to be generated does not have empty rows and columns and all the components are rectangles. However, we do not assume that the components are in a special configuration, we only suppose that the sets of their row and column indices are disjoint. Let $R^{(t)}_{m,n}$ denote the number of such discrete sets of size $m \times n$ having exactly t components. For each i and j there exists exactly one discrete rectangle of size $i \times j$, i.e., $R^{(1)}_{i,j} = 1$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Moreover, $R^{(t)}_{i,j} = 0$ if $i < t$ or $j < t$, and for $t > 1$ the following recursive formula holds

$$R^{(t)}_{m,n} = \sum_{i<m,\ j<n} R^{(t-1)}_{i,j} \cdot t \tag{7}$$

where the factor t represents the proper weights for describing the possible permutations of the column sets of the t components. Then, the total number of such discrete sets of size $m \times n$ is $\sum_{t=1}^{\min\{m,n\}} R^{(t)}_{m,n}$ and the given algorithm can be modified in a straightforward way (see [3] for further details).

In particular, the above method can also be extended to the whole class of hv-convexes, too (see again [3]). Let $Q^{(t)}_{m,n}$ denote the number of arbitrary hv-convex discrete sets with minimal bounding rectangle of size $m \times n$ with nonempty rows and columns and having exactly t components. Then $Q^{(t)}_{i,j} = 0$ if $i < t$ or $j < t$, and $Q^{(1)}_{i,j} = P_{i,j}$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Finally, for $t > 1$ the following recursive formula holds

$$Q^{(t)}_{m,n} = \sum_{k<m,\ l<n} P_{k,l} \cdot Q^{(t-1)}_{m-k,n-l} \cdot t\ . \tag{8}$$

Then, we get that the total number of arbitrary hv-convex discrete sets of size $m \times n$ with nonempty rows and columns is $\sum_{t=1}^{\min\{m,n\}} Q^{(t)}_{m,n}$. However, due to its huge computational complexity this generation method is applicable for discrete sets of moderate sizes only. Although several useful statistics could be done even in the classes $\mathcal{S}$ and $\mathcal{S}'$ it is an important open question whether more sophisticated and more efficient generation techniques for the whole class of *hv*-convexes can be developed.

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