On the Realizable Weaving Patterns of Polynomial Curves in $\mathbb{R}^3$

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Abstract. We prove that the number of distinct weaving patterns produced by n semi-algebraic curves in $\mathbb{R}^3$ defined coordinate-wise by polynomials of degrees bounded by some constant d, is bounded by $2^{O(n \log n)}$, where the implied constant in the exponent depends on d. This generalizes a similar bound obtained by Pach, Pollack and Welzl [3] for the case when $d = 1$.

1 Introduction

In [3], Pach, Pollack and Welzl considered weaving patterns of n lines in $\mathbb{R}^3$ and showed that asymptotically only a negligible fraction of possible weaving patterns are realizable by straight lines in $\mathbb{R}^3$ (see Remark 2 below). In this paper, we consider weaving patterns produced by polynomial curves in $\mathbb{R}^3$. Since, such curves are much more flexible than lines, it is reasonable to expect a much bigger number of realizable weaving patterns. In this paper, we prove that the number of distinct weaving patterns, realized by polynomial curves with degrees bounded by some constant d, is still asymptotically negligible.

Crossing patterns of semi-algebraic sets of fixed description complexity were considered in [1], where Ramsey type results are proved for such arrangements. However, since semi-algebraic curves in $\mathbb{R}^3$ (unlike lines) need not satisfy simple above-below relationships and can intertwine in complicated ways, it is not immediately clear whether the framework in [1] is applicable in our setting.

The rest of the paper is organized as follows. In Section 2, we define weaving patterns for polynomial curves and state the main result of the paper (Theorem 1). Since, the projections to the plane of curves defined by polynomials in $\mathbb{R}^3$ can have complicated patterns of intersection, defining what is meant by a weaving pattern for such curves requires some care. In Section 3, we recall some basic facts from [2]. The main tools used in the proof of Theorem 1, are Cylindrical Algebraic Decomposition, and a bound on the number of connected components of the realizations of all realizable sign conditions on a family of polynomials

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(see Theorem 2). We give here the basic definitions, and state the results that we need, referring the reader to [2] for details. In Section 4 we prove Theorem 1. Finally, in Section 5 we compare the number of weaving patterns realizable by polynomial curves of fixed degrees with the total number of weaving patterns.

2 Weaving Patterns in $\mathbb{R}^3$

Let $\gamma_1, \ldots, \gamma_n : (-\infty, \infty) \to \mathbb{R}^3$ be n semi-algebraic curves given by

$$\gamma_i(s) = (x_i(s), y_i(s), z_i(s)),\ 1 \leq i \leq n,$$

where x_i, y_i, z_i are polynomials whose degrees are bounded by d. We will assume that the curves are not self-intersecting in $\mathbb{R}^3$ and the images of γ_i and γ_j do not intersect, unless $i = j$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection sending $(x, y, z) \mapsto (x, y)$. For $1 \leq i < j \leq n$, let

$$M_{ij} = \{\pi(\gamma_i(s_{ij}^1)), \ldots, \pi(\gamma_i(s_{ij}^{\ell_{ij}}))\} \subset \mathbb{R}^2,\ \ s_{ij}^1 < \cdots < s_{ij}^{\ell_{ij}},$$

denote the finite set of ℓ_{ij} isolated points of intersections of $\pi(\text{image}(\gamma_i))$ and $\pi(\text{image}(\gamma_j))$. Also, let

$$M_{ii} = \{\pi(\gamma_i(s_{ii}^1)), \ldots, \pi(\gamma_i(s_{ii}^{\ell_{ii}}))\} \subset \mathbb{R}^2,\ \ s_{ii}^1 < \cdots < s_{ii}^{\ell_{ii}},$$

and such that $\pi(\gamma_i(s_{ii}^k)) = \pi(\gamma_i(s)), s \neq s_{ii}^k \Rightarrow s > s_{ii}^k$.

We assume that each of the intersection points correspond to a normal crossing. In particular, for $p \in M_{ij}$ (respectively, $p \in M_{ii}$) $\pi^{-1}(p) \cap (\text{image}(\gamma_i)\ \cup \text{image}(\gamma_j))$ (respectively, $\pi^{-1}(p)\ \cap\ \text{image}(\gamma_i)$) consists of exactly two points. This is not a very strong assumption, since for every finite family of smooth algebraic curves, almost all linear projections, π, satisfy these assumptions. The set of bad projections is a Zariski closed subset in the space of all linear projections.

For $1 \leq i < j \leq n$, and $1 \leq k \leq \ell_{ij}$, we define $V_{ij}^k \in \{+1, -1\}$ in the following way.

$$\begin{aligned} V_{ij}^k &= +1 \ \text{ if } z_i(s_{ij}^k) > z_j(s) \text{ where } s \in \mathbb{R} \text{ is such that } \pi(\gamma_j(s)) = \pi(\gamma_i(s_{ij}^k)), \\ &= -1 \ \text{ else.} \end{aligned}$$

In other words, V_{ij}^k is $+1$ if $\text{image}(\gamma_i)$ lies *above* $\text{image}(\gamma_j)$ over $\pi(\gamma_i(s_{ij}^k))$, which is a point of intersection of the projections of the images of the two curves, γ_i, γ_j, to the XY-plane.

Similarly, we define for each $1 \leq i \leq n$, and $1 \leq k \leq \ell_{ii}$, $V_{ii}^k \in \{+1, -1\}$ as follows.

$$\begin{aligned} V_{ii}^k &= +1 \ \text{ if } z_i(s_{ii}^k) > z_j(s) \text{ where } s \neq s_{ii}^k \text{ is such that } \pi(\gamma_i(s)) = \pi(\gamma_i(s_{ij}^k)), \\ &= -1 \ \text{ else.} \end{aligned}$$

Now consider the union of the projections of the images of the curves, namely

$$\pi(\text{image}(\gamma_1)), \ldots, \pi(\text{image}(\gamma_n)),$$

as a planar embedding of a planar graph (self loops allowed), whose vertices are at the points, $M_{ij}^k, 1 \leq i \leq j \leq n, 1 \leq k \leq \ell_{ij}$, and whose edges are the various curve segments joining the vertices. Two such graph embeddings are said to be equivalent, if one can be mapped to the other by a homeomorphism of the plane. Given an ordered set of curves, $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, satisfying the assumptions stated above, we denote by $G(\Gamma)$ the equivalence class of the corresponding embedded graph in the XY-plane. Finally, we call $G(\Gamma)$ along with the labeling of each of its vertex, M_{ij}^k by $V_{ij}^k \in \{+1, -1\}$, to be the weaving pattern produced by Γ.

In this paper we address the following question. How many distinct weaving patterns can be produced by n algebraic curves, $\gamma_1, \ldots, \gamma_n : (-\infty, \infty) \to \mathbb{R}^3$ where $\gamma_i(s) = (x_i(s), y_i(s), z_i(s))$, and x_i, y_i, z_i are polynomials whose degrees are bounded by some constant d ?

We prove the following theorem.

Theorem 1. *The number of distinct weaving patterns produced by Γ is bounded by $2^{O(n \log n)}$, where the constant in the exponent depends on d.*

This generalizes the bound proved in [3], which is the special case when $d = 1$. Also, note that $\pi(\text{image}(\gamma_1)), \ldots, \pi(\text{image}(\gamma_n))$, can have $\binom{n}{2}d^2$ crossing points and hence the number of possible weaving patterns could be potentially as large as $2^{\binom{n}{2}d^2}$. However, its clear from Theorem 1 only a negligible fraction of these are realizable by curves defined by polynomials with degrees bounded by d.

3 Preliminaries

In this section, we recall a few notions from semi-algebraic geometry that we will need in the proof of Theorem 1. More details, including proofs of the results stated below, can be found in [2].

3.1 Realizable Sign Conditions and Associated Bounds

A *sign condition* is an element of $\{0, 1, -1\}$. We denote for $x \in \mathbb{R}$

$$\begin{cases} \text{sign}(x) = 0 & \textbf{iff } x = 0, \\ \text{sign}(x) = 1 & \textbf{iff } x > 0, \\ \text{sign}(x) = -1 & \textbf{iff } x < 0. \end{cases}$$

Let $\mathcal{Q} \subset \mathbb{R}[X_1, \ldots, X_k]$, A *sign condition* on $\mathcal{Q}$ is an element of $\{0, 1, -1\}^{\mathcal{Q}}$. We say that $\mathcal{Q}$ *realizes* the sign condition σ at $x \in \mathbb{R}^k$ if

$$\bigwedge_{Q \in \mathcal{Q}} \text{sign}(Q(x)) = \sigma(Q).$$

The *realization of the sign condition* σ is

$$\mathcal{R}(\sigma) = \{x \in \mathbb{R}^k \mid \bigwedge_{Q \in \mathcal{Q}} \text{sign}(Q(x)) = \sigma(Q)\}.$$

The sign condition σ is *realizable* if $\mathcal{R}(\sigma)$ is non-empty. The set $\mathrm{Sign}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$ is the set of all realizable sign conditions for $\mathcal{Q}$ over $\mathbb{R}^k$.

For $\sigma \in \mathrm{Sign}(\mathcal{Q})$, let $b_0(\sigma)$ denote the number of connected components of

$$\mathcal{R}(\sigma) = \{x \in \mathbb{R}^k \mid \bigwedge_{Q \in \mathcal{Q}} \mathrm{sign}(Q(x)) = \sigma(Q)\}.$$

Let $b_0(\mathcal{Q}) = \sum_\sigma b_0(\sigma)$. We write $b_0(d, k, s)$ for the maximum of $b_0(\mathcal{Q})$ over all $\mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}[X_1, \ldots, X_k]$ whose elements have degree at most d, $\#(\mathcal{Q}) = s$.

The following theorem [2] gives an upper bound on $b_0(d, k, s)$ which we will use later in the paper.

Theorem 2.

$$b_0(d, k, s) \leq \sum_{1 \leq j \leq k} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

3.2 Cylindrical Decomposition

Cylindrical Algebraic Decomposition is a classical tool used in the study of, as well as in algorithms for computing, topological properties of semi-algebraic sets. We give here the basic definitions and properties of Cylindrical Algebraic Decomposition referring the reader to [2] for greater details.

A *cylindrical decomposition* of $\mathbb{R}^k$ is a sequence $\mathcal{S}_1, \ldots, \mathcal{S}_k$ where, for each $1 \leq i \leq k$, $\mathcal{S}_i$ is a finite partition of $\mathbb{R}^i$ into semi-algebraic subsets, called the *cells of level* i, which satisfy the following properties:

Each cell $S \in \mathcal{S}_1$ is either a point or an open interval.

For every $1 \leq i < k$ and every $S \in \mathcal{S}_i$, there are finitely many continuous semi-algebraic functions

$$\xi_{S,1} < \ldots < \xi_{S,\ell_S} : S \longrightarrow \mathbb{R}$$

such that the cylinder $S \times \mathbb{R} \subset \mathbb{R}^{i+1}$ is the disjoint union of cells of $\mathcal{S}_{i+1}$ which are:

either the graph of one of the functions $\xi_{S,j}$, for $j = 1, \ldots, \ell_S$:

$$\{(x', x_{j+1}) \in S \times \mathbb{R} \mid x_{j+1} = \xi_{S,j}(x')\} ,$$

or a band of the cylinder bounded from below and from above by the graphs of the functions $\xi_{S,j}$ and $\xi_{S,j+1}$, for $j = 0, \ldots, \ell_S$, where we take $\xi_{S,0} = -\infty$ and $\xi_{i,\ell_S+1} = +\infty$:

$$\{(x', x_{j+1}) \in S \times \mathbb{R} \mid \xi_{S,j}(x') < x_{j+1} < \xi_{S,j+1}(x')\} .$$

A *cylindrical decomposition adapted to a finite family of semi-algebraic sets* $T_1, \ldots, T_\ell$ is a cylindrical decomposition of $\mathbb{R}^k$ such that every T_i is a union of cells.

Given a finite set $\mathcal{P}$ of polynomials in $\mathbb{R}[X_1,\ldots,X_k]$, a subset S of $\mathbb{R}^k$ is $\mathcal{P}$-*semi-algebraic* if S is the realization of a quantifier free formula with atoms $P=0$, $P>0$ or $P<0$ with $P\in\mathcal{P}$. A subset S of $\mathbb{R}^k$ is $\mathcal{P}$-*invariant* if every polynomial $P\in\mathcal{P}$ has a constant sign (>0, <0, or $=0$) on S. A *cylindrical decomposition of* $\mathbb{R}^k$ *adapted to* $\mathcal{P}$ is a cylindrical decomposition for which each cell $C\in\mathcal{S}_k$ is $\mathcal{P}$-invariant. It is clear that if S is $\mathcal{P}$-semi-algebraic, a cylindrical decomposition adapted to $\mathcal{P}$ is a cylindrical decomposition adapted to S.

Given a family of polynomials $\mathcal{P}\subset\mathbb{R}[X_1,\ldots,X_k]$, there exists another family of polynomials $\mathrm{Elim}_{X_k}(\mathcal{P})$ (see [2], page 145, for the precise definition of Elim) having the following property.

We denote, for $i=k-1,\ldots,1$,

$$\mathrm{C}_i(\mathcal{P})=\mathrm{Elim}_{X_{i+1}}(\mathrm{C}_{i+1}(\mathcal{P})),$$

with $\mathrm{C}_k(\mathcal{P})=\mathcal{P}$, so that

$$\mathrm{C}_i(\mathcal{P})\subset\mathbb{R}[X_1,\ldots,X_i].$$

The semi-algebraically connected components of the sign conditions on the family,

$$\mathrm{C}(\mathcal{P})=\cup_{i\le k}\mathrm{C}_i(\mathcal{P})$$

are the cells of a cylindrical decomposition adapted to $\mathcal{P}$. We call $\mathrm{C}(\mathcal{P})$ the *cylindrifying family of polynomials associated to* $\mathcal{P}$.

Moreover, if s is a bound on $\#(\mathcal{P})$, and d a bound on the degrees of the elements of $\mathcal{P}$, $\#(\mathrm{Elim}_{X_k}(\mathcal{P}))$ is bounded by $O(s^2d^3)$. Moreover, the the degrees of the polynomials in $\mathrm{Elim}_{X_k}(\mathcal{P})$ with respect to $X_1,\ldots,X_{k-1}$ is bounded by $2d^2$.

Remark 1. The set $\mathrm{C}_i(\mathcal{P})$ has the following additional property. For $\sigma\in\mathrm{sign}(\mathrm{C}_i(\mathcal{P}))$ and C a connected component of $\mathcal{R}(\sigma,\mathbb{R}^i)$, for each $x=(x_1,...,x_i)\in C$, the family $\displaystyle\bigcup_{i<j\le k}\mathrm{C}_j(\mathcal{P})(x_1,...,x_i)$ is the cylindrifying family of polynomials associated to $\mathcal{P}(x_1,...,x_i)$, and moreover the induced cylindrical decompositions *have the same structure.* More precisely, this means that there is a 1-1 correspondence between the cylindrical cells as x varies over C.

4 Proof of the Main Result

For $1\le i\le n$, let

$$P_i=\sum_{j=0}^{d}A_{i,j}T_i^j\in\mathbb{R}[\bar{A}_i,T_i],$$

$$Q_i=\sum_{j=0}^{d}B_{i,j}T_i^j\in\mathbb{R}[\bar{B}_i,T_i],$$

$$R_i=\sum_{j=0}^{d}C_{i,j}T_i^j\in\mathbb{R}[\bar{C}_i,T_i],$$

where we denote by $\bar{A}_i$ (respectively, $\bar{B}_i, \bar{C}_i$) the vector of variables, $(A_{i,0}, ..., A_{i,d})$ (respectively, $(B_{i,0}, \ldots, B_{i,d}), (C_{i,0}, \ldots, C_{i,d})$).

Similarly, we denote by $\bar{A}$ (respectively, $\bar{B}, \bar{C}$) the vector of variables,

$$(A_{1,0}, \ldots, A_{1,d}, \ldots, A_{n,0}, \ldots, A_{n,d})$$

(respectively, $(B_{1,0}, ..., B_{1,d}, ..., B_{n,0}, ..., B_{n,d}), (C_{1,0}, ..., C_{1,d}, ..., C_{n,0}, ..., C_{n,d})$). We denote by γ_i the triple (P_i, Q_i, R_i). For fixed values $(\bar{a}_i, \bar{b}_i, \bar{c}_i)$, the triples $\gamma_i(\bar{a}_i, \bar{b}_i, \bar{c}_i) = (P_i(\bar{a}_i, T_i), Q_i(\bar{b}_i, T_i), R_i(\bar{c}_i, T_i)), 1 \leq i \leq n$ gives rise to an ordered set of curves in $\mathbb{R}^3$, which we denote by $\Gamma(\bar{a}, \bar{b}, \bar{c})$. Let $WP(\bar{a}, \bar{b}, \bar{c})$ denote the weaving pattern produced by $\Gamma(\bar{a}, \bar{b}, \bar{c})$. We want to bound the cardinality of the set,

$$\{WP(\bar{a}, \bar{b}, \bar{c}) \mid (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^{3(d+1)n}\}.$$

Now, consider the following family of polynomials:

$$\mathcal{A}_i = \{X - P_i(\bar{A}_i, T_i), Y - Q_i(\bar{B}_i, T_i), Z - R_i(\bar{C}_i, T_i)\} \subset \mathbb{R}[\bar{A}_i, \bar{B}_i, \bar{C}_i, X, Y, Z, T_i].$$

Let $\mathcal{B}_i = \mathrm{Elim}_{T_i}(\mathcal{A}_i) \subset \mathbb{R}[\bar{A}_i, \bar{B}_i, \bar{C}_i, X, Y, Z]$, and let

$$\mathcal{B} = \bigcup_{1 \leq i \leq n} \mathcal{B}_i.$$

Notice, if we specialize $(\bar{A}_i, \bar{B}_i, \bar{C}_i)$ to some $(\bar{a}_i, \bar{b}_i, \bar{c}_i) \in \mathbb{R}^{3(d+1)}$, the image of the curve $\gamma_i(\bar{a}_i, \bar{b}_i, \bar{c}_i) \in \mathbb{R}^3$ is a $\mathcal{B}_i(\bar{a}_i, \bar{b}_i, \bar{c}_i)$-semi-algebraic set.

The following proposition relates the weaving pattern, $WP(\bar{a}, \bar{b}, \bar{c})$ to a cylindrical decomposition of R^3 adapted to the family $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$.

Proposition 1. *Let $(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^{3(d+1)n}$. The weaving pattern, $WP(\bar{a}, \bar{b}, \bar{c})$ is determined by the cylindrical decomposition induced by the cylindrifying family of polynomials associated to $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$.*

In particular, if two points $(\bar{a}, \bar{b}, \bar{c}), (\bar{a}', \bar{b}', \bar{c}') \in \mathbb{R}^{3(d+1)n}$, are such that the cylindrical decompositions induced by the cylindrifying families of polynomials associated to $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$ and $\mathcal{B}(\bar{a}', \bar{b}', \bar{c}')$ have the same structure, then $WP(\bar{a}, \bar{b}, \bar{c}) = WP(\bar{a}', \bar{b}', \bar{c}')$.

Proof. The proposition is a consequence of the definition of weaving pattern, the definition of cylindrifying families of polynomials, and the fact that the images of the curves, $\gamma_i(\bar{a}_i, \bar{b}_i, \bar{c}_i)$, are all $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$-semi-algebraic sets.

We now prove Theorem 1.

Proof. Let $\mathcal{C}_1 = \mathrm{Elim}_Z(\mathcal{B}), \mathcal{C}_2 = \mathrm{Elim}_Y(\mathcal{C}_1),$ and $\mathcal{C}_3 = \mathrm{Elim}_X(\mathcal{C}_2)$.

The set $\mathcal{C}_3$ has the following property which is a consequence of Remark 1 in Section 3. Let C be a connected component of the realization of a realizable sign condition of $\mathcal{C}_3$. Then, for each $(\bar{a}, \bar{b}, \bar{c}) \in C$, $\mathcal{B}(\bar{a}, \bar{b}, \bar{c}) \cup \mathcal{C}_1(\bar{a}, \bar{b}, \bar{c}) \cup \mathcal{C}_2(\bar{a}, \bar{b}, \bar{c})$ is the cylindrifying family of polynomials associated to $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$ and moreover the cylindrical decompositions induced have the same structure as $(\bar{a}, \bar{b}, \bar{c})$ varies over C.

Since, by Proposition 1, for fixed $(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^{3(d+1)n}$ the weaving pattern of $\Gamma(\bar{a}, \bar{b}, \bar{c})$ is determined by any Cylindrical Decomposition of $\mathbb{R}^3$ adapted to $\mathcal{B}(\bar{a}, \bar{b}, \bar{c})$, then by the previous observation, the number of distinct weaving patterns is clearly bounded by $b_0(\mathcal{C}_3)$, which we now proceed to bound from above.

From the bounds stated in Section 3, we have that for $1 \leq i \leq n$, $\#(\mathcal{B}_i) = O(d^3)$ and the degrees of the polynomials in $\mathcal{B}_i$ are bounded by $O(d^2)$. Hence, $\#(\mathcal{B}) = O(nd^3)$. Since, $\mathcal{C}_3$ is obtained from $\mathcal{B}$ after three successive Elim operations, we get that, $\#\mathcal{C}_3 = (nd)^{O(1)}$ and the degrees of the polynomials in $\mathcal{C}_3$ is bounded by $d^{O(1)})$. The number of variables in the polynomials in $\mathcal{C}_3$ is $3(d+1)n$. Using the bound in Theorem 2, we get that $b_0(\mathcal{C}_2)$ is bounded by

$$(nd)^{O(dn)} = 2^{O(n \log n)}.$$

5 Most Weaving Patterns Are Not Realizable

We have the following theorem which generalizes Theorem 3 in [3].

Theorem 3. *The number of weaving patterns realizable by polynomial curves of degrees bounded by a constant, divided by the total number of weaving patterns of n curves whose projections are allowed to intersect at most a constant number of times, tends to 0 exponentially fast, as $n \to \infty$.*

Proof. By Theorem 1, the number of distinct weaving patterns produced by such curves is bounded by $2^{O(n \log n)}$. On the other hand, considering n lines in the plane in general position, and counting all possible ways of labeling the $\binom{n}{2}$ crossings, we see that there are at least $2^{\binom{n}{2}}$ possible weaving patterns.

Remark 2. The proof of the upper bound in Theorem 3 in [3] does not seem to consider the fact that the projections of different sets of n lines in $\mathbb{R}^3$ to the plane, can produce arrangements which are combinatorially distinct, and these would produce distinct weaving patterns by definition. In fact, obtaining good control on this number complicates the proof of Theorem 1 in this paper. However, since the number of combinatorially distinct arrangements of n lines in $\mathbb{R}^2$ is still bounded by $2^{O(n \log n)}$, the proof of the theorem in [3] is still valid.

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