Accounting for Nonlinear Aspects in Multiphysics Problems: Application to Poroelasticity

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Abstract. Multiphysics phenomena lead to computationally intensive structural analyses. Recently, a new strategy derived from the LATIN method was described and successfully applied to the consolidation of saturated porous soils.
One of the main achievements was the use of the LATIN method to take into account the different time scales which usually arise from the different physics: a multi-time-scale strategy was proposed.
Here we go one step further and use the LATIN method to deal with some of the classical nonlinearities of poroelasticity problems (such as non-constant stiffness and permeability) and we show that these phenomena do not result in a significant increase of the computation cost.

1 Introduction

For coupled multiphysics problems such as fluid-structure interaction, partitioned procedures and staggered algorithms are often preferred, from the point of view of computational efficiency, to direct analysis (also called the monolithic approach). Moreover, partitioning strategies enable one to use different analyzers for different subsystems, and help keep the software manageable.

Recently, an approach suitable for multiphysics problems was developed based on the LArge Time INcrement method (LATIN) [1] and applied to the consolidation of saturated porous soils, which is a typical example of a highly coupled fluid-structure interaction problem. The term *consolidation* designates the slow deformation of the solid phase accompanied by flow of the pore fluid. One of the consequences of natural consolidation is surface subsidence, i.e. the lowering of the Earth's surface. The consolidation analysis of soils has long been recognized as an important problem in civil engineering design [2].

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The principles of the LATIN method and examples of its applicability to such coupled multiphysics problems were given in [3]. This strategy was compared to the Iterated Standard Parallel Procedure (ISPP) [4], which is one of the standard partitioning schemes. In [5], a multi-time-scale strategy was proposed in order to improve the LATIN procedure by taking into account the different time scales. An *ad hoc* radial loading approximation for both kinematic and static quantities was also set up in order to increase the modularity of the approach and to reduce the storage cost. Here, we go one step further and use the LATIN method to deal with some of the classical nonlinearities of poroelasticity problems (such as non-constant stiffness and permeability).

2 The Reference Problem

Let us briefly describe a typical consolidation problem [6]. A structure Ω is made of a saturated porous material undergoing small perturbations and isothermal evolution over the time interval $[0,T]$ being studied.

The loading consists of a prescribed displacement $\underline{U}_d$ on a part $\partial_1\Omega$ of the boundary, a traction force $\underline{F}_d$ on the complementary part $\partial_2\Omega$ of $\partial_1\Omega$, a fluid flux w_d on another part $\partial_3\Omega$ of the boundary and, finally, a prescribed pore pressure p_d on the complementary part $\partial_4\Omega$ of $\partial_3\Omega$. For the sake of simplicity, we assume that there are no body forces.

For solid quantities, strain and stress are denoted $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ respectively; for fluid quantities, the pore pressure gradient is denoted $\underline{Z}$ and the opposite of Darcy's velocity $\underline{W}$; finally, q denotes the rate of fluid mass accumulation in each representative elementary volume.

The state of the structure is given by the set of the fields $\mathbf{s} = (\boldsymbol{\varepsilon}, p, \underline{Z}, \boldsymbol{\sigma}, q, \underline{W})$ defined on the whole structure Ω and over the time interval $[0,T]$ being considered. The problem consists in finding $\mathbf{s}$ in the corresponding space $\mathbf{S}^{[0,T]}$ which verifies at each time step the following equations:

- in the solid, compatibility of strains $\boldsymbol{\varepsilon}$ and equilibrium of stresses $\boldsymbol{\sigma}$:

$$\begin{aligned} \underline{U} \in \mathcal{U}^{[0,T]} \quad \text{and} \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\underline{U}) \quad \text{on } \Omega \\ \underline{\operatorname{div}}\boldsymbol{\sigma} = \underline{0} \quad \text{on } \Omega \quad \text{and} \quad \boldsymbol{\sigma}\underline{n} = \underline{F}_d \quad \text{on } \partial_2\Omega \end{aligned} \tag{1}$$

  $\mathcal{U}^{[0,T]}$ being the set of the finite-energy displacement fields on $\Omega\times[0,T]$ equal to $\underline{U}_d$ on $\partial_1\Omega$;
- in the fluid, flow conservation for Darcy's velocity $-\underline{W}$:

$$\begin{aligned} p \in \mathcal{P}^{[0,T]} \quad \text{and} \quad \underline{Z} = \underline{\operatorname{grad}}\, p \quad \text{on } \Omega \\ q = \operatorname{div}\underline{W} \quad \text{on } \Omega \quad \text{and} \quad \underline{W}\cdot\underline{n} = w_d \quad \text{on } \partial_4\Omega \end{aligned} \tag{2}$$

  $\mathcal{P}^{[0,T]}$ being the set of the finite-energy pressure fields on $\Omega\times[0,T]$ equal to p_d on $\partial_3\Omega$;
- the constitutive relations:

- Hooke's law, which relates the macroscopic stress $\boldsymbol{\sigma}$ to the strain $\boldsymbol{\varepsilon}$ and the pore pressure p so that:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} - bp\mathbf{I} \tag{3}$$

- Darcy's law, which relates Darcy's velocity to the pore pressure gradient:

$$\underline{W} = \frac{K}{\mu_w}\underline{Z} \tag{4}$$

- compressibility, which relates the fluid accumulation rate to the pressure rate and couples it with the rate of volume modification:

$$q = \frac{1}{Q}\dot{p} + b\,\mathrm{Tr}\,\dot{\boldsymbol{\varepsilon}} \tag{5}$$

$\mathbf{D}$ is Hooke's tensor of the drained skeleton, b is Biot's coefficient, K is the intrinsic macroscopic permeability and μ_w is the dynamic viscosity of the saturation fluid. Throughout the following sections, the operator $\frac{K}{\mu_w}\mathbf{I}$ will be designated by $\mathbf{H}$. Finally, Q is Biot's modulus.

3 Nonlinear Behavior

Most of the consolidation problems which have been analyzed so far are limited to the assumption of linear elastic constitutive behavior and constant permeability, but in most geotechnical situations the behavior of the soil is nonlinear. Following Kondner and his co-workers [7], the stress-strain curves for both clay and sand in a conventional triaxial compression test (constant σ_3) may be approximated by a hyperbolic equation of the form:

$$\sigma_1 - \sigma_3 = \frac{\varepsilon_1}{A + B\varepsilon_1} \tag{6}$$

which relates the difference between the major principal stress σ_1 and the minor principal stress σ_3 to the major principal strain ε_1. A and B are material constants which can be determined experimentally. Then, Hooke's law is defined by:

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon})\boldsymbol{\varepsilon} - bp\mathbf{I} \tag{7}$$

However, Kondner's model (6) is available only for one-dimensional analysis. This is the case of the following numerical test. There is also evidence that the intrinsic permeability is not constant, even in the case of full saturation. It seems reasonable [2] to assume a dependency of the permeability on the void ratio (or porosity) as well as on the deformation. We propose to test the LATIN method on a variation of one of the laws given in [8] for the intrinsic permeability:

$$K(\boldsymbol{\varepsilon}) = K_0 \frac{n_0}{1+n_0}\left(1 + \frac{1}{n_0}\left\langle \frac{\mathrm{Tr}\,\boldsymbol{\varepsilon} - \mathrm{Tr}\,\boldsymbol{\varepsilon}_0}{-\mathrm{Tr}\,\boldsymbol{\varepsilon}_0}\right\rangle_+^{\alpha}\right) \tag{8}$$

where $\langle\cdot\rangle_+$ denotes the positive part, K_0 and n_0 the initial intrinsic permeability and porosity, $\boldsymbol{\varepsilon}_0$ the strain below which the intrinsic permeability cannot decrease (typically $\mathrm{Tr}\,\boldsymbol{\varepsilon}_0 = -n_0$), and α a material constant. Darcy's law is then defined by:

$$\underline{W} = \mathbf{H}(\varepsilon)\underline{Z} \tag{9}$$

Thus, the consolidation problem which is to be simulated is nonlinear.

4 The LATIN Method for Multiphysics Problems

The LATIN method is a nonincremental iterative approach originally designed for nonlinear time-dependent problems [1]. For coupled multiphysics problems, the method consists in extending the notion of *material interface* (between substructures) [9] to that of *multiphysics interface*. Such an interface must take into account the coupling between the constitutive relations. The development of this strategy was described in [3] and only the main principles will be reviewed here.

At each iteration, the LATIN method produces an approximation of the solution over the whole domain and over the entire time interval being studied. The method is based on three principles:

- **The first principle** consists in separating the difficulties. For coupled field problems, a first set of equations, $\mathbf{A_d}$, containing the so-called admissibility conditions is defined. In order to avoid dealing with both a global and a coupled problem simultaneously, the remaining equations are grouped into a second set of equations, $\boldsymbol{\Gamma}$; these equations, which are local in the space variables, are the constitutive relations.
- **The second principle** of the method consists in using search directions to build approximate solutions of $\mathbf{A_d}$ and $\boldsymbol{\Gamma}$ alternatively until a sufficient convergence level has been reached. Each iteration consists of 2 stages: once an element $\mathbf{s}_n \in \mathbf{A_d}$ is known, the *local stage* of iteration $n+1$ uses an initial search direction E^+ to provide an element $\hat{\mathbf{s}}_{n+1/2} \in \boldsymbol{\Gamma}$:

$$\begin{aligned}(\hat{\boldsymbol{\sigma}}_{n+1/2} - \boldsymbol{\sigma}_n) + \mathbf{L}_{n-1/2}(\hat{\dot{\boldsymbol{\varepsilon}}}_{n+1/2} - \dot{\boldsymbol{\varepsilon}}_n) &= 0\\ (\hat{q}_{n+1/2} - q_n) + r(\hat{p}_{n+1/2} - p_n) &= 0\\ (\hat{\underline{W}}_{n+1/2} - \underline{W}_n) + \mathbf{H}_{n-1/2}(\hat{\underline{Z}}_{n+1/2} - \underline{Z}_n) &= 0\end{aligned} \tag{10}$$

$\mathbf{L}_{n-1/2}$, $\mathbf{H}_{n-1/2}$ and r are three parameters of the method; they do not influence the solution once convergence has been reached. However, their values modify the convergence rate of the algorithm. In dimensional analysis, r can be chosen in the form $r = \frac{1}{Qt_h}$, where t_h is an arbitrary characteristic time. The choice of $\mathbf{L}_{n-1/2}$ and $\mathbf{H}_{n-1/2}$ will be discussed below.

At each integration point, using the constitutive relations (3,4,5), the local stage leads to the resolution of a small system of ordinary differential equations in the local space variables:

$$\begin{aligned}\mathbf{L}_{n-1/2}\hat{\hat{\boldsymbol{\varepsilon}}}_{n+1/2} + \mathbf{D}(\hat{\boldsymbol{\varepsilon}}_{n+1/2})\hat{\boldsymbol{\varepsilon}}_{n+1/2} - b\hat{p}_{n+1/2}\mathbf{I} &= \mathbf{A}_n \\ \frac{1}{Q}\hat{\dot{p}}_{n+1/2} + r\hat{p}_{n+1/2} + b\,\mathrm{Tr}\,\hat{\dot{\boldsymbol{\varepsilon}}}_{n+1/2} &= \alpha_n \\ (\mathbf{H}_{n-1/2} + \mathbf{H}(\hat{\boldsymbol{\varepsilon}}_{n+1/2}))\hat{\underline{Z}}_{n+1/2} &= \underline{\beta}_n\end{aligned} \tag{11}$$

where $\mathbf{A}_n = \boldsymbol{\sigma}_n + \mathbf{L}_{n-1/2}\dot{\boldsymbol{\varepsilon}}_n$, $\alpha_n = q_n + rp_n$ and $\underline{\beta}_n = \underline{W}_n + \mathbf{H}_{n-1/2}\underline{Z}_n$ are known quantities from local stage $n+1$, and with the initial conditions on the pressure and strain fields. This nonlinear system (11) is solved using a Newton-type scheme.

Once an element $\hat{\mathbf{s}}_{n+1/2} \in \boldsymbol{\Gamma}$ is known, the *linear stage* provides an element $\mathbf{s}_{n+1} \in \mathbf{A}_{\mathbf{d}}$. $\mathbf{s}_{n+1}$, which must satisfy the admissibility relations, is sought along a search direction E^- conjugate of the previous one, so that the mechanical and hydraulic problems remain uncoupled:

$$\begin{aligned}(\boldsymbol{\sigma}_{n+1} - \hat{\boldsymbol{\sigma}}_{n+1/2}) - \mathbf{L}_{n+1/2}(\dot{\boldsymbol{\varepsilon}}_{n+1} - \hat{\dot{\boldsymbol{\varepsilon}}}_{n+1/2}) &= 0 \\ (q_{n+1} - \hat{q}_{n+1/2}) - r(p_{n+1} - \hat{p}_{n+1/2}) &= 0 \\ (\underline{W}_{n+1} - \underline{\hat{W}}_{n+1/2}) - \mathbf{H}_{n+1/2}(\underline{Z}_{n+1} - \underline{\hat{Z}}_{n+1/2}) &= 0\end{aligned} \tag{12}$$

One can note that the search directions in linear stage n and local stage $n+1$ are conjugates if the parameters of these directions are kept constant.

In order to use a finite element approach, the admissibility of $\mathbf{s}_{n+1}$ is expressed using a variational formulation. On the one hand, this admissibility condition consists in $\underline{U} \in \mathcal{U}^{[0,T]}$ and $\boldsymbol{\sigma} \in \mathcal{S}^{[0,T]}$ such that:

$$\forall t \in [0,T], \quad \forall \underline{U}^\star \in \mathcal{U}_0, \quad \int_\Omega \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}(\underline{U}^\star)]d\Omega = \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^\star dS \tag{13}$$

where $\mathcal{U}_0$ is the set of the finite-energy displacement fields on Ω which vanish on $\partial_1\Omega$. On the other hand, the admissibility condition also consists in $p \in \mathcal{P}^{[0,T]}$, $q \in \mathcal{Q}^{[0,T]}$ and $\underline{W} \in \mathcal{W}^{[0,T]}$ such that:

$$\forall t \in [0,T], \quad \forall p^\star \in \mathcal{P}_0, \quad \int_\Omega (qp^\star + \underline{W} \cdot \underline{\mathrm{grad}}\, p^\star)d\Omega = \int_{\partial_4\Omega} w_d p^\star dS \tag{14}$$

where $\mathcal{P}_0$ is the set of the finite-energy pressure fields on Ω which vanish on $\partial_3\Omega$. Equations (13) and (14) define two uncoupled global problems parameterized by time t.

The convergence of this approach is guaranteed provided that $\mathbf{L}_{n+1/2}$, $\mathbf{H}_{n+1/2}$ and r are positive definite operators which remain constant throughout the iterations [1].

– **The third principle** uses the fact that the successive approximations are defined over both the entire domain and the entire time interval to represent the solution on a radial loading basis. This last point was detailed in [1] and developed, for this particular case, in [3,5]. Briefly, this approach enables one to reduce the number of space fields generated and, therefore, the number of global systems to be solved.

A multi-time-scale strategy was also described in [5]. This strategy enables one to use different time steps for the solid and fluid parts of the problem. In particular, in order to perform an iso-quality simulation (i.e. with identical contributions to the global error) the fluid part requires a smaller time step than the solid.

Choice of the search direction $(\mathbf{L}_{n+1/2}, \mathbf{H}_{n+1/2})$ Many choices, all of which ensure the convergence of the LATIN method, are available [1]. The easiest way is to take a constant search direction:

$$\forall n, \quad \mathbf{L}_{n+1/2} = t_m \mathbf{D}(\hat{\boldsymbol{\varepsilon}} = \mathbb{O}) \quad \text{and} \quad \mathbf{H}_{n+1/2} = \mathbf{H}(\hat{\boldsymbol{\varepsilon}} = \mathbb{O}) = \frac{K_0}{\mu_w}\mathbf{I} \tag{15}$$

where t_m is an arbitrary characteristic time. This choice allows one to assemble operators $\mathbf{L}_{n+1/2}$ and $\mathbf{H}_{n+1/2}$ only once at the beginning of the algorithm. In [1], it was shown that optimal convergence of the method is achieved by using a tangent search direction. In the present case of a multiphysics problem, an approximation of such a tangent direction is:

$$\begin{aligned} (\boldsymbol{\sigma}_{n+1} - \hat{\boldsymbol{\sigma}}_{n+1/2}) - \mathbf{D}(\hat{\boldsymbol{\varepsilon}}_{n+1/2})(\dot{\boldsymbol{\varepsilon}}_{n+1} - \hat{\dot{\boldsymbol{\varepsilon}}}_{n+1/2}) &= 0 \\ (q_{n+1} - \hat{q}_{n+1/2}) - r(p_{n+1} - \hat{p}_{n+1/2}) &= 0 \\ (\underline{W}_{n+1} - \underline{\hat{W}}_{n+1/2}) - \mathbf{H}(\hat{\boldsymbol{\varepsilon}}_{n+1/2})(\underline{Z}_{n+1} - \underline{\hat{Z}}_{n+1/2}) &= 0 \end{aligned} \tag{16}$$

which is equivalent to $\mathbf{L}_{n+1/2}(t) = \mathbf{D}(\hat{\boldsymbol{\varepsilon}}_{n+1/2}(t))$ and $\mathbf{H}_{n+1/2}(t) = \mathbf{H}(\hat{\boldsymbol{\varepsilon}}_{n+1/2}(t))$. Such a choice requires the assembly and factorization of the operators not only at each iteration, but also at each time step. A new approximation consists in defining an average of the operators over the time interval $[0, T]$:

$$\mathbf{L}_{n+1/2} = \frac{1}{T}\int_0^T \mathbf{D}(\hat{\boldsymbol{\varepsilon}}_{n+1/2}(t))dt \quad \text{and} \quad \mathbf{H}_{n+1/2} = \frac{1}{T}\int_0^T \mathbf{H}(\hat{\boldsymbol{\varepsilon}}_{n+1/2}(t))dt \tag{17}$$

5 Numerical Results

The proposed test case concerns the consolidation of a Berea sandstone soil. The geometry is shown in Fig. 1 and the material characteristics are given in Table 1. The simulation was performed for the one-dimensional case, since the law (6) is defined only in that case.

Table 1. Water-saturated Berea sandstone poroelastic material

Initial porosity	$n_0 = 0.19$	Initial Young's modulus	$E_0 = 14.4$ GPa
Poisson's coeff.	$\nu = 0.2$	Biot's modulus	$Q = 13.5$ GPa
Biot's coeff.	$b = 0.78$	Initial permeability	$\frac{K_0}{\mu_w} = 2\ 10^{-10}$ m^3.s.kg^{-1}

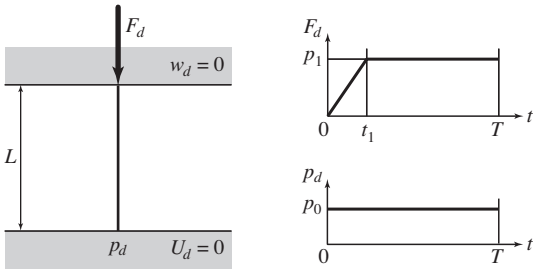


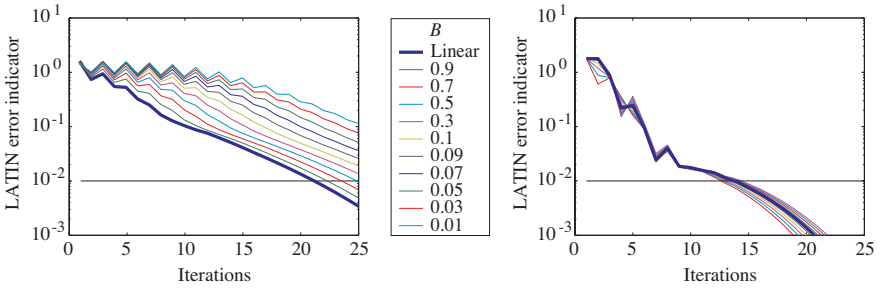
Fig. 1. The force-driven test problem

The time interval was $T = 1$ s with $t_1 = T/2$ and the pressures were $p_1 = 10$ MPa and $p_0 = 0.1$ MPa; the initial condition was $p(t = 0) = p_0$; the height of the structure was $L = 5$ m, discretized into 100 elements (quadratic interpolation for displacements and linear interpolation for pore pressures). The search direction parameters were set to $t_m = 9\ 10^{-3}\ t_c$ and $t_h = 8\ 10^{-3}\ t_c$, where $t_c = 9.3$ s.

Two simulations were performed to illustrate the behavior of the method when nonlinearity increases. The first test was dedicated to the evaluation of the influence of stiffness: in (6), the value of A and σ_3 were set to $A = \frac{1}{E_0}$ and $\sigma_3 = 0$ while the value of B increased from 0 (which corresponds to the linear case) to 1 GPa^{-1}. The second test concerned the evaluation of the influence of permeability: in (8), the value of α was set to $\alpha = 3$ while the initial porosity $n_0 = -\operatorname{Tr} \boldsymbol{\varepsilon}_0$ was no longer that of the Berea sandstone, but was assumed to decrease from 0.9 to 0.01. (The linear case was recovered by taking $n_0 \to +\infty$.)

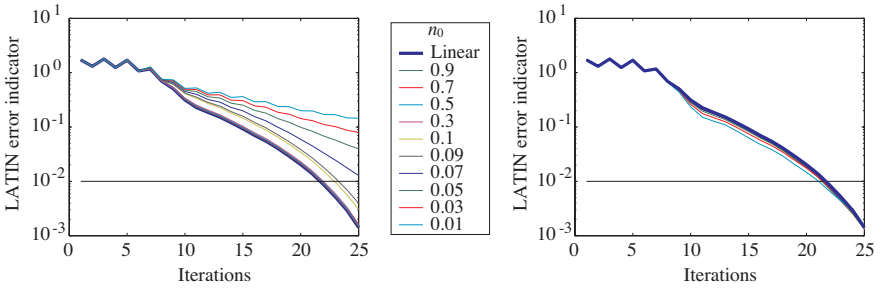
From here on, the error indicator based on the difference between an element $\mathbf{s}$ of $\mathbf{A_d}$ and an element $\hat{\mathbf{s}}$ of $\boldsymbol{\Gamma}$ will be used: $\hat{\eta} = e(\hat{\mathbf{s}} - \mathbf{s})/e(\frac{1}{2}(\hat{\mathbf{s}} + \mathbf{s}))$ with $e^2(\hat{\mathbf{s}} - \mathbf{s}) = \int_0^T (\frac{1}{2}\|\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}\|^2_{\mathbf{D}} + \frac{1}{2}\|\hat{p} - p\|^2_{Q^{-1}})dt$, $\|\boldsymbol{\varepsilon}\|^2_{\mathbf{D}} = \int_\Omega \operatorname{Tr}[\boldsymbol{\varepsilon}\mathbf{D}(\hat{\boldsymbol{\varepsilon}} = \mathbb{O})\boldsymbol{\varepsilon}]d\Omega$ and $\|p\|^2_{Q^{-1}} = \int_\Omega pQ^{-1}pd\Omega$.

Fig. 2**(a)** and Fig. 3**(a)** show that if constant search directions, such as (15), are used (as in [3,5]) the convergence rate is very dependent on the degree of nonlinearity. One can see in Fig. 2**(b)** and Fig. 3**(b)** that if updated average search directions, such as (17), are used at each iteration the convergence rate becomes nearly independent, but in that case, even if the number of iterations is smaller, the strategy could become very expensive because it requires the assembly and factorization of the operators at each iteration. However, one can note that nearly identical results can be obtained by using updated search directions only during the first iterations (usually 4 or 5). This reduces the computational cost significantly. Let us observe that nonlinearities do not increase the number of iterations needed to reach a given error.



(a) Constant search direction **(b)** Variable search direction

Fig. 2. Variable rigidity



(a) Constant search direction **(b)** Variable search direction

Fig. 3. Variable permeability

6 Conclusions

In this paper, we described a partitioned strategy based on the LATIN approach which enables one to take into account some of the classical nonlinearities of consolidation problems. The numerical tests showed that if updated search directions are used during the first iterations, the convergence rate is nearly independent of the level of nonlinearity. Thus, these nonlinear phenomena do not result in a significant increase in the computational costs.

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