Improved A-P Iterative Algorithm in Spline Subspaces*

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Abstract. In this paper, we improve A-P iterative algorithm, and use the algorithm to implement the reconstruction from weighted samples, and obtain explicit convergence rate of the algorithm in spline subspaces.

1 Introduction

For a bandlimited signal of finite energy, it is completely described by the famous classical Shannon sampling theorem. This classical theorem has broad application in signal processing and communication theory and has been generalized to many other forms. However, in many real applications sampling points are not always regularly. It is well-known that in the sampling and reconstruction problem for non-bandlimited spaces, signal is often assumed to belong to a shift-invariant spaces[1, 2, 4, 5, 7, 8, 9, 10]. As the special shift-invariant spaces, spline subspaces yield many advantages in their generation and numerical treatment so that there are many practical applications for signal or image processing[1, 2, 3, 9].

For practical application and computation of reconstruction, Goh et al., show practical reconstruction algorithm of bandlimited signals from irregular samples in [11], Aldroubi et al., present a A-P iterative algorithm in [5]. In this paper, we improve the A-P iterative algorithm in spline subspaces. The improved algorithm occupies better convergence than the old one.

2 Improved A-P Iterative Algorithm in V_N

Aldroubi presented A-P iterative algorithm in [5]. In this section, we will improve the algorithm. The improved algorithm occupies faster convergence. We will discuss the cases of non-weighted samples and weighted samples, respectively.

We define some symbols. $V_N = \{\sum_{k\in Z} c_k\varphi_N(\cdot - k) : \{c_k\} \in \ell^2\}$ is spline subspace generated by $\varphi_N = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ (N convolutions), $N \geq 1$.

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Let $\chi_{j,1}(x) = \chi_{[x_j, \frac{x_j+x_{j+1}}{2})}(x), \chi_{j,2}(x) = \chi_{[\frac{x_j+x_{j+1}}{2}, x_{j+1})}(x), \forall x_j \in X$ (X is sampling set),

$$W(L^p) = \{f \in L^p : \| f \|_{W(L^p)} = (\sum_{k \in Z^d} esssup_{x\in[0,1]^d} |f(x+k)|^p)^{(1/p)} < \infty\}$$

if $1 \le p < \infty$,

$$W(L^\infty) = \{f \in L^\infty : \| f \|_{W(L^p_v)} = \sup_{k\in Z^d} esssup_{x\in[0,1]^d} |f(x+k)| < \infty\}$$

if $p = \infty$. Let oscillation $osc_\delta(f)(x) = \sup_{|y|\le\delta} |f(x+y) - f(x)|$.

We show some lemmas that will be used in the proof of Theorem 2.1 , 2.2, and 2.3.

Lemma 2.1[5] *If φ is continuous and has compact support, then for any$f \in V^p(\varphi) = \{\sum_{k\in\mathbb{Z}} c_k \varphi(\cdot - k) : (c_k) \in \ell^p\}$ the conclusions (i)-(ii) hold:*

i.$\|f\|_{L^p} \approx \|c\|_{\ell^p} \approx \|f\|_{W(L^p)}$,

ii.$V^p(\varphi) \subset W_0(L^p) \subset W_0(L^q) \subset W(L^q) \subset L^q(\mathbb{R}) (1 \le p \le q \le \infty)$.

Lemma 2.2 *If $f \in V_N$, then for any $0 < \delta < 1$ we have* $\|osc_\delta(f)\|^2_{L^2} \le (3N\delta)^2 \sum_{k\in\mathbb{Z}} |c_k|^2$.

Proof. We have the following equalities and inequalities:

$$\|osc_\delta(f)\|_2^2 = \int_{-\infty}^{\infty} |\sup_{|y|\le\delta} |\sum_{n\in\mathbb{Z}} c_n(\varphi_N(x+y-n) - \varphi_N(x-n))||^2 dx$$

$$\le \int_{-\infty}^{\infty} (\sum_{n\in\mathbb{Z}} |c_n| \sup_{|y|\le\delta} |\varphi_N(x+y-n) - \varphi_N(x-n)|)^2 dx$$

$$= \int_0^1 \sum_{k\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |c_{k-m}| \sup_{|y|\le\delta} |\varphi_N(x+y+m) - \varphi_N(x+m)|)^2 dx.$$

By induction method and properties of φ_N, we can easily check that

$$\sum_{k\in\mathbb{Z}} \sup_{x\in[0,1]} |\varphi'_N(x+k)| \le N \qquad (\triangle)$$

From $(\triangle)$ and properties of φ_N, we can obtain the following estimate:

$$\|osc_\delta(f)\|_2^2 \le \int_0^1 \sum_{k\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |c_{k-m}|^2 \sup_{|y|\le\delta} |\varphi_N(x+y+m) - \varphi_N(x+m)|^2) dx$$

$$\le \sum_{k\in\mathbb{Z}} |c_k|^2 \int_0^1 (\sum_{m\in\mathbb{Z}} \sup_{|y|\le\delta} |\varphi_N(x+y+m) - \varphi_N(x+m)|)^2 dx$$

$$\leq \delta^2 \sum_{k\in\mathbb{Z}} |c_k|^2 \int_0^1 (\sum_{m\in\mathbb{Z}} \sup_{|y|\leq\delta} |\varphi_N'(x+\theta y+m)|)^2 dx$$

$$\leq \delta^2 \sum_{k\in\mathbb{Z}} |c_k|^2 \int_0^1 (\sum_{m\in\mathbb{Z}} \sup_{t\in[-1,2]} |\varphi_N'(t+m)|)^2 dx$$

$$\leq (3N\delta)^2 \sum_{k\in\mathbb{Z}} |c_k|^2.$$

The last inequality of the above inequalities derives from ($\triangle$). From $x \in [0,1]$,$0 < \theta < 1$ and $0 < \delta < 1$, we have the forth inequality of the above inequalities.

Lemma 2.3[5] *For any $f \in V^p(\varphi)$ the following conclusions hold:*

1.$\|osc_\delta(f)\|_{W(L^p)} \leq \|c\|_{\ell^p} \|osc_\delta(\varphi)\|_{W(L^1)}$,

2.$\| \sum_{k\in\mathbb{Z}} c_k\varphi(\cdot - k)\|_{W(L^p)} \leq \|c\|_{\ell^p}\|\varphi\|_{W(L^1)}$.

Lemma 2.4 *If $X = \{x_n\}$ is real sequence with $\sup_i(x_{i+1} - x_i) = \delta < 1$, then for any $f = \sum_{k\in\mathbb{Z}} c_k\varphi_N(\cdot - k) \in V_N$ we have $\|Qf\|_{L^2} \leq \|Qf\|_{W(L^2)} \leq (3+\delta)\|c\|_{\ell^2}\|\varphi\|_{W(L^1)}$.*

Proof. : For $f = \sum_{k\in\mathbb{Z}} c_k\varphi_N(\cdot - k)$ we have

$$|f(x) - (Qf)(x)| = |\sum_{j\in\mathbb{Z}}(f(x) - f(x_j))\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}}(f(x) - f(x_{j+1}))\chi_{j,2}(x)|$$

$$\leq \sum_{j\in\mathbb{Z}} |f(x) - f(x_j)|\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} |f(x) - f(x_{j+1})|\chi_{j,2}(x)$$

$$\leq \sum_{j\in\mathbb{Z}} osc_{\frac{\delta}{2}}(f)(x)\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} osc_{\frac{\delta}{2}}(f)(x)\chi_{j,2}(x)$$

$$= osc_{\frac{\delta}{2}}(f)(x).$$

From this pointwise estimate and Lemma 2.3 we get

$$\|f - Qf\|_{W(L^2)} \leq \|osc_{\frac{\delta}{2}}(f)\|_{W(L^2)}$$

$$\leq \|c\|_{\ell^2}\|osc_{\frac{\delta}{2}}(\varphi_N)\|_{W(L^1)}.$$

And by the results of [7] or [8] we know

$$\|osc_{\frac{\delta}{2}}(\varphi_N)\|_{W(L^1)} \leq (2+\delta)\|\varphi_N\|_{W(L^1)}.$$

Putting the above discuss together, we have

$$\|Qf\|_{L^2} \leq \|Qf\|_{W(L^2)} \leq \|f - Qf\|_{W(L^2)} + \|f\|_{W(L^2)}$$

$$\leq (2+\delta)\|c\|_{\ell^2}\|\varphi_N\|_{W(L^1)} + \|\sum_{k\in\mathbb{Z}} c_k\varphi_N(\cdot - k)\|_{W(L^2)}$$

$$\leq (2+\delta)\|c\|_{\ell^2}\|\varphi_N\|_{W(L^1)} + \|c\|_{\ell^2}\|\varphi_N\|_{W(L^1)}$$

$$\leq (3+\delta)\|c\|_{\ell^2}\|\varphi_N\|_{W(L^1)}.$$

The following Theorem 2.1 is one of our main theorems in this paper.

Theorem 2.1 *Let P be an orthogonal projection from $L^2(\mathbb{R})$ to V_N and $Q[f(x)] = \sum_j f(x_j)\chi_{j,1}(x) + \sum_j f(x_{j+1})\chi_{j,2}(x)$. If sampling set $X = \{x_n\}$ is a real sequence with $\sup_i(x_{i+1} - x_i) = \delta < 1$ and $\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi+2k\pi)|^2}} < 1$, then any $f \in V_N$ can be recovered from its samples $\{f(x_j) : x_j \in X\}$ on sampling set X by the iterative algorithm*

$$\begin{cases} f_1 = PQf, \\ f_{n+1} = PQ(f - f_n) + f_n. \end{cases}$$

The convergence is geometric, that is,

$$\|f_{n+1} - f\|_{L^2} \le \left(\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\right)^n \|f_1 - f\|_{L^2}.$$

Proof. By $Pf = f$ and $\|P\|_{op} = 1$, for any $f = \sum_{k\in\mathbb{Z}} c_k\varphi_N(\cdot - k) \in V_N$ we have

$$\begin{aligned}
\|(I - PQ)f\|_{L^2}^2 &= \|Pf - PQf\|_{L^2}^2 \le \|P\|_{op}^2\|f - Qf\|_{L^2}^2 = \|f - Qf\|_{L^2}^2 \\
&= \int |\sum_{j\in\mathbb{Z}} f(x)\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} f(x)\chi_{j,2}(x) - \sum_{j\in\mathbb{Z}} f(x_j)\chi_{j,1}(x) \\
&\qquad - \sum_{j\in\mathbb{Z}} f(x_{j+1})\chi_{j,2}(x)|^2 dx \\
&\le \int (\sum_{j\in\mathbb{Z}} |f(x) - f(x_j)|\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} |f(x) - f(x_{j+1})|\chi_{j,2}(x))^2 dx \\
&= \int \sum_{j\in\mathbb{Z}} |f(x) - f(x_j)|^2\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} |f(x) - f(x_{j+1})|^2\chi_{j,2}(x) dx \\
&\le \int \sum_{j\in\mathbb{Z}} |osc_{\frac{\delta}{2}}(f)(x)|^2\chi_{j,1}(x) + \sum_{j\in\mathbb{Z}} |osc_{\frac{\delta}{2}}(f)(x)|^2\chi_{j,2}(x) dx \\
&= \|osc_{\frac{\delta}{2}}(f)\|_{L^2}^2 \le (3N\frac{\delta}{2})^2 \sum_{k\in\mathbb{Z}} |c_k|^2 = (3N\frac{\delta}{2})^2\|c\|_{\ell^2}^2 \\
&\le \left(\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\right)^2 \|f\|_{L^2}^2.
\end{aligned}$$

The third equality is from property $\sum_j(\chi_{j,1}(x) + \chi_{j,2}(x)) = 1$. The forth equality derives from property $supp\chi_{j,i} \cap supp\chi_{k,l} = \emptyset (j \ne k, i \ne l)$. By Lemma 2.2, the forth inequality holds. And we have

$$\|f_{n+1} - f\|_{L^2} = \|f_n + PQ(f - f_n) - f\|_{L^2} = \|PQ(f - f_n) - (f - f_n)\|_{L^2}$$

$$\leq \|I - PQ\| \|f - f_n\|_{L^2} \leq \cdots \leq \|I - PQ\|^n \|f - f_1\|_{L^2}.$$

Combining with the estimate of $\|I - PQ\|$, we can imply

$$\|f_{n+1} - f\|_{L^2} \leq \left(\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\right)^n \|f_1 - f\|_{L^2}.$$

Taking assumption $\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi+2k\pi)|^2}} < 1$, we know the algorithm is convergent.

In the following, we will show improved A-P iterative algorithm from weighted samples.

Theorem 2.2 *Let P be an orthogonal projection from $L^2(\mathbb{R})$ to V_N and weight function satisfy the following three conditions (i)-(iii):*
(i)$supp\varphi_{x_j} \subset [x_j - \frac{a}{2}, x_j + \frac{a}{2}]$
(ii) *there exist $M > 0$ such that $\int |\varphi_{x_j}(x)|dx \leq M$,*
(iii)$\int \varphi_{x_j}(x)dx = 1$.
Let $Af(x) = \sum_j \langle f, \varphi_{x_j}\rangle \chi_{j,1}(x) + \sum \langle f, \varphi_{x_{j+1}}\rangle \chi_{j,2}(x)$. If sampling set $X = \{x_n\}$ is a real sequence with $0 < \sup_i(x_{i+1} - x_i) = \delta < 1$ and we choice proper δ and a such that $\frac{3N}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi+2k\pi)|^2}}(\delta + a(3+a)M) < 1$, then any $f \in V_N$ can be recovered from its weighted samples $\{\langle f, \varphi_{x_j}\rangle : x_j \in X\}$ on sampling set X by the iterative algorithm

$$\begin{cases} f_1 = PAf, \\ f_{n+1} = PA(f - f_n) + f_n. \end{cases}$$

The convergence is geometric, that is,

$$\|f_{n+1} - f\|_{L^2} \leq \left(\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}(\delta + a(3+a)M)\right)^n \|f_1 - f\|_{L^2}.$$

Proof. By $Pf = f$ and $\|P\|_{op} = 1$, for any $f = \sum_{k\in\mathbb{Z}} c_k \varphi_N(\cdot - k) \in V_N$ we have

$$\|f - PAf\|_{L^2} = \|f - PQf + PQf - PAf\|_{L^2}$$

$$\leq \|f - Qf\|_{L^2} + \|Qf - Af\|_{L^2} \qquad (1).$$

From the proof of Theorem 3.1, we have the following estimate for $\|f - Qf\|_{L^2}$:

$$\|f - Qf\|_{L^2} \leq \left(\frac{3N\delta}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\right)\|f\|_{L^2} \qquad (2).$$

For the second term $\|Qf - Af\|_{L^2}$ of (1) we have the pointwise estimate

$$|(Qf - Af)(x)|$$

$$= |\sum_j (f(x_j) - \langle f, \varphi_{x_j} \rangle)\chi_{j,1}(x) + \sum_j (f(x_{j+1}) - \langle f, \varphi_{x_{j+1}} \rangle)\chi_{j,2}(x)|$$

$$= |\int \sum_j (f(x_j) - f(\xi))\varphi_{x_j}(\xi)\chi_{j,1}(x)$$

$$+ \sum_j (f(x_{j+1}) - f(\xi))\varphi_{x_{j+1}}(\xi)\chi_{j,2}(x) d\xi|$$

$$\leq M(\sum_j osc_{\frac{a}{2}}(f)(x_j)\chi_{j,1}(x) + \sum_j osc_{\frac{a}{2}}(f)(x_{j+1})\chi_{j,2}(x))$$

$$= MQ(\sum_{k\in\mathbb{Z}} |c_k| osc_{\frac{a}{2}}(\varphi_N)(x-k)).$$

The above second equality derives from $\int \varphi_{x_j}(x)dx = 1$. By $\int |\varphi_{x_j}(x)|dx \leq M$ and $supp\varphi_{x_j} \subset [x_j - \frac{a}{2}, x_j + \frac{a}{2}]$, we know the above first inequality.

From this pointwise estimate and Lemma 2.4, it follows that:

$$\|Qf - Af\|_{L^2} \leq M(3+a)\|c\|_{\ell^2}\|osc_{\frac{a}{2}}(\varphi_N)\|_{W(L^1)}$$

$$\leq M(3+a)\frac{\|osc_{\frac{a}{2}}(\varphi_N)\|_{W(L^1)}}{\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\|f\|_{L^2}$$

$$\leq M(3+a)\frac{3Na}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}\|f\|_{L^2} \qquad (3).$$

By combining (1),(2) and (3), we can obtain

$$\|I - PA\|_{L^2} \leq \frac{3N}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}(\delta + a(3+a)M).$$

Similar to the procedure in the proof of Theorem 3.1, we have

$$\|f_{n+1} - f\|_{L^2} \leq (\frac{3N}{2\sqrt{\sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2}}(\delta + a(3+a)M))^n \|f_1 - f\|_{L^2}.$$

Remark 1. Term $(\frac{1}{2})^n$ is added in the expression of convergence rate. This improves the velocity of convergence. From the construction of operator Q and A, we know why it appears in the expression of convergence rate.

The reconstruction algorithm in Theorem 2.1 and 2.2 require the existence of orthogonal projection from L^2 onto V_N. For this purpose, the following Theorem 2.3 will construct the orthogonal projection. We can find the similar proof of Theorem 2.3 in [5, 10].

Theorem 2.3 *Let* $X = \{x_n\}$ *be a real sequence such that* $0 < \sup_i(x_{i+1} - x_i) = \delta < 1$. *Then*

$$Pf = \sum_{x_j \in X} \langle f, k_{x_j} \rangle \tilde{k}_{x_j}$$

is orthogonal projection from L^2 *onto* V_N, *where* $\{k_{x_j}\}$ *and* $\{\tilde{k}_{x_j}\}$.

Remark 2. : The above improved A-P iterative algorithm maybe be generalized to the case of $L^p(\mathbb{R})$ and $V^p(\varphi)$ whenever generator φ belongs to $W_0(L^1)$. We will study it in future work.

3 Conclusion

In this paper we pay main attention on the weighted sampling and reconstruction in spline subspaces. We give some reconstruction methods from different weighted sampling in spline subspaces. The improved A-P iterative algorithm performs better than the old A-P algorithm. And we obtain the explicit convergence rate of the improved A-P iterative algorithm in spline subspaces.

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