No-Bend Orthogonal Drawings of Subdivisions of Planar Triconnected Cubic Graphs
(Extended Abstract)

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Abstract. A plane graph is a planar graph with a fixed embedding. In a no-bend orthogonal drawing of a plane graph, each vertex is drawn as a point and each edge is drawn as a single horizontal or vertical line segment. A planar graph is said to have a no-bend orthogonal drawing if at least one of its plane embeddings has a no-bend orthogonal drawing. In this paper we consider a class of planar graphs, called subdividions of planar triconnected cubic graphs, and give a linear-time algorithm to examine whether such a planar graph G has a no-bend orthogonal drawing and to find one if G has.

1 Introduction

An *orthogonal drawing* of a plane graph G is a drawing of G with the given embedding such that each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end [RN02,RNN02,RNN99,S84,T87]. A *bend* is a point where an edge changes its direction in a drawing. If G has a vertex of degree five or more, then G has no orthogonal drawing. On the other hand, if G has no vertex of degree five or more, that is, the maximum degree Δ of G is at most four, then G has an orthogonal drawing, but may need bends. Some plane graphs have an orthogonal drawing without bends, in which each edge is drawn by a single horizontal or vertical line segment. We call such a drawing a *no-bend drawing*. Figure 1(a) depicts a no-bend drawing of the plane graph in Fig. 1(b). Rahman *et al.* [RNN02] obtained a necessary and sufficient condition for a plane graph G with $\Delta \leq 3$ to have a no-bend drawing, and gave a linear-time algorithm to find a no-bend drawing if G has.

We say that *a planar graph G has a no-bend drawing* if at least one of the plane embeddings of G has a no-bend drawing. Figures 1(b), (c) and (d) depict three of the different plane embeddings of the same planar graph G. Among them only the embedding in Fig. 1(b) has a no-bend drawing as illustrated in Fig. 1(a). Thus the *planar* graph G has a no-bend drawing. It is a NP-complete problem to examine whether a planar graph G with $\Delta \leq 4$ has a no-bend drawing

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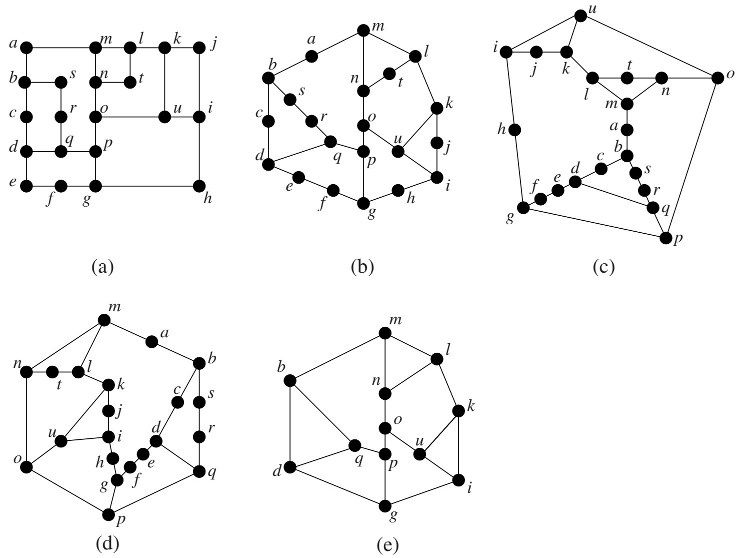

Fig. 1. A no-bend drawing (a), and three different embeddings (b), (c) and (d) of the same graph which is a subdivision of the triconnected cubic planar graph in (e).

[GT01]. However, for a planar graph G with $\Delta \leq 3$, Di Battista *et al.* [DLV98] gave an $O(n^5 \log n)$ time algorithm to find an orthogonal drawing of G with the minimum number of bends. Hence, by their algorithm, one can examine in time $O(n^5 \log n)$ whether a planar graph with $\Delta \leq 3$ has a no-bend drawing.

In this paper we consider a class of planar graphs, called subdivisions of planar triconnected cubic graphs, and give a linear-time algorithm to examine whether such a planar graph G has a no-bend drawing and to find a no-bend drawing if G has. The graph in Fig. 1(b) is obtained from the planar triconnected cubic graph in Fig. 1(e) by inserting vertices of degree two into some edges, and hence is a subdivision of the cubic graph.

The rest of the paper is organized as follows. Section 2 describes some definitions and presents preliminary results. Section 3 presents a necessary and sufficient condition for a subdivision G of a planar triconnected cubic graph to have a no-bend drawing; the condition leads to a linear-time algorithm. Finally Section 4 is a conclusion.

2 Preliminaries

In this section we give some definitions and present preliminary results.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. The *degree* $d(v)$ of a vertex v is the number of neighbors of v in G. We call a vertex of degree two in G a *2-vertex* of G. A graph G is called *cubic* if $d(v) = 3$ for every vertex v. For $V' \subseteq V$, $G - V'$ denotes a graph obtained from

G by deleting all vertices in V' together with all edges incident to them. For a subgraph G' of G, we denote by $G - G'$ the graph obtained from G by deleting all vertices in G'. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$.

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \cdots, w_k, v(= w_{k+1})$ passing through new 2-vertices $w_1, w_2, \cdots, w_k, k \geq 1$. A graph G is said to be a *subdivision* of a graph G' if G is obtained from G' by subdividing some of the edges of G'. A subdivision of a triconnected cubic graph is biconnected, and the degree of any vertex is either 2 or 3.

Let $P = w_0, w_1, w_2, \cdots, w_{k+1}, k \geq 1$, be a path of a graph G such that $d(w_0) \geq 3, d(w_1) = d(w_2) = \cdots = d(w_k) = 2$, and $d(w_{k+1}) \geq 3$. Then we call the subpath $P' = w_1, w_2, \cdots, w_k$ of P a *chain* of G, and we call vertices w_0 and w_{k+1} the *supports* of the chain P'. If G is a subdivision of a triconnected graph, then any 2-vertex of G is contained in exactly one of the chains of G.

Let G be a planar biconnected graph, and let Γ be a plane embedding of G. The contour of a face of Γ is a cycle of G, and is simply called a *face* or a *facial cycle* of Γ. We denote by $F_o(\Gamma)$ the *outer face* of Γ. For a cycle C of Γ, we call the plane subgraph of Γ inside C (including C) the *inner subgraph* $\Gamma_I(C)$ for C, and call the plane subgraph of Γ outside C (including C) the *outer subgraph* $\Gamma_O(C)$ for C. Any face of Γ is either in $\Gamma_I(C)$ or in $\Gamma_O(C)$. An edge of G is called a *leg* of C if it is incident to exactly one vertex of C and located outside C. A cycle C of Γ is called a *k-legged cycle* if C has exactly k legs in Γ and there is no edge which joins two vertices on C and is located outside C. An edge of G is called a *hand* of C if it is incident to exactly one vertex of C and located inside C. A cycle C of Γ is called a *k-handed cycle* if C has exactly k hands in Γ and there is no edge which joins two vertices on C and is located inside C. Both the set of k legs of a k-legged cycle and the set of k hands of a k-handed cycle in G correspond to a "cutset" of k edges in G.

Rahman *et al.* [RNN02] obtained a necessary and sufficient condition for a plane graph G with $\Delta \leq 3$ to have a no-bend drawing, and gave a linear time algorithm to find a no-bend drawing of G if G has, as follows.

Lemma 1. [RNN02] *A plane embedding Γ of a planar biconnected graph G with $\Delta \leq 3$ has a no-bend drawing if and only if Γ satisfies the following three conditions:*

(a) *there are at least four 2-vertices on the outer face $F_o(\Gamma)$;*
(b) *every 2-legged cycle in Γ contains at least two 2-vertices; and*
(c) *every 3-legged cycle in Γ contains at least one 2-vertex.*

Furthermore one can examine in linear time whether Γ satisfies the conditions above, and one can find a no-bend drawing of Γ in linear time if Γ has.

Although the results above for a plane graph is known, it is difficult to examine whether a planar graph has a no-bend drawing or not, since a planar graph may have an exponential number of plane embeddings in general. However, the following fact is known for subdivisions of planar triconnected graphs.

Fact 2 [NC88] *Let G be a subdivision of a planar triconnected graph. Then there is exactly one embedding of G for each face embedded as the outer face. Furthermore, for any two plane embeddings Γ and Γ' of G, any facial cycle of Γ is a facial cycle of Γ'.* ■

Fact 2 implies that a subdivision G of a planar triconnected graph has an $O(n)$ number of embeddings, one for each chosen outer face. Examining by the linear algorithm in [RNN02] whether the three conditions in Lemma 1 hold for each of the $O(n)$ embeddings, one can examine in time $O(n^2)$ whether the planar graph G has a no-bend drawing. In Section 3 we present a necessary and sufficient condition for G to have a no-bend drawing. Our condition leads a linear-time algorithm to examine whether G satisfies the condition and to find a no-bend drawing if G has.

We have the following lemma and facts on a subdivision of a planar triconnected cubic graph, which will be useful in Section 3.

Lemma 3. [RNG02] *Let G be a subdivision of a planar triconnected cubic graph, and let Γ be an arbitrary plane embedding of the planar graph G. Then the following* (a) and (b) *hold.*

(a) *For any 2-legged cycle C of Γ, the set of all vertices not in $\Gamma_I(C)$ induces a chain of G on $F_o(\Gamma)$.*
(b) *For any chain P on $F_o(\Gamma)$, the outer face of the plane graph $\Gamma - P$ is a 2-legged cycle in Γ.* ■

A cycle of Γ violating Condition (b) or (c) in Lemma 1 is called a *bad cycle* of Γ: a 2-legged cycle is *bad* if it contains at most one 2-vertex; a 3-legged cycle is *bad* if it contains no 2-vertex. Let C be a 3-handed cycle of a plane embedding Γ of G, let F be a face in $\Gamma_I(C)$, and let Γ' be a plane embedding of G in which F is embedded as $F_o(\Gamma')$. Then C is a 3-legged cycle of Γ'. If C contains no 2-vertex, then C is a bad cycle of Γ'. We thus call a 3-handed cycle C of Γ a *bad 3-handed cycle* of Γ if it contains no 2-vertex, otherwise we call C a *good 3-handed cycle.*

3 No-Bend Drawings of Planar Graphs

We obtain the following necessary and sufficient condition for a subdivision G of a planar triconnected cubic graph to have a no-bend drawing.

Theorem 1. *Let G be a subdivision of a planar triconnected cubic graph, and let Γ be an arbitrary plane embedding of G. Then the planar graph G has a no-bend drawing if and only if Γ has a face F satisfying the following conditions* (i)–(v)*:*

(i) *there are at least four 2-vertices on F;*

(ii) *F is contained in $\Gamma_I(C)$ for any bad 3-legged cycle C in Γ;*
(iii) *F is contained in $\Gamma_O(C)$ for any bad 3-handed cycle C in Γ;*
(iv) *if there is exactly one chain P on F, then the face F' which contains P and is different from F contains at least two 2-vertices which are not on P;and*
(v) *if there are exactly two chains P_1 and P_2 on F, and one of them, say P_1, contains exactly one vertex, then the face F' which contains P_2 and is different from F contains at least one 2-vertex which is not on P_2.*

Proof. Necessity: Assume that G is a subdivision of a planar triconnected cubic graph and has a no-bend drawing. Then G has a plane embedding Γ' which has a no-bend drawing. Let F be the face of Γ such that $F = F_o(\Gamma')$. Then we can show that F satisfies (i)–(v), as follows.

(i) Since Γ' has a no-bend drawing, by Lemma 1(a) there are at least four 2-vertices on $F_o(\Gamma') = F$.

(ii) Suppose for a contradiction that F is not contained in $\Gamma_I(C)$ for a bad 3-legged cycle C in Γ. Then F is contained in $\Gamma_O(C)$. One can observe that C is a bad 3-legged cycle of Γ', and hence by Lemma 1 Γ' does not have a no-bend drawing, a contradiction.

(iii) Suppose for a contradiction that F is not contained in $\Gamma_O(C)$ for a bad 3-handed cycle C in Γ. Then F is contained in $\Gamma_I(C)$. One can observe that C is a bad 3-legged cycle of Γ', and hence by Lemma 1 Γ' does not have a no-bend drawing, a contradiction.

(iv) Assume that there is exactly one chain P on F. By Lemma 3(b) Γ' has a 2-legged cycle $C = F_o(\Gamma' - P)$. Γ' does not have any 2-legged cycle other than C; otherwise, by Lemma 3(a) $F_o(\Gamma')$ has a chain other than P, and hence $F = F_o(\Gamma')$ would have two chains. Thus Γ' has exactly one 2-legged cycle C. Since Γ' has a no-bend drawing, by Lemma 1(b) C contains at least two 2-vertices. Since F has exactly one chain P, all 2-vertices on $F_o(\Gamma')$ are on P and hence the maximal subpath Q_1 of C that is on $F_o(\Gamma')$ contains no 2-vertex. Thus all 2-vertices on C are contained in the maximal subpath Q_2 of C that is not on $F_o(\Gamma')$. Clearly Q_2 is on F' and is not on P. Hence F' contains at least two 2-vertices which are not on P.

(v) Assume that F has exactly two chains P_1 and P_2 and P_1 contains exactly one vertex. Since $F_o(\Gamma') = F$ contains at least four 2-vertices by Lemma 1(a), P_2 contains at least three 2-vertices. By Lemma 3(b) Γ' has two 2-legged cycles $C_1 = F_o(\Gamma' - P_1)$ and $C_2 = F_o(\Gamma' - P_2)$. By Lemma 3(a) Γ' does not have any 2-legged cycle other than C_1 and C_2. By Lemma 1(b) C_2 contains at least two 2-vertices. The maximal subpath Q_1 of C_2 that is on $F_o(\Gamma')$ contains exactly one 2-vertex, and hence the maximal subpath Q_2 of C_2 that is not on $F_o(\Gamma')$ contains at least one 2-vertex. Clearly Q_2 is on F' and is not on P_2, and hence F' contains at least one 2-vertex which is not on P_2.

Sufficiency: Assume that Γ has a face F satisfying Conditions (i)–(v). Let Γ' be an embedding of G such that $F = F_o(\Gamma')$. We can prove that that Γ' satisfies Conditions (a)–(c) in Lemma 1 and hence Γ' has a no-bend orthogonal drawing. The detail of the proof is omitted in this extended abstract. $\mathcal{Q.E.D.}$

Theorem 1 immediately leads to a linear algorithm to examine whether a subdivision G of a planar triconnected cubic graph has a no-bend drawing and to find a no-bend drawing of G if it exists, as follows. Let Γ be an arbitrary plane embedding of G. Using a method similar to one in [RNN00,RNG02], one can examine in linear time whether Γ has a face F satisfying the conditions in Theorem 1. If Γ has such a face F, a plane embedding Γ' such that $F = F_o(\Gamma')$ can be obtained in linear time [NC88]. Finding a no-bend drawing of Γ' takes linear time [RNN02].

4 Conclusions

In this paper we presented a necessary and sufficient condition for a subdivision G of a planar triconnected cubic graph to have a no-bend drawing, and gave a linear-time algorithm to examine whether G satisfies the condition and to find a no-bend drawing if G has.

It is left as a future work to find a linear-time algorithm to find a no-bend drawing for a larger class of planar graphs.

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