# Chapter 8
# Introducing History of Mathematics Education Through Its Actors: Peter Treutlein's Intuitive Geometry

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**Abstract** This paper deals with the questions why and how to introduce into teacher education the history of teaching practices and educational reforms. In particular, we are interested in the developments of curricular school geometry during the 19th century and the reforms at the beginning of the last century in Germany. The life and work of Peter Treutlein—a contemporary of Felix Klein—and a conceptual reformist of geometry instruction, schoolbook author, committed teacher and school principal with educational experience of many years opens to us many opportunities to link present teaching practices in Geometry to its traditions, some of which we will discuss.

**Keywords** Felix Klein · Teacher education · History of mathematics education
Reforms in geometry teaching · Peter Treutlein

## 8.1 Introduction

Are Felix Klein's ideas and the European didactic and German speaking tradition of *Intuitive thinking and visualizations* still important for present and future theoretical considerations as well as the practice of the teaching and learning of mathematics? In the contribution, we study this question with respect to mathematics teacher education.

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The translation of Felix Klein's *Elementary Mathematics from an Advanced Standpoint* (Volume II: Geometry)[1] by Hedrick and Noble from 1949 published by Dover (Klein 2004) and earlier by Macmillan (Klein 1939) finishes after the historical excursus to Euclid's Elements. The last chapter [i.e. the fourth so-called *Schlußkapitel* (Final Chapter)], of Klein's book (Klein 1925) is missing. The title of the missing chapter is "Observations of the Teaching of Geometry".[2]

Here, Felix Klein gives a short overview of the development of the teaching of geometry from the perspective of the goals of the Meraner Reform in England, France, Italy and Germany. Felix Klein's cultural-historical approach to development already appears in the structure of his overviews (Klein 2016, pp. 226–231):

- Importance of the Historical Background
- Contrasting Modern Requirements
- Criticism of the Traditional Teaching.

The last section of the fourth chapter, "Teaching practice in Germany", is dedicated to a critical reflection on the curricular development of geometry and to geometry instruction in Germany. Here, Felix Klein mainly discusses the work of Julius Henrici (1841–1910) and Peter Treutlein (1845–1912). His esteem for the work of Henrici and Treutlein (1883, 1896, 1897) becomes evident in the quote "Henrici-Treutlein is an extremely noteworthy book" (Klein 1925, p. 261) and in the subsequent description of recent developments in geometry. As early as 1911, Felix Klein wrote the introduction to Treutlein's (1911a) theoretical work on the reform of the traditional Euclidean curriculum in school geometry: "G*eometrical intuitive teaching (Anschauungsunterricht) as a first step in a two-stage geometrical instruction at our higher schools*".

In this introduction, Felix Klein wrote: "However, I am pleased to express that I mean to recognize everywhere the same principles that I follow in my own teaching. Such a coincidence finally is a matter of personal disposition." (Treutlein 1911a, p. III, translation by the author).

In 1925, after the publication of various textbooks by different authors implementing reforms of Euclidean school geometry, Felix Klein recommends Treutlein's conceptual work to teachers: "For this teaching, the following works will be of great use for many teachers: (a) The book, that arose from a mature pedagogical experience of P. Treutlein "*The geometrical intuitive teaching (Anschauungsunterricht) as a first step in a two-stage geometrical instruction at our higher schools*", Leipzig 1911" (Klein 1925, p. 292). The second textbook recommended by Felix Klein is Heinrich Emil Timerding's "The education of intuition (Anschauung)" (Timerding 1912).

Peter Treutlein was not only a conceptual reformist of geometry instruction and a schoolbook author, he was also a committed teacher and school principal with many

[1]The translated version of the second volume of Klein's book was published in 1939, later than the book on arithmetic, algebra, analysis, which was published as translation in 1932.

[2]In the meantime, Gert Schubring published a new translation (Klein 2016) of Felix Klein's *Elementary Mathematics from a Higher Standpoint* with a translation of the "Schlußkapitel" in Volume II.

years' educational experience, who implemented the realizations of his theoretical concepts in everyday school life, resulting in his three-volume textbook with Julius Henrici.

The study of the history of educational reforms and their theoretical foundation is particularly relevant today. German students have experienced several reforms during their school time, the theoretical foundation of which has not yet taken place. Keywords for these reforms are *output* and *competence orientation*, the introduction of educational standards and central tests, the abolition of the orientation classes and pre-school education, the reduction of upper secondary classes by one year, the digitalization of learning environments, the restructuring of secondary schools, and the overall present *inclusion*. In particular, regarding the tremendous speed with which these political reforms are pushed through, it is certainly worthwhile to engage in reform, which had been prepared for half a century; the *Meraner Reform* was discussed widely and implemented in small steps.

In this paper, we show how some activities related to the reform of geometrical instruction, and in particular, historical collections of mathematical models for use in schools could be used in current university training for mathematics teachers.

First, we take a look at some of the reasons why the history of mathematics and the history of mathematics education not only *can* but *should* be included in university teacher education. Hereby, we are particularly interested in the history of mathematics education at the end of the 19th century and at the beginning of the 20th century—a fascinating period of reforms of mathematical curricula as well as of the German school system.

The second question in our paper is how the history of the teaching of mathematics can be introduced in university education for future mathematics teachers and how the aforementioned school reforms could be meaningful for today's students. Here, we briefly glance at the history of mathematical models and instruments and their industrial production as teaching tools. In doing so, we will not concentrate on historical mathematical models for the visualization of higher mathematics and their use in teaching mathematics at university, but on mathematical models which illustrate and visualize school mathematics. This again leads us to Peter Treutlein and the school models that he invented.

In the last section, we outline how Peter Treutlein's collection of school models, his three-volume joint school textbook with Henrici (1881–1883) and Treutlein's conceptual theoretical work on reforms in geometry teaching can be used in the university education of mathematics teachers.

## 8.2 History of Mathematics in Mathematics Education

The topic of "the use of the history of mathematics in mathematics teaching" is gaining popularity in the international discourse of mathematics education. Possible reasons for this might be the introduction of topics from the history mathematics in the curricula in such countries as Denmark, Austria or Great Britain, or likewise the new interest in concept development from a mathematical perspective and the historical genetic method in the style of Toeplitz (1949).

There are numerous publications on the variety of ways of using the history of mathematics in teaching mathematics as well as on the promotion of mathematical interests which range from theoretical material for classification (e.g. Jankvist 2009; Kronfellner 1998) to concrete descriptions of their realizations in teaching practices (e.g. Fauvel and van Maanen 2000; Katz and Michalowicz 2000; Jahnke 2006; Shell-Gellasch 2007; Glaubitz 2011). It is important to note whether the history of mathematics and mathematics education is used to inspire new perspectives on the teaching of mathematics and concept development or whether the history of a mathematical idea, an authority or an institution are the subject of investigation. The latter especially presumes methodology and knowledge of the history of mathematics.

A special challenge is presented by the use of original historical sources in order to teach a mathematical idea as well as its historical conceptual development. In this case, on top of the use of representations and mathematical languages that are different to those the students are used to, problems may also occur due to a foreign or ancient language or to different translations of the original source. On the other hand, working with artefacts bearing contemporary witness to a past period is particularly attractive. The mathematical exploration of historical sources is often easier if related to models, instruments, equipment or also mathematical toys, and even better if accompanied by appropriate texts and descriptions of the experiments. The mathematical concept development also benefits from the use of mathematical instruments and visualizing models. Vollrath illustrates how this can be done with a non-historical approach to historical drawing and measuring instruments in his book "Verborgene Ideen" (en.: hidden ideas) (Vollrath 2013).

There is no doubt that it is important for future mathematics teachers to know some of the history of their own discipline, that is, the history of mathematics, but also the history of their own profession, i.e. the history of mathematics education. Both topics are scientific disciplines with specialized knowledge and methodology. Do student teachers have the prerequisites and capabilities to look into the history of a mathematical concept in order to teach it later on with historical awareness?

To include the contents of the history of mathematics in their teaching, teachers are required to have developed a particular interest in this field and to therefore gladly accept the challenge of teaching mathematics in a way that also embraces perspectives of the humanities. Hardly any German university offers canonical lectures on the cultural history of mathematics or on selected topics on the history of mathematics. Therefore, an introduction to the history of *school* mathematics and its instruction should not require any substantiated knowledge of *history of mathematics* as an own scientific discipline.

The choice of topics related to the history of mathematical models and instruments and their collections allows for rich access to the history of mathematics and its education. The development and production of mathematical models was already used in the 19th and early 20th centuries for the training of student mathematics teachers. Nowadays this is an episode in the history of European science. The use of historical mathematical models and their digital images in the study, teaching and development of mathematics allow us to relate historical, technical, educational and information technology aspects to each other.

One opportunity to link today's university teacher education to the last century could be to complete similar tasks, for example to design and produce a mathematical model. In the framework of a project at the Georg-August-University Göttingen to introduce the *Göttinger Sammlung historischer mathematischer Modelle* (Göttingen's collection of historical mathematical models) into teaching, students designed and produced their own mathematical models. In this project, the digitally available historical collection in Göttingen (Göttinger Sammlung mathematischer Modelle und Instrumente 2017) became the subject of several teaching activities in the study of mathematics as well as in mathematics education (see also Weiss-Pidstrygach 2015). Among the models of this collection are those produced by students a century ago. The collection of historical mathematical models in Göttingen is closely connected with the name and activities of Felix Klein (Rowe 2013). In 1880, when appointed Professor of Geometry at the University of Leipzig, Felix Klein suggested in his inauguration speech (Klein 1895, p. 538) acquiring a collection of mathematical models "to reduce the gap which already separates the theoretical mathematician from math's applications" and to improve teaching at the university. However, these models visualize higher mathematics and are quite demanding in terms of hidden mathematics. Understanding the background of most of the Brill and Schilling collection's models (Polo-Blanco 2007; Schilling 1903) is a very challenging mathematical task for students.

Another approach to link modern teaching with historical collections of models is the pedagogical perspective. There is a variety of literature with historical and pedagogical perspectives on the development of mathematical models, which can constitute the content framework for historical research (for instance Bartolini Bussi et al. 2010).

Contemporary student mathematics teachers can have varied experiences with mathematical models. Pedagogical reform is an important topic in educational studies. Fröbel, Pestalozzi, Kerschensteiner and Dewey (Führer 2000; Klafki 2000) explore sense perception and activity orientation in mathematics education and their approaches are part of the curriculum in educational science. Students are also familiar with modern mathematical hands-on exhibitions, as most schools have their own mathematical models and toys.

## 8.3 Treutlein's Models and Textbooks in the University Education of Mathematics Teachers

An extremely suitable introduction from a pedagogical perspective—but with a mathematician's eye—is given by Treutlein in his book *Intuitive Geometry as a First Level of Two-Level Geometry Courses at Our Secondary Schools* (Treutlein 1911a). The first chapter is dedicated to the history of intuitive geometry and forms a solid foundation for the historical contextualization of his concept of a geometrical instruction starting with intuitive geometry. From the second chapter, he moves from the historical perspective to a mathematical and a pedagogical one. In this theoretically conceptual work, he also incorporates two of his other works: A collection of cataloged school models (Wiener 1912) and a school textbook of three volumes, which he composed between 1881 and 1883 with Julius Henrici—a school principal in Heidelberg.

The study of this work is made significantly easier through its digital availability. A historical excursus based on a few historical mathematical school models provides students with an opportunity to develop their own questions in a field particularly interesting to them and to discuss them afterwards.

The study of Treutlein is highly relevant with regard to curricular and methodical reforms at the turn of the 19th century. His theoretical foundation, which includes examples in his *Intuitive Geometry Lesson*, forms the basis of the discussion of Treutlein's concept of a reform of geometry instruction. Treutlein suggests a division so that one should start with intuitive spatial geometry (*anschauliche Raumlehre*) with geometrical object lessons of two or three years, which are then followed by (academic) geometry lessons with the duration of five to six years. After a historical contextualization of intuitive geometry, Treutlein lists the requirements and theoretical principles for an intuitive spatial theory as well as the respective methods of instruction according to these principles. The last part of this work deals with practical exemplary lesson planning.

Treutlein's geometrical school models (Treutlein 1911b) put his concepts of reform into practice as they result from his education experience of more than forty years. The implementation and usage of his models are evident in some examples of the third paragraph (Treutlein 1911a, p. 109) as well as in his school geometry textbooks.

Treutlein's programme contains three essential 19th century thoughts:

- The idea of an intuitive geometry lesson as a preliminary step for subsequent formal deductive geometry instruction.
- The implementation of the most recent mathematical developments such as projective geometry and transformation geometry into mathematical instruction.
- The training of the spatial imagination and the fusion of both spatial and plain geometry.

The developments, implementations and generalizations of the Pythagorean Theorem (see Treutlein 1911a, pp. 183–184) can be used to compare assignments and exercises in modern textbooks with historical ones from Treutlein (see also Henrici and Treutlein 1897). Treutlein's book starts with a step-by-step development of the necessary technical terms through a comparison of areas which he constructs by means of gnomones, in the spirit of Euclid. For the conversion of surfaces, he uses self-constructed models while his arguments and demonstrations often derive from abstract geometry. The arithmetic examples often explain the algebraic terms (Figs. 8.1 and 8.2).

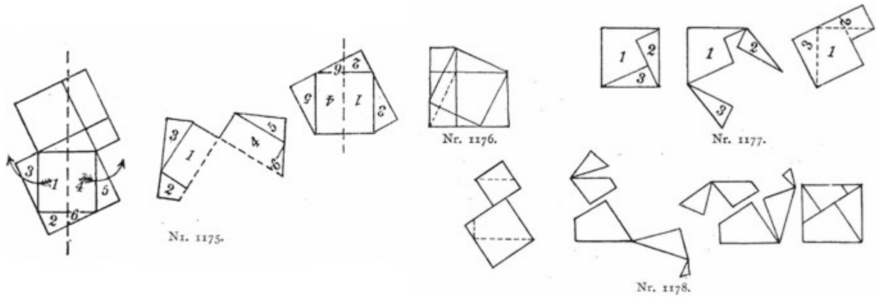


**Fig. 8.1** Examples of plain moveable models from Treutlein's collection. The model 1175 is used as a solution sketch (see Treutlein 1911a, p. 183) and as an important example in the first volume of the tenth chapter "Comparison of areas" (see Henrici and Treutlein 1897, p. 88)

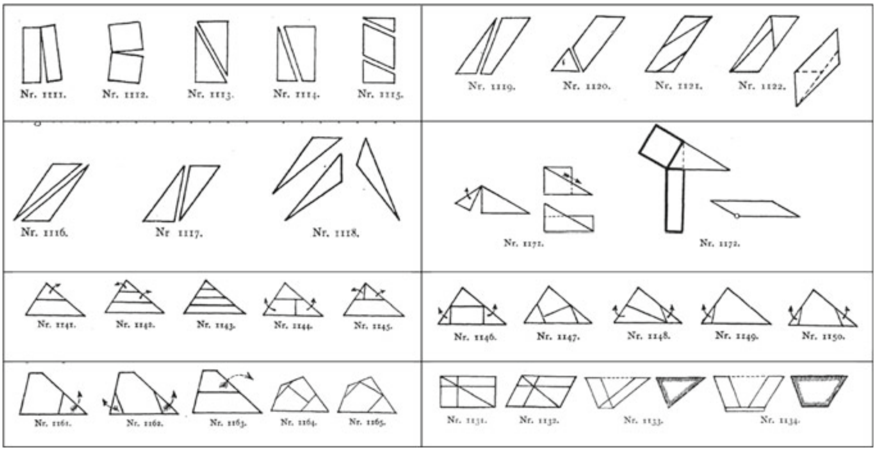


**Fig. 8.2** Samples from Treutleins catalogue (Wiener 1912, pp. 50–53)

Students can choose for themselves a model accompanied by the respective problem provided in Treutlein's and Henrici's geometry book in order to compare the development of concepts and demonstrations in modern schoolbooks and those written in Treutlein's era. The students can also experience the differences between modern teaching methods and those of Treutlein's time by planning a unit using one or more of Treutlein's models. The comparison should clarify the difference between the student's idea of a concept and the one Treutlein describes in his book.

Another possible way to link today's teaching practice with that of a century ago comes from Treutlein's criticism of the geometry classes of his time. He criticizes the following four features of geometry teaching practice (Treutlein 1911a, p. 71):

1. The well-known, much celebrated, and often infrequently hard-bred, strictly dogmatic teaching,
2. The sharp separation of general space geometry from plane geometry,

3. The retraction of the space considerations towards the very end of the usual course,
4. Beginning with more abstract doctrines, on straight lines and planes, before exploring the geometry of bodies.

One can similarly question the features of modern geometry teaching and whether these criticisms have been overcome.

A historical contextualization of Peter Treutlein's work and life gives Schönbeck (1994), a historical classification and reception of Treutlein's textbook in three volumes with Julius Henrici can be found for example in Gerhard Becker's paper (1994).

One of the most surprising discoveries in my studies of Treutlein's concept of intuitive geometry was the activity orientation of his exercises and tasks. He already uses paper folding, outdoor mathematics and construction of models to develop space intuition and to teach modern approaches to geometry like transformation geometry. A more detailed description of various possibilities how to relate Treutlein's models to present-day teacher university education the interested reader finds in Weiss-Pidstrygach (2015, 2016).

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