Chapter 6
Coherence and Fidelity of the Function Concept in School Mathematics

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Abstract We define notions of mathematical coherence and mathematical fidelity and apply them to a study of the function concept in school mathematics, as represented by the results of image searches on the word "function" in various languages. The coherence and fidelity of the search results vary with the language. We study this variation from a mathematical viewpoint and distill dimensions which characterize that variation, and which can provide insights into the characteristics of professional communities of school educators and into principles for selection and evaluation of curriculum resources.

Keywords Klein · Function · Mathematical coherence · Mathematical fidelity

6.1 Introduction

> We, who use to be called the reformers, would put the function concept at the very center of teaching, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into teaching as early as possible with constant use of the graphical method, the representation of functional relations in the xy system, which is used today as a matter of course in every practical application of mathematics.
>
> —Klein, 1908 (Klein et al. 2016)

Klein's vision of the "function concept at the center of instruction" was part of a broader school reform movement at the beginning of the 20th century, as recounted by Krüger (2018) in this volume. Today many aspects of that vision have been realized in mathematics education. The function concept is firmly embedded in the school curriculum, with an explicit definition of the concept usually occurring in later grades. In earlier grades the influence of the function concept on curriculum varies. In many curricula, for example the reform curricula of the 90s in the US, functional thinking in early grades can be seen in the use of patterns and tables, even if the concept

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was not explicitly defined. Recent standards in the US have partially reversed this trend, with a greater emphasis on arithmetic and the properties of operations in the early grades (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010).

In this paper it is our purpose to analyze the function concept as it appears in school mathematics from a mathematical point of view. We intend this to be a work of mathematical analysis, not a work of educational research. It is not our purpose, nor our expertise, to investigate the relationship between the function concept on the one hand and students, teachers, and classrooms on the other hand. We focus on two aspects of the function concept in school mathematics: mathematical coherence and mathematical fidelity. By mathematical coherence we mean the strength and consistency of mathematical connections, the tendency of a curriculum, or of a collection of curriculum materials, to form a mathematically logical and consistent whole. By mathematical fidelity we mean the extent to which a curriculum, or a collection of curriculum materials, faithfully presents the underlying mathematical concept as it is situated in the discipline of mathematics. Note that mathematical fidelity is not the same as mathematical formality; a mathematical concept can be presented in a way that is appropriate for the age of the students, while still being presented with fidelity. We hope that the mode of mathematical analysis presented here may be useful to the producers of curriculum and other resources for students, teachers, and classrooms.

We illustrate the concepts of mathematical coherence and mathematical fidelity with an analysis of the results of internet image searches on the word "function" in various languages. One reason for doing this is simply that it is a convenient way of producing raw material with which to illustrate the mode of analysis. However, there is some reason to believe that the collection of images so obtained is telling us something about how the function concept is presented in schools. It is common to think of the internet as a network somewhat similar to the network of neurons in the brain (see, for example, Woodford 2017). Furthermore, searching on the word "function" in various languages leads to results that are largely related to school mathematics. The algorithms behind search engines are designed to give prominence to results that are highly interconnected with other sites on the internet; thus they might be seen as giving results that are prominent in the community using the network. Putting these observations together suggests that structural features of the results of an image search bear some relation to structural features of school mathematics in the language of the search. We do not, however, investigate that relationship, nor is such an investigation necessary for our purpose.

The paper is organized as follows. We start with a brief survey of various school mathematics definitions of the function concept, on order to ground the analysis that takes place in later sections. We then present the results of image searches in various languages, and discuss the mathematical coherence and mathematical fidelity of the results. We conclude with some thoughts on a possible connection between the analysis and the idea of concept image in mathematics education, and on implications of the analysis for the selection and evaluation of curriculum materials.

A note on terminology: We follow Zimba (2017) in using the word "presentation" rather than "representation." The concept of representation in mathematics education research is complex. It is linked to thoughts and processes which are not recoverable from mere contemplation of presentations. A presentation is an artifact—a written definition, an image, a physical model—that is the result of someone saying "here is a function." Our goal in this paper to observe presentations with a mathematical eye, attempting to excavate their mathematical structure and meaning.

6.2 The Definition of Function in School Mathematics

We present seven definitions from various sources: three from widely used US high school textbooks, three from highly ranked internet search results on "definition of function," and one provided by one of the lead authors of the US Common Core State Standards in Mathematics. In choosing these definitions, we have tried to illustrate a range, and have avoided choosing definitions that were badly flawed or wrong. We have also limited the choice to definitions that were intended for school students, not university students. Since the search was conducted in English, it naturally reflects a bias to sources from English speaking countries. Later in this article we look at image searches in other languages.

1. "In mathematics, relations like these—where each possible value of one variable is associated with exactly one value of another variable—are called functions.. . ." (Core-Plus Mathematics 2015).
2. "A relationship between inputs and outputs is a function if there is no more than one output for each input." (Core Connections Algebra 2013).
3. "A function is a rule that assigns to each value of one quantity a single value of a second quantity." (EngageNY/Eureka Math, Grade 8, Module 5 2017).
4. "A variable quantity regarded in relation to one or more other variables in terms of which it may be expressed or on which its value depends." (Top ranked result from a Google search performed on 11 December 2017).
5. "A mathematical correspondence that assigns exactly one element of one set to each element of the same or another set." (Merriam-Webster 2017).
6. "[A] function is a relation in which each element of the domain is paired with exactly one element of the range." (iCoachMath.com 2017).
7. "A function is any sort of process, or calculation, or lookup table, or rule, that links input quantities and output quantities in a repeatable way." (Jason Zimba 2017).

Compare these definitions with two definitions that bracket the period from the beginning of Klein's "past two centuries" to the current day:

> Those quantities that depend on others in this way, namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others. (Euler, 1755, Euler and Blanton 2000)

> A function from a set A to a set B is a relation $R \subset A \times B$ with the property that if $(a, b) \in R$ and $(a, b') \in R$ then $b = b'$. (Standard modern definition)

The first of these definitions, which we call the dynamic definition, conceives of a function as a dynamic object, a coordination of two varying quantities. In the second definition, which we call the static definition, there is no movement, only logic. An important difference between the two is that in the dynamic definition the condition on outputs—that no input produces more than one output—goes without saying; how could it be otherwise in a situation where one quantity changes in response to changes in another quantity? Whereas in the static definition, situated as it is in the wider context of relations on sets and subject to the demands of precision needed for formal mathematical proofs, it is necessary to state the condition explicitly. (See the chapter by Thompson and Milner in this volume for a comparison of dynamic and static definitions as they relate to teachers' mathematical meanings, Thompson and Milner 2018.)

The definitions in (1)–(7) feel the pull of both these bracketing definitions. On the one hand, the dynamic definition is the most natural one for a school context. Many early examples of functions that students encounter are functions of time representing moving objects. The definition (4) is close to the dynamic definition. On the other hand, with the possible exception of (7), the remaining definitions pay homage to the static definition by presenting the logical condition on outputs in some form (in (7) this is captured naturally and implicitly by the idea of a lookup table and by the word "repeatable"). One is compelled to wonder how this looks to the student, whose initial experience with functions rarely involves situations where the issue of having more than one output for a given input arises naturally, except in artificial examples constructed for the exact purpose of emphasizing that point. It might make sense to wait until the issue arises naturally, such when we consider implicit functions defined by equations in two variables, or when we consider scatter plots of statistical data. At that point, the idea of an input-output process might automatically suggest that only one output can be chosen.

6.3 Probing the Image of Function in the Internet Brain

We now move from definitions to images. We conducted searches using Google Image Search in various languages on the word "function" or on its translation into another language. Figures 6.1, 6.2, 6.3, 6.4 and 6.5 show the results for English, French, German, Japanese, and Spanish respectively.

Our purpose in conducting searches in different languages is not to attempt any international comparisons. There are many reasons why such comparisons are likely to be invalid. Some languages are spoken in many countries around the world, others are concentrated in a small number of countries. It would not be surprising if searches in a concentrated language showed less variation. Also, the proportion of search results that pertain to school mathematics could vary widely from country to country, depending on the extent to which school teachers in the country rely on the internet for materials.

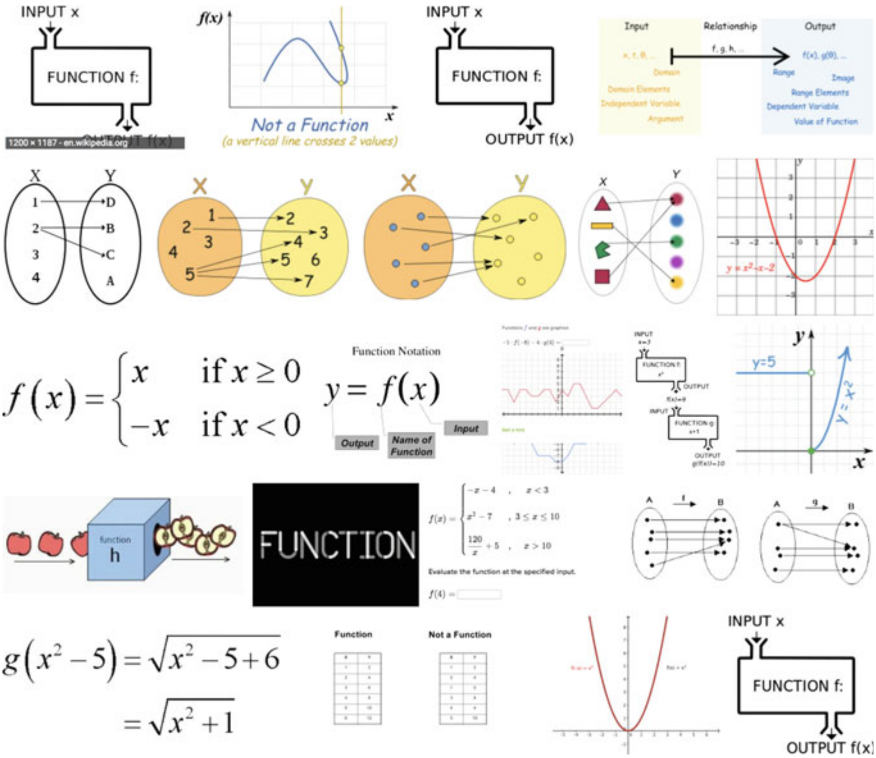

Fig. 6.1 Image search on "function", 11 December 2017

Nonetheless, different languages partition the internet into distinct networks, and therefore the comparison of different language searches using the same search engine is a source of variation that can reveal structure in the underlying network. This helps determine whether the search results are merely random noise, or whether they are revealing a discernible artifact of school mathematics amenable to analysis.

In Fig. 6.1 we see a profusion of ways in which the function is presented. There are graphs, tables, and algebraic expressions. There are input-output machines. There are arrow diagrams with two sets representing domain and codomain and arrows between them showing the function. What does this collection of images tell us about the concept of function in school mathematics in English speaking countries?

For one thing, it reveals the attractive force of the static definition noted in the previous section. We see a graphical example in the top row (thus a highly ranked search); examples using arrow diagrams in the second row; and an example comparing two tables in the bottom row, one of which is a function and one not. None of these are natural examples of relations, such as equivalence or congruence, but rather examples of relations that almost satisfy the condition to be a function—every input has a unique output—but that fail this condition with one or two inputs. Note,

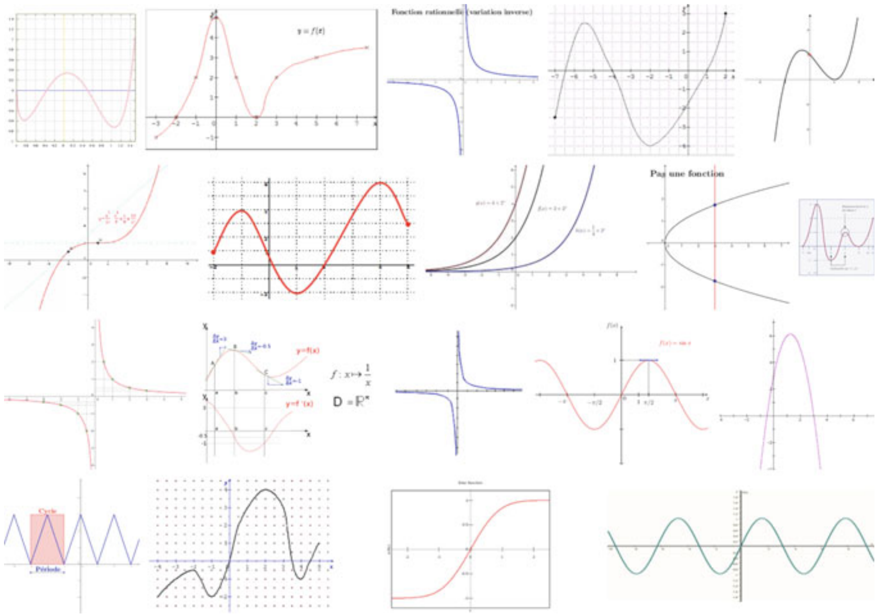

Fig. 6.2 French image search at google.fr, 11 December 2017

for example, the two arrows emanating from the number 2 in the second row, or that the x-value 1 unnaturally occurs twice in the table on the right in the bottom row. One would think that the world was full of impostors pretending to be functions and that it is the role of education to train students in ceaseless vigilance against them.

The Spanish language search yields results similar to English. (Note that both Spanish and English are much more widely spoken languages than the other three.) The searches in French, German, and Japanese produce a narrower range of results, focused mainly on graphs. Thus there are discernible differences in structure between the searches on different networks. In the next two sections we analyze these differences through the lenses of mathematical coherence and mathematical fidelity.

6.3.1 Mathematical Coherence

We have defined mathematical coherence to be the strength of mathematical connections. There are five major ways in which functions are presented in Fig. 6.1: table, graph, expression or equation, input-output process, and arrow diagram. Each of these is useful for presenting a different aspect of the function concept. We assess the strength of connections by observing the extent to which one mode of presentation contains signposts to another.

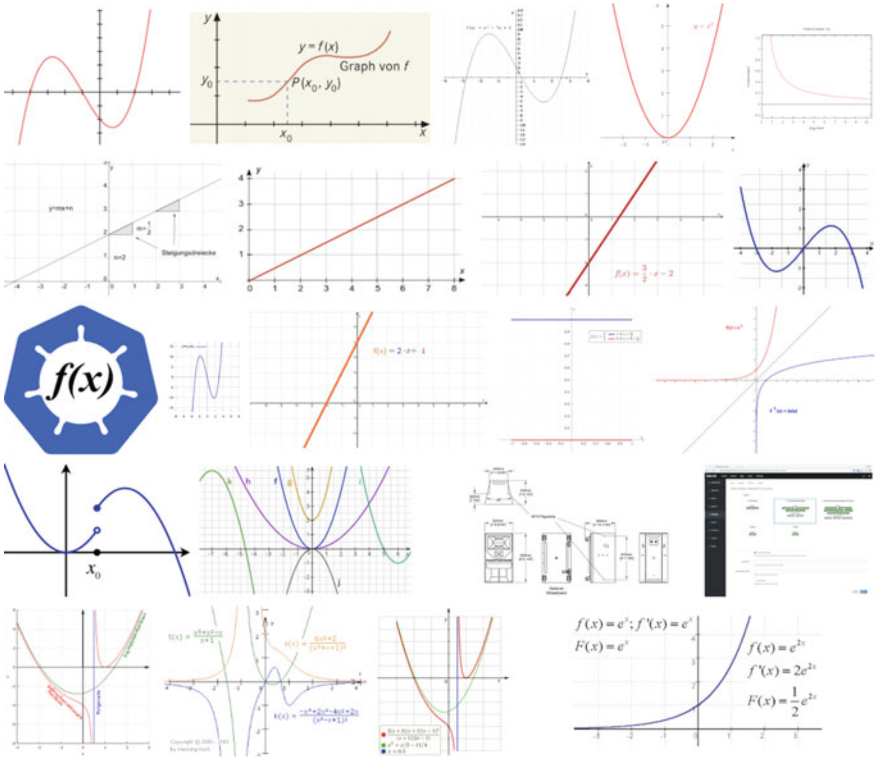

Fig. 6.3 German image search at google.de, 11 December 2007

Take for example the connection between a graph and an input-output machine. The input-output nature of a graph is not clearly visible, in that the process of reading outputs from inputs involves seeing hidden lines from the axes. Thus a possible sign that the graphical presentations are well connected mathematically to the input-output diagrams is the presence on the graph of auxiliary vertical and horizontal lines to a particular point on the graph. One sees such lines in the bottom left image in Fig. 6.4 or the second image in the top row in Fig. 6.3. One might also consider the presence or absence of grids on graphs as an indicator of the strength of this connection. There are marked differences on these indicators between the various searches.

Another possible connection to look for is between tables and input-output processes. A naked table depends on a convention that the column on the left is the input and the column on the right is the output, a convention that may or may not be strong in the mind of the student. However, some of the table images, for example the one on the bottom right in Fig. 6.4, are presented in spreadsheet form, which indicates an approach to tables that explicitly builds the input-output process into the production of the table.

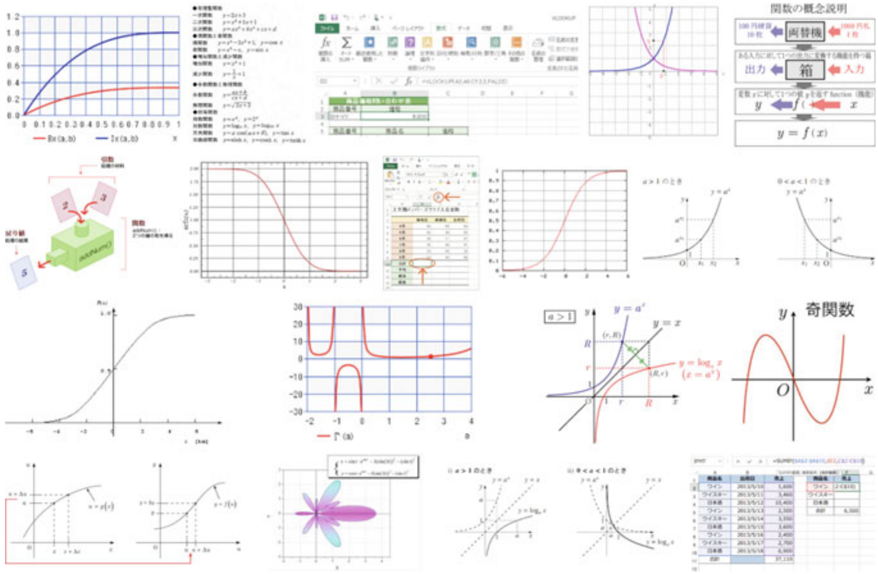

Fig. 6.4 Japanese image search at google.co.jp, 11 December 2007

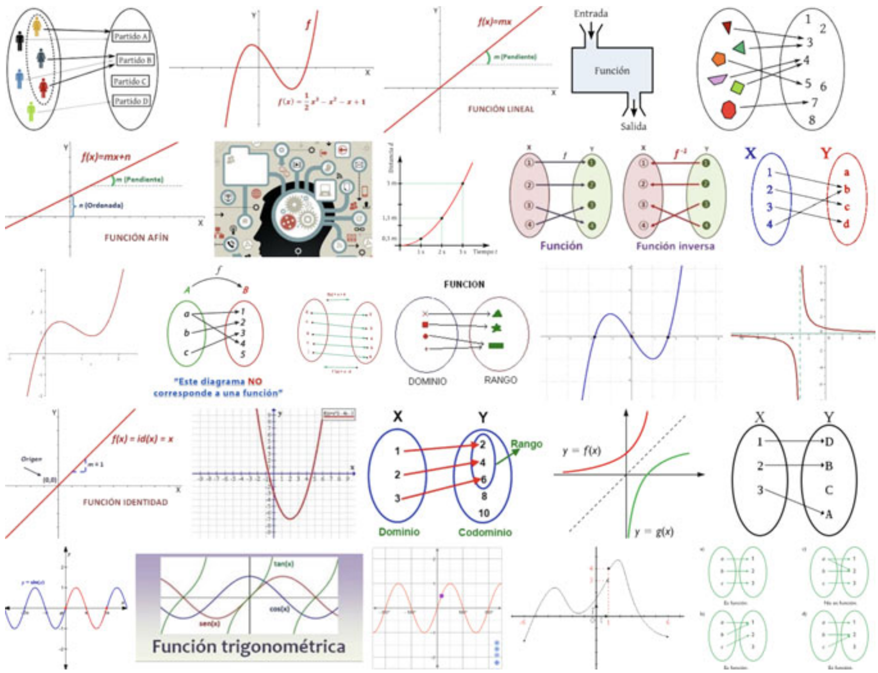

Fig. 6.5 Spanish image search at google.es, 11 December 2007

There are other important connections between ways of presenting functions that cannot reasonably be expected to be fully revealed in a search on images alone. For example, the connection between graphs and expressions or equations depends on written texts. However, there are some indications of the strength of that connection here. Consider, for example, the graph on the right of the third row in Fig. 6.1. Although the scale on the axes is not given, the graph of $y = x^2$ suggests that a grid square represents a unit in this graph. But then the graph labeled $y = 5$ is in fact the graph of $y = 6$. It is not particularly striking that among the millions of graphs on the internet there is one with a a mistake, but it is worth noting that a mistaken graph receives a rank of 15 among those millions. The rank is determined by the density of connections to the image. The internet brain in this case seems to be treating the equation as merely a label for the graph, rather than a related way of describing it. Contrast this image with the image second from the left in the bottom row of Fig. 6.3. The graphs are color coded with the equations and the scale is indicated on the axes. The high rank given to this graph by the German internet brain suggests a greater appreciation of the connection.

6.3.2 *Mathematical Fidelity*

Mathematical fidelity refers to the extent to which a presentation is faithful to the concept as it is situated in the discipline of mathematics. The static definition is the canonically accepted one in the discipline, and the presence of arrow diagrams, with their representation of the two sets A and B and arrows linking elements of those sets, might at first sight be seen as an indicator of fidelity to the static definition.

However, fidelity is not the same thing as formality. A question that must be considered is the meaning and function of arrow diagrams in school mathematics. In practice, in their later lives using mathematics, students rarely if ever encounter such diagrams. Even in school mathematics, they only encounter them in the environment of tasks designed specifically to see if students understand the logical condition in the static definition. But, as we have seen, that condition goes without saying in the dynamic definition, which is more suited to the real functions most students encounter in their learning pathways during and after school. Understanding the static definition seems to be an isolated learning goal with little relation to other areas of school mathematics, and it is, furthermore, a goal which is not usually pursued to any great depth in modern curricula. Arrow diagrams of the sort seen here seem to be a non-functional stub in the web of school mathematical knowledge. It is worth noting that a different type of arrow diagram that uses two parallel number lines, which does not appear in our searches, is potentially much richer. See for example Gilbey (2017).

Although the connection between the arrow diagrams and the static definition is obvious to an advanced observer, it is worth wondering what it conveys to the novice. From an advanced point of view it is clear that they are intended to indicate an association between inputs and outputs. But arrows are also used to indicate movement, so the diagrams could be construed as suggesting that the inputs are somehow moved to the outputs. This could be problematic when compared with other ways of presenting functions. In a graph, for example, the inputs and outputs remain firmly fixed on the axes.

The input-output machines in our search results vary in their degree of fidelity to the function concept. Look particularly at the apple slicing machine on the left of the fourth row in Fig. 6.1. This would seem to violate the fundamental property that every input should have a unique output. Of course, it is not surprising that flawed images of functions exist on the internet, but it is striking that they receive such a high ranking, indicating that many of the neurons in the English internet brain have found this image valuable. Compare the apple-slicing machine with the image on the left of row 2 in Fig. 6.4, which carefully labels the inputs, the machine, and the outputs, indicating that the output is the result of adding the inputs.

Finally, consider the variation in the graphs across the different image searches. From a mathematical point of view the graph is possibly the richest way of presenting a function, as Klein suggested in the quotation at the beginning of this article. On the one hand it is faithful to the static definition, in that it shows a subset of the Cartesian product of the domain and codomain. On the other hand it can capture the dynamic quality of Euler's definition and the school definitions if it is used to visualize the coordination of two varying quantities by means of an imagined or digital moving image where a point moves along the graph linked to inputs and outputs on the axes. If equipped with a grid and a scale on the axes it captures the numerical information in a table. The degree of fidelity is captured by the extent to which the graph presents all these features, and one sees considerable variation in that extent as one looks at the different graphical images. Some are annotated in a way that is dense with meaning, others appear to mere pictorial images.

6.4 Concluding Thoughts

The collections of images from the internet brain considered in this paper bring to mind the idea of concept image in the sense of Tall and Vinner (1981):

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

It would be interesting to know to what extent the digital images considered here reflect the concept images in the minds of individual students and teachers, or in the collective consciousness of the community of K–12 mathematics educators, if such a thing exists. Is it possible to investigate the coherence and fidelity of the entire collection of "mental pictures and associated properties and processes" related to the function concept in a particular community of practice in school mathematics?

Whether or not the idea of concept image can be extended that far, the exploratory analysis presented here shows significant variation between different networked communities along the dimensions of mathematical coherence and mathematical fidelity. At a more fine-grained level, for the function concept, we see variation in

- the extent to which a way of presenting functions makes visible connections to a different way
- the density of meaningful annotation in a presentation
- the degree to which components of a presentation are semantic rather than pictorial
- the extent to which extraneous features of the presentation violate mathematical properties.

The variation in these dimensions suggest that they are viable candidates for parameters of change in professional communities. Paying attention to them could lead to improved judgement and evaluation of curricular materials, to better curation of large resource collections, and to the creation of sub-communities with concept images that are more coherent and faithful than the larger communities to which they belong.

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