Chapter 12
Klein's Conception of 'Elementary Mathematics from a Higher Standpoint'

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Abstract This chapter studies Klein's conception of elementarisation; it is first put into the context of other approaches for mathematics teacher education in Germany. Then, approaches in mathematics education and in history of education to conceive of the relation between academic knowledge and school disciplines are discussed. The wrong translation of Klein's German term "höher" in the long time prevailing American translation is commented on, in preparation for the analysis of the concept of "element" in the history of science. Klein's practice and his introduction of the term "hysteresis" to emphasise the independence of school mathematics are discussed. The last section reflects the consequences of the hysteresis notion for integrating recent scientific advances into school curricula.

Keywords Felix klein · Elementarisation · Elements · Apollonius · d'Alembert Hysteresis · Set theory

12.1 Introduction

My main issue here is the notion of elementarisation. There is a widespread misunderstanding to conceive of this term within connotations like "simple", merely a didactical category as the exact opposite to scientific and academic knowledge. For Klein, however, these are completely misleading connotations; rather, deep philosophical and epistemological meanings are revealed to be implied.

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12.2 A Differing View of Elementary Mathematics

Actually, the term "elementary mathematics" was not new in Klein's times; there existed a well-known publication for mathematics teachers and mathematics students—likewise in three volumes—that used this term and Klein expressed his profound disagreement with this work. It was the *Encyclopaedia of Elementary Mathematics,* published from 1903 with various re-editions, by Heinrich Weber and Josef Wellstein, both mathematics professors at the University of Straßburg.

As Klein pointed out, the Weber-Wellstein Encyclopaedia gave a systematic presentation of the various parts relevant to the school curriculum (Klein 2016a, b, third preface) whereas he was highlighting those issues that deserved methodological comments. More importantly, Klein criticised that 'elementary' meant for them "fundamental for higher mathematics" (see on Epstein's 4th revised edition 1922: Klein 2016a, b, 300). And, that as a consequence, Weber-Wellstein did not address the relevance for teaching in schools:

> Thus, we find in his book detailed and abstract discussions of the concept of number, limit concept, number theoretic issues, etc., while the elements of calculus remain disregarded, although the author supports their teaching in schools. (ibid., p. 300)

And Klein criticised that they did not reflect the pedagogical dimension of teaching, neglecting in particular Klein's plea for intuitive approaches (p. 33):

> I shall indicate at once certain differences between this work and the plan of my lecture course. In Weber-Wellstein, the entire structure of elementary mathematics is built up systematically and logically in the mature language accessible to the advanced student. No account is taken of how these things actually may come up in school teaching. The presentation in the schools, however, should be *psychological* – to use a 'catch word' - and not *systematic*. (ibid., p. 4)

Even worse, Weber-Wellstein did not follow Klein's conception of elementarisation by reconstructing school mathematics via the function concept:

- Another difference between Weber-Wellstein and myself has to do with *delimiting the content of school mathematics*. Weber and Wellstein are disposed to be "*conservative"*, while I am "*progressive*". […] We, who use to be called the "*reformers*", would put the *function concept* at the very centre of teaching, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used.
- As opposed to these comparatively recent ideas, Weber-Wellstein adhere essentially to the traditional limitations of the subject matter. In this lecture course I shall of course be a protagonist of the new conception (ibid., pp. 4–5).

12.3 Differing Views of the Relation Between Academic Mathematics and School Mathematics

In fact, the basic epistemological issue implied by the concept of elementarisation is the relation between school mathematics and academic mathematics. This relation is far from being evident or easily resolvable. This is documented by two extreme recent positions in mathematics education and history of education about this relation. Both poles are represented by French researchers: Yves Chevallard and André Chervel.

Chevallard's conception of *transposition didactique* is well known: the conception of the didactic transposition proposes to examine how academic knowledge of mathematics ("savoir savant") becomes school mathematical knowledge. For this, the concept distinguishes three types of knowledge:

– "Objet de savoir"—subject of knowledge;
– "Objet d'enseigner"—subject to be taught: the academic knowledge becomes teachable knowledge by the efforts of mathematics educators (their community being called "noosphère");
– "Objet d' enseignement"—teaching subject (Chevallard 1985, 39).

As has been criticized by several researchers, the explanation offered by the transposition notion conceives of a unilateral process: it has as its starting point a pole designed as advanced, the academic or university knowledge and as its final point another pole inferior to it, made at school and involving the teacher in the classroom.

The other extreme is represented by the research area *history of school disciplines*. In fact, researchers of this area typically work on subjects such as literature, the humanities, the native language, history and geography, religion, and even philosophy. Thus, the focus of their approach is the socializing function of school and hence, in particular, of school disciplines. The school culture and school subjects are thus characterized by autonomies: it is believed that school disciplines enjoy autonomy with respect to the other disciplines (Chervel 1988, 73), while the example of mathematics shows that the set of all disciplines influence strongly the status, the level and the views of school mathematics (Schubring 2005). Moreover, Chervel emphasises the generative nature of the school, which results in creating, due to its character understood as relatively autonomous, school disciplines (see Vinao 2008).

12.4 Implications of the Term "Advanced"

Kilpatrick had emphasised in his lecture at ICME-11, 2008, in Mexico that the term "advanced" used by the American translators in the title is profoundly misleading and does not correspond to Klein's conceptions of elementarisation (Kilpatrick 2008). In fact, the term "advanced" corresponds best to Chevallard's conception of *transposition.*

The term “advanced” implies a fundamental misunderstanding of Klein’s notion of *elementary* and of *Elementarmathematik.* The term “advanced” means that elementary mathematics is somewhat delayed, being of another nature. It means exactly the contrary of what Klein was intending. By contrasting two poles, ‘elementary’ versus ‘advanced’, one would admit just that discontinuity between school mathematics and academic mathematics that Klein wanted to eliminate.

For Klein, there was no separation between elementary mathematics and academic mathematics. His conception for training teachers in higher education departed from a holistic vision of mathematics: mathematics, steadily developing and reforming itself within this process, leading to ever new restructured elements, and provides therefore new accesses to the elements. There is a widespread understanding of the term “elementary” as meaning it something ‘simple’ and not loaded with a conceptual dimension—perhaps even approaching ‘trivial’. Connected, in contrast, with the notion of element, ‘elementary’ means for Klein to unravel the fundamental conception. What is at stake, hence, is the concept of *elements*.

12.5 The Concept of Elements

Beyond mere factual information, with his lecture notes Klein led the students to gain a more comprehensive and methodological point of view on school mathematics. The three volumes thus enable us to understand Klein’s far-reaching conception of *elementarisation*, of the “elementary from a higher standpoint”, in its implementation for school mathematics: the elements are understood as the fundamental concepts of mathematics, as related to the whole of mathematics and according to its restructured architecture.

Clearly, Klein was not the first to reflect about the concept of elements. It is in particular in mathematics that one finds reflections about its meaning and its use. This strand of reasoning was brought about by the very title of Euclid’s paradigmatic geometry textbook. The analysis of a masterpiece of Hellenistic mathematics has also given rise to a revealing discussion of “elements” with regard to compendium, encyclopaedia and textbook: Fried and Unguru, in the introduction to their new edition of the *Conica* by Apollonius of Perga, discuss the division made by Apollonius himself in his presentation of his work to the mathematician Eudemus “regarding the contents of the **Conica**, namely, that the first four books ‘belong to a course in the elements,’ while the latter four ‘are fuller in treatment’” (Fried et al. 2001, p. 58). They understand Apollonius’ comments as implying that an “elementary treatment” did not mean for him rather trivial parts, which can be omitted, but instead essential conceptual expositions (ibid.). To approach Apollonius’s meaning of “elementary”, they refer to the analyses by historians of mathematics of the difference between the first four books of the *Conica* and the further books. They refer in particular to Heath who had published in 1896 an edition of Apollonius’ *Conica*:

> According to Heath, the elementary nature of the first four books distinguishes them from the rest by the "fact that the former contain a connected and scientific exposition of the general theory of conic sections as the indispensable basis for further extensions of the subject in certain special directions, while the fifth book is an instance of such specialization…". Heath also calls the first four books a "text-book or compendium of conic sections," and the last four books "a series of monographs on special portions of the subject." (ibid., pp. 58–59)[1]

And they largely agree with Gerald Toomer in his entry "Apollonius of Perga" in the *Dictionary of Scientific Biography*:

> Toomer adopts the same kind of image when he writes, "[Apollonius'] aim was not to compile an encyclopedia of all possible theorems on conic sections, but to write a systematic textbook on the 'elements' and to add some more advanced theory which he happened to have elaborated." (ibid., p. 59)

Fried and Unguru agree in particular with the distinction of "elements" from an encyclopedia but disagree with Heath's assessment as a compendium. They insist on the *systematic* character of exposition and on the *connected and scientific exposition as indispensable basis* for refinements and extensions.

In Modern Times, probably the most profound reflection on the concept of elements has been undertaken in the wake of Enlightenment, among the first approaches to making science generally accessible.[2] It was Jean le Rond d'Alembert (1717–1783) who conceptualized what he called to "elementarise" the sciences. It was his seminal and extensive entry "élémens des sciences" in the *Encyclopédie*, the key work of the Enlightenment, where he gave this analysis of and reflection on how to elementarise a science, that is how to connect the elements with the whole of that science. This procedure is to be able to identify the elements of a science, or in other words, rebuild in a new coherent way all parts of a science that may have accumulated independently and not methodically:

> "On appelle en général élémens d'un tout, les parties *primitives & originaires* dont on peut supposer que ce tout est formé". (d'Alembert 1755, 491 e)[3]

In this sense, there is no qualitative difference between the elementary parts and the higher parts. The elements are considered as the "germs" of the higher parts:

> "Ces propositions réunies en un corps, formeront, à proprement parler, les élémens de la science, puisque ces *élémens* seront comme un germe qu'il suffiroit de développer pour connoitre les objets de la science fort en détail". (d'Alembert 1755, 491 d)[4]

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[1] Quotes from ibid, p. lxxvi and lxxvi–lxxvii.

[2] Alain Trouvé has studied contributions to the notion of element by philosophers, scientists and pedagogues, since Antiquity until the early 19th century (Trouvé 2008). Essentially, it is a documentation of positions taken, without a deeper analysis. Trouvé understood "élémenter" in the traditional sense, as "simplifying the contents of teaching", a first form of what would later be called "transposition didactique" (ibid., p. 93).

[3] In general, one calls elements of a whole the primitive and original parts, of which one might suppose that this whole is formed.

[4] These propositions, united in one body, will properly constitute the elements of science, since these *elements* will be like a germ, which it would be sufficient to develop in order to know the objects of science in great detail.

An extensive part of the entry is dedicated to the reflection on elementary books—*livres élémentaires*, such as schoolbooks, which are essential, on the one hand, to disseminate the sciences and, on the other, to make progress in the sciences, that is, to obtain new truths. In his reflection on elementary books, d'Alembert emphasised another aspect of great importance regarding the relationship between the elementary and the higher: he underlined that the key issue for the composition of good elementary books consists in investigating the "metaphysics" of propositions—or in terms of today, the epistemology of science.

In the first phase of the French Revolution, the composition and publication of *livres élémentaires* constituted a key issue of the concerns for building a new society. The elaboration of *livres élémentaires* was conceived of as essential for instituting the new system of public education; practically the first measure for this task was to organise a *concours* for composing these textbooks (Schubring 1984, pp. 363 f.; Schubring 1988).[5] It is highly characteristic that in the later Napoleonic period the emphasis on *livres élémentaires* was replaced by a policy of creating *livres classiques*, focussing on the humanities (Schubring 1984, p. 371).

12.6 Klein's Practice

In fact, Klein's work can be understood exactly as providing an epistemological, or methodological access to mathematics as analysed and propagated by d'Alembert. It was not to provide factual knowledge—Klein presupposed it to have already been studied:

> I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the most important disciplines of mathematics. (Klein 2016a, b, p. 1 ff.)

Whereas he outlined as his goal:

> And it is precisely in such summarising lecture courses as I am about to deliver to you that I see one of the most important tools. (ibid., p. 1)

Indeed, Klein explicitly exposed the epistemological aspect of his work: explaining the connections, in particular the connections between sub-disciplines, which normally are treated separately, and pointing out the links of particular mathematical issues and questions with a synthetic view of the whole of mathematics. Thus, future teachers would achieve to deepening of their understanding of the basic concepts of mathematics and appreciate the nature of mathematical concepts:

> My task will always be to show you the *mutual connection between problems in the various disciplines*, these connections use not to be sufficiently considered in the specialised lecture

[5]Recently, Barbin has proposed a seemingly related notion: *élémentation*. It means, according to her, "the process by which a science is organized in view of its presentation or its teaching, and especially in the case oft he writing of a textbook" (Barbin 2015, p. 41). This notion is rather near to transposition.

> courses, and I want more especially to emphasize the relation of these problems to those of school mathematics. In this way, I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you here vivid stimuli for your teaching. (ibid., p. 2)

I should remark here that, given the methodological task of these lecture courses, Klein evidently did not aspire to elaborate any teaching unit or to propose a didactical sequence—as he always emphasised, this should be the exclusive task of the teacher, given his autonomy with regard to teaching methods. Klein was therefore always distant from what became later the dominant practice of so-called Subject didactics (*Stoff-Didaktik*) in Western Germany (see Schubring 2016).

There is a decisive difference between d'Alembert's and Klein's notion of elementarisation. Basically, d'Alembert's notion was not a historical one; he did not reflect the effect of scientific progress on the elements. But this was exactly Klein's notion. He emphasised:

> The normal process of development […] of a science is the following: higher and more complicated parts become gradually more elementary, due to the increase in the capacity to understand the concepts and to the simplification of their exposition ("law of historical shifting"). It constitutes the task of the school to verify, in view of the requirements of general education, whether the introduction of elementarised concepts into the syllabus is necessary or not. (Klein and Schimmack 1907, p. 90)

The historical evolution of mathematics entails therefore a process of restructuration of mathematics where new theories, which at first might have been somewhat isolated and poorly integrated, become well connected to other branches of mathematics and effect a new architecture of mathematics, based on re-conceived elements, and thus on a new set of elementarised concepts.

For Klein's conception of elementarisation there is a second concept, complementing the notion of "historical shifting": it is the notion of hysteresis. Hysteresis is a term from physics and since it is not so well known, here a definition within physics:

> A retardation of the effect when the forces acting upon a body are changed (as if from viscosity or internal friction); *esp.*: a lagging in the values of resulting magnetization in a magnetic material (such as iron) due to a changing magnetizing force.

Klein continued in applying this term to the relation between scholarly mathematics and school mathematics:

> In this connection I should like to say that it is not only excusable but even desirable that the schools should always lag behind the most recent advances of our science by a considerable space of time, certainly several decades; that, so to speak, a certain **hysteresis** should take place. But the hysteresis, which actually exists at the present time is in some respects unfortunately much greater. It embraces more than a century, in so far as the schools, for the most part, ignore the entire development since the time of Euler. (Klein 2016a, b, pp. 220–221; my emphasis, G.S.) (Fig. 12.1).

Klein's conception of elementarisation thus implied, regarding the curriculum, that new discoveries and developments in scholarly mathematics should have reached

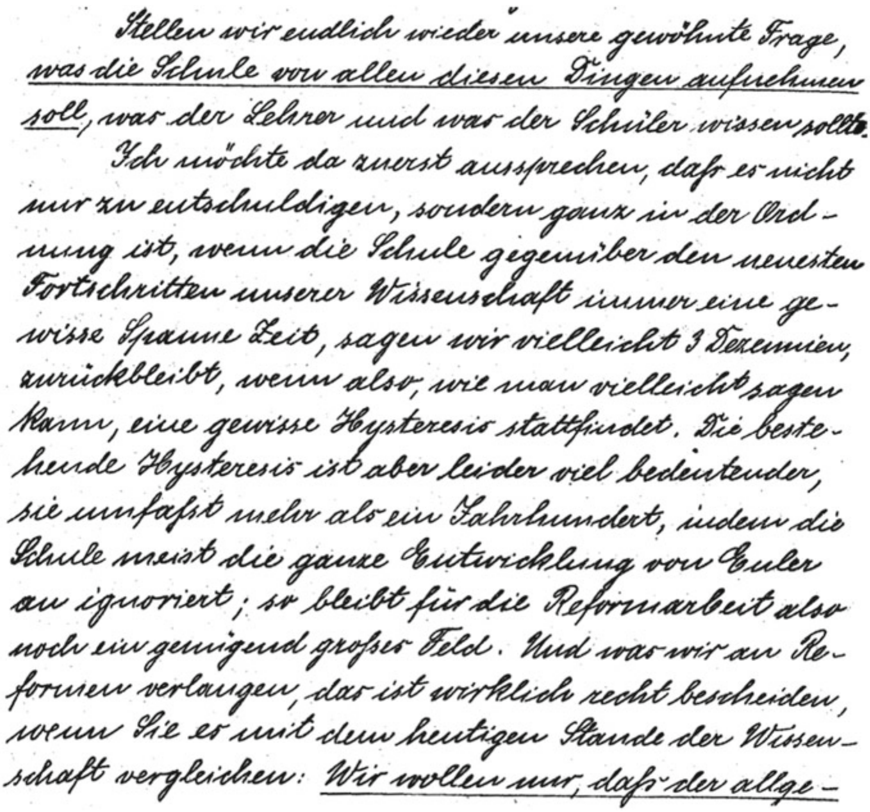
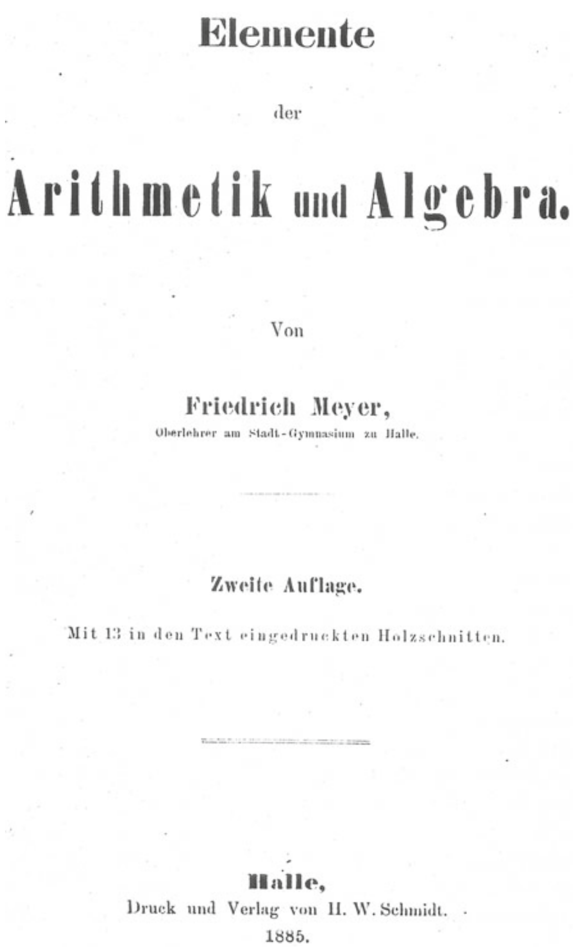

Stellen wir endlich wieder unsere gewöhnte Frage, was die Schule von allen diesen Dingen aufnehmen soll, was der Lehrer und was der Schüler wissen sollte.
Ich möchte da zuerst aussprechen, daß es nicht nur zu entschuldigen, sondern ganz in der Ordnung ist, wenn die Schule gegenüber den neuesten Fortschritten unserer Wissenschaft immer eine gewisse Spanne Zeit, sagen wir vielleicht 3 Dezennien, zurückbleibt, wenn also, wie man vielleicht sagen kann, eine gewisse Hysteresis stattfindet. Die bestehende Hysteresis ist aber leider viel bedeutender, sie umfaßt mehr als ein Jahrhundert, indem die Schule meist die ganze Entwicklung von Euler an ignoriert; so bleibt für die Reformarbeit also noch ein genügend großes Feld. Und was wir an Reformen verlangen, das ist wirklich recht bescheiden, wenn Sie es mit dem heutigen Stande der Wissenschaft vergleichen: Wir wollen nur, daß der allge-

Fig. 12.1 Original handwriting of the lithographic second edition

a certain maturity, and integration with the remainder of mathematics—in other words, a restructuration of mathematics from newly conceived elements of science. The concept of hysteresis thus meant that new developments should enter *after* this process of renewed elementarisation. It is evident that Klein did apply the concept of hysteresis in particular to set theory (Fig. 12.2) .

12.7 Modernism and the Challenge by Set Theory

Set theory was a case for Klein where this theoretical development was too fresh, and not yet accomplished and even further from having matured to the point of having induced an intra-disciplinary process of integration and restructuration. The concepts of set theory did not (yet) provide new elements for mathematics—hence Klein's polemic against Friedrich Meyer's schoolbook of 1885 who's intention had been, in fact, to use set theory as new elements for teaching arithmetic and algebra


Elemente

der

Arithmetik und Algebra.

Von

Friedrich Meyer,

Oberlehrer am Stadt-Gymnasium zu Halle.

Zweite Auflage.

Mit 13 in den Text eingedruckten Holzschnitten.

Halle,

Druck und Verlag von H. W. Schmidt.

1885.

Fig. 12.2 Title page of Meyer's schoolbook

(see Klein 2016a, b, p. 289, note 181). Meyer, mathematics teacher at a Gymnasium in Halle and friend of Cantor, introduced there the notions of set theory—not yet fully developed then by Cantor—as foundations for the number concept. Klein had sharply criticised this schoolbook in his first edition, but softened his critique in subsequent editions. In Klein's times, mathematics had not achieved the level of architecture established by Bourbaki—and hence not of "modern math".

Given that set theory has been almost identified with modernism in mathematics, I need to comment somewhat on the book by Herbert Mehrtens: *Modeme—Sprache—Mathematik* (1990), where he models Göttingen mathematics as bi-polar: Hilbert representing "modernism" and Klein representing "counter-modernism". I was always critical of this book and Mehrtens' assessments, since he misrepresents both mathematicians: Hilbert was not that theoretician and formalist who freely created abstract theories whom Mehrtens compared with the artists of that time as no longer bound by any claim to represent reality. And depicting Hilbert as “anti-intuitive” (1996, p. 521) deeply misunderstands Hilbert’s vision and practice of mathematics. Klein is, on the other hand, denounced by Mehrtens to be tied to reality and to intuition largely because German Nazi mathematicians later abused these notions.

Yet, one has to admit that Klein showed scepticism and reservation regarding set theory and axiomatics. On the one hand, he praised the progress in function theory brought about by Cantor’s new theories:

> The investigations of George Cantor, the founder of this theory, had their beginning precisely in considerations concerning the existence of transcendental numbers. They permit one to view this matter in an entirely new light.

On the other hand, Klein warned against the abstractness of set theory. Thus, he showed misgivings when he spoke of the “modern” function concept launched by Cantor:

> In connection with this, there has arisen, finally, a *still more far-reaching entirely modern generalisation of the function concept*. Up to this time, a function was thought of as always defined at every point in the continuum made up of all the real or complex values of x, or at least at every point in an entire interval or region. But since recently the concept of sets, created by Georg Cantor, has made its way more and more to the foreground, in which the continuum of all x is only an obvious example of a “set” of points. From this new standpoint functions are being considered, which are defined only for the points x of some *arbitrary set*, so that in general y is called a *function of x when to every element of a set x of things (numbers or points) there corresponds an element of a set y*. (Klein 2016a, b, p. 220)

Clearly, this abstract function concept was not at all adapted for Klein’s curricular reform programme with a function concept as its kernel, which could interrelate analysis and geometry. His misgivings were even stronger concerning his doubts as to whether all this might have applications:

> Let me point out at once a difference between this newest development and the older one. The concepts considered under headings 1. to 5. have arisen and have been developed with reference primarily to applications in nature. We need only think of the title of Fourier’s work! But the newer investigations mentioned in 6. and 7. are the result purely of the drive for mathematical research, which does not care for the needs of exploring the laws of nature, and the results have indeed found as yet no direct application. The optimist will think, of course, that the time for such application is bound to come. (ibid.)

Given Klein’s intense plea for applications, one should remark, furthermore, that he not only alerted, in the first volume in the context of the emergence of set theory, against pushing a formalist programme for the foundations of mathematics too far, but he also had taken up the issue again in volume III of his *Elementarmathematik*, advising against searching for the New only for the sake of doing it:

> Provided that a deep epistemological need exists, which will be satisfied by the study of a new problem, then it is justified to study it; but if one does it only to do something new, then the extension is not desirable. (Klein 2016a, p. 157)

Klein did even not exempt Hilbert from his critical scepticism: he commented upon Hilbert's research on the foundations of arithmetic to establish the consistency of operating with numbers:

> Obviously one can then operate with $a, b, c, \ldots$, precisely as one ordinarily does with actual numbers. (Klein 2016a, b, p. 14)

commenting:

> The tendency to crowd intuition completely off the field and to attain to really *pure* logical investigations seems to me not completely realisable. It seems to me that *one must retain a remainder*, *albeit a minimum*, *of intuition*. (ibid., p. 15)

Klein added, however, a cautious remark that he did not want to criticise Hilbert, albeit in a rather implicit manner:

> I have felt obliged to go into detail here very carefully, in as much as misunderstandings occur so often at this point, because people simply overlook the existence of the second problem. This is by no means the case with Hilbert himself, and neither disagreements nor agreements based on such an assumption can hold. (ibid., p. 16)

The second problem, which Klein is emphasising here, was put by him as the epistemological aspect of the task of justification of arithmetic—and his intention was to say that it should not be overlooked when researching upon the logical aspect of justification of arithmetic.

12.8 Concluding Remarks

In fact, school mathematics will always be confronted with the tension between logical and epistemological aspects; there can be no definite solution. But Klein's concept of hysteresis offers a viable approach to realising an elementarisation that puts school mathematics into a productive relation with the progress of mathematics.

The attractiveness of Klein's lecture notes is due to his epistemological understanding of the elements, and to not falling into the trap of practicing elementarisation as a simplification but as the challenge to understanding the connectivity and coherence of the branches and specialities of mathematics.

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