The Nonlocal *p*-Laplacian Evolution Problem on Graphs: The Continuum Limit

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Abstract. The non-local p-Laplacian evolution equation, governed by given kernel, has various applications to model diffusion phenomena, in particular in signal and image processing. In practice, such an evolution equation is implemented in discrete form (in space and time) as a numerical approximation to a continuous problem, where the kernel is replaced by an adjacency matrix of graph. The natural question that arises is to understand the structure of solutions to the discrete problem, and study their continuous limit. This is the goal pursued in this work. Combining tools from graph theory and non-linear evolution equations, we give a rigorous interpretation to the continuous limit of the discrete p-Laplacian on graphs. More specifically, we consider a sequence of deterministic simple/weighted graphs converging to a so-called *graphon*. The continuous p-Laplacian evolution equation is then discretized on this graph sequence both in space and time. We therefore prove that the solutions of the sequence of discrete problems converge to the solution of the continuous evolution problem governed by the graphon, when the number of graph vertices grows to infinity. We exhibit the corresponding convergence rates for different graph models, and point out the role of the graphon geometry and the parameter p.

Keywords: Nonlocal diffusion · p-Laplacian · Graphs · Graph limits
Numerical approximation

1 Introduction

In its continuous form, the nonlocal p-Laplacian problem with homogeneous Neumann boundary conditions governed by a given kernel K corresponds to the following nonlinear evolution equation

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = -\boldsymbol{\Delta}_p^K u(x,t), & (x,t) \in \Omega \times]0,T], \\ u(x,0) = g(x), & x \in \Omega, \end{cases} \tag{1}$$

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where

$$\boldsymbol{\Delta}_p^K = -\int_{\Omega} K(x,y)\big|u(y,t)-u(x,t)\big|^{p-2}(u(y,t)-u(x,t))dy.$$

$\Omega = [0,1]$ (without loss of generality), $K(\cdot,\cdot)$ is a symmetric, nonnegative and bounded mapping and $p \in [1,+\infty]$. The problem of existence and uniqueness of a solution to (1) is non-trivial. Relying on the theory of nonlinear semi-groups [1], we have the following theorem

Theorem 1. *Suppose $p \in]1,+\infty[$ and let $g \in L^p(\Omega)$.*

(i) For any $T > 0$, there exists a unique strong solution in $[0,T]$ of (1).
(ii) Moreover, for $q \in [1,+\infty]$, if $g_i \in L^q(\Omega)$, $i = 1,2$, and u_i is the solution of 1 with initial condition g_i, then

$$\big\|u_1(t)-u_2(t)\big\|_{L^q(\Omega)} \le \big\|g_1-g_2\big\|_{L^q(\Omega)}, \quad \forall t \in [0,T]. \tag{2}$$

There are many applications that integrate equation (1) to model nonlocal diffusion processes. It appears as the flow of gradient associated with a particular case of a functional non-local introduced in [6].

For $p \neq 2$, the discrete p-Laplacien on graphs was studied for the semi-supervised classification, as well as for various image processing applications such as simplification and unsupervised segmentation (see Figs. 1 and 2 for some illustrations). Indeed, the data in practice being discrete, graphs constitute a natural structure adapted to their representation. The nodes of this graph represent the data and the edges represent the interactions between these data. These interactions can then model a geometric proximity of the data but also other measures of similarities, depending on the application. For example, for images, we can find different types of interaction (local or non-local), according to construction of the graph, which makes it easy to find methods for processing local or non-local images.These practical considerations naturally lead to a discrete time and space approximation of (1). To do this, we fix $n \in \mathbb{N}$ and consider a partition $\mathcal{Q}_n$ on Ω

$$[(i-1)/n, i/n[, \quad i \in [n], \quad \mathcal{Q}_n = \{\Omega_i^{(n)}, i \in [n]\},$$

where $[n] = \{1,\cdots,n\}$. Let $\tau_{h-1} := \big|t_h - t_{h-1}\big|, h \in [N]$,the time steps corresponding to a division of the interval of time $[0,T]$ of maximum size $\tau = \max \tau_h$. The discrete form in time (explicit) and space of (1) is thus written

$$\begin{cases} \dfrac{u_i^h - u_i^{h-1}}{\tau_{h-1}} = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}(K_n)_{ij}\big|u_j^{h-1}-u_i^{h-1}\big|^{p-2}(u_j^{h-1}-u_i^{h-1}), \\ u_i(0) = g_i^0, \quad i \in [n]. \end{cases} \tag{3}$$

$(K_n)_{ij}$ represents the adjacency matrix of a given convergent graph sequence $\{G_n\}$ converging to a limit object called graphon $K(\cdot,\cdot)$ (see [2] for more details about graph limits). Our goal is to study the continuum limit of the discrete p-Laplacian on graphs and quantify the rate convergence and the error estimates. All the proofs of the results can be found in the long version [3].

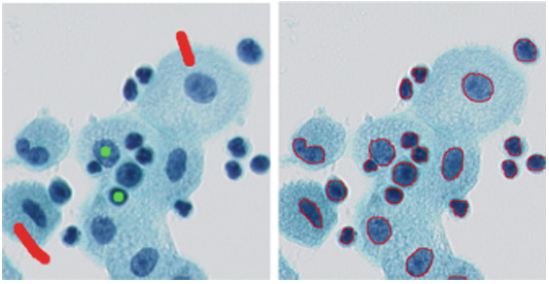

Fig. 1. Semi-supervised segmentation. Left : image with labeled vertices. Right : classified graph.

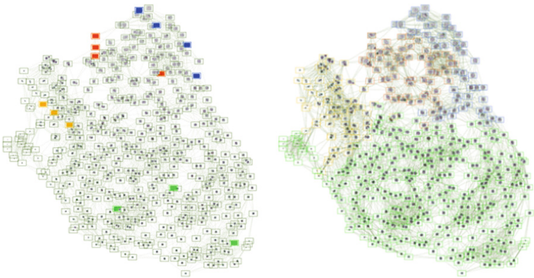

Fig. 2. Semi-supervised classification. Left : graph with initial labels. Right : segmented image.

Objectives and Contributions. The discrete formulation (3) is only an approximation of the underlying continuous problem (1). Several questions then arise: what is the structure of the solutions of the discrete problem (3)? A continuous limit, i.e. when $n \to +\infty$, does it exist? If so, what is the rate of convergence towards this limit and what is the relationship of the latter with the strong single solution of (1)? What are the parameters involved in this rate of convergence and their influence?

It is to all these questions that this article brings answers. More precisely, by combining tools from graph theory and nonlinear evolution equations, we give a rigorous interpretation to the continuous limit of the p-Laplacian discrete graph problem. To do this, we consider a sequence of graphs with n vertices whose limit object is a graphon. (1) is then discretized according to (3) on this sequence of graphs. Thus, we prove the consistency of the discrete problem (3), i.e. the convergence of the solutions of the sequence of the problems discredited towards the solution of the problem of continuous evolution governed by the graphon when $n \to +\infty$. We give the corresponding convergence rates for different graph models (simple and weighted), and we highlight the influence of graphon geometry. All the proofs of results can be found in the long version [3].

2 Prerequisites on Graphs

An undirected graph $G = (V(G), E(G))$, where $V(G)$ stands for the set of nodes and $E(G) \subset V(G) \times V(G)$ denotes the edges set, without loops and parallel edges is called simple.

A weighted graph G is a graph with weight $\beta((i,j))$ associated to each edge (i,j). The adjacency matrix of a weighted graph is obtained by replacing the 1's in the adjacency matrix by the weights of the edges. An unweighted graph is a weighted graph where all the edge weights are 1.

Let $G_n = (V(G_n), E(G_n))$, $n \in \mathbb{N}$, be a sequence of dense, finite, and simple graphs, i.e.; $\left|E(G_n)\right| = O(\left|V(G_n)\right|^2)$, where $\left|.\right|$ denotes the cardinality of a set. For two simple graphs F and G, hom(F,G) indicates the number of homomorphisms (adjacency-preserving maps) from $V(F)$ to $V(G)$. Then, it is worthwhile to normalize the homomorphism numbers and consider the homomorphism densities

$$t(F,G) = \frac{hom(F,G)}{\left|V(G)\right|^{\left|V(F)\right|}}.$$

(Thus $t(F,G)$ is the probability that a random map of $V(F)$ into $V(G)$ is a homomorphism).

Definition 1 *(cf. [2]). The sequence of graphs $\{G_n\}_n$ is called convergent if $t(F,G_n)$ is convergent for every simple graph F.*

This notion is extended to weighted graphs. To every $\phi : V(F) \to V(G)$, we have

$$hom_\phi(F,G) := \prod_{(i,j)\in E(F)} \beta_G(\phi(i), \phi(j)).$$

Then the homomorphism function is defined by

$$hom(F,G) = \sum_{\phi:V(F)\to V(G)} hom_\phi(F,G)$$

and the homomorphism density as defined for simple graphs

$$t(F,G) = \frac{hom(F,G)}{\left|V(G)\right|^{\left|V(F)\right|}}.$$

Convergent graph sequences have a limit object, which can be represented as a measurable symmetric function $K : \Omega^2 \to \Omega$, here Ω stands for $[0,1]$. Such functions are called graphons. Let $\mathcal{K}$ denote the space of all bounded measurable functions $K : \Omega^2 \to \mathbb{R}$ such that $K(x,y) = K(y,x)$ for all $x, y \in [0,1]$. We also define $\mathcal{K}_0 = \{K \in \mathcal{K} : 0 \le K \le 1\}$ the set of all graphons.

Proposition 1 **([4, Theorem 2.1]).** *For every convergent sequence of simple graphs, there is $K \in \mathcal{K}_0$ such that*

$$t(F, G_n) \to t(F, K) := \int_{\Omega^{|V(F)|}} \prod_{(i,j)\in E(F)} K(x_i, x_j)dx. \tag{4}$$

for every simple graph F. Moreover, for every $K \in \mathcal{K}_0$, there is a sequence of graphs $\{G_n\}_n$ satisfying (4).

3 Networks on Simple Graphs

We consider first the case of a sequence of simple graphs converging to $\{0,1\}$ graphon. Briefly speaking, we define a sequence of simple graphs $G_n = (V(G_n), E(G_n))$ such that $V(G_n) = [n]$ and

$$E(G_n) = \left\{(i,j) \in [n]^2 : \ \Omega_{ij}^{(n)} \cap \overline{\mathrm{supp}(K)} \neq \emptyset\right\},$$

where

$$\mathrm{supp}(K) = \left\{(x,y) \in \Omega^2 : \ K(x,y) \neq 0\right\}. \tag{5}$$

As we have mentioned before, the kernel K represents the corresponding graph limit, that is the limit as $n \to \infty$ of the function $K_{G_n} : \Omega^2 \to \{0,1\}$ such that

$$K_{G_n}(x,y) = \begin{cases} 1, & if \quad (i,j) \in E(G_n) \quad and \quad (x,y) \in \Omega_{ij}^{(n)}, \\ 0 & otherwise. \end{cases}$$

As $n \to \infty$, $\{K_{G_n}\}_n$ converges to the $\{0,1\}$-valued mapping $K(\cdot,\cdot)$ whose support is defined by (5).

Let us recall that our main goal is to compare the solutions of the discrete and continuous models and establish some consistency results. Since the solutions do not live in the same spaces, it is convenient to represent some intermediate model that is the continuous extension of the discrete problem, using the vector $U^h = (u_1^h, u_2^h, \cdots, u_n^h)^T$ whose components solve the previous system to obtain the following linear interpolation on Ω, for $x \in \Omega_i^{(n)}$, $t \in]t_{h-1}, t_h]$

$$\check{u}_n(x,t) = \frac{t_h - t}{\tau_{h-1}} u_i^{h-1} + \frac{t - t_{h-1}}{\tau_{h-1}} u_i^h \tag{6}$$

and

$$\bar{u}_n(x,t) = \sum_{h=1}^{N} u_i^{h-1} \chi_{]t_{h-1},t_h]}(t) \chi_{\Omega_i^{(n)}}(x).$$

So that $\check{u}_n(x,t)$ uniquely solves the following problem

$$\begin{cases} \frac{\partial}{\partial t}\check{u}_n(x,t) = -\boldsymbol{\Delta}_p^{K_n^s}(\bar{u}_n(x,t)), & (x,t) \in \Omega\times]0,T], \\ \check{u}_n^0(x) = g_n^0(x), & x \in \Omega, \end{cases} \tag{7}$$

As where

$$g_n^0(x) = g_i := n \int_{\Omega_i^{(n)}} g(x)dx \quad if \quad x \in \Omega_i^{(n)}, i \in [n],$$

$K_n^s(x, y)$ is the piecewise constant function such that for $(x, y) \in \Omega_{ij}^{(n)}$, $(i, j) \in [n]^2$

$$\begin{cases} n^2 \int_{\Omega_{ij}^{(n)}} K(x,y)dxdy & if \quad \Omega_i^{(n)} \times \Omega_j^{(n)} \cap \overline{\mathrm{supp}(K)} \neq \emptyset, \\ 0 & otherwise. \end{cases}$$

By analogy of what was done in [5], the rate of convergence of the solution of the discrete problem to the solution of the limiting problem depends on the regularity of the boundary $\mathrm{bd}(\overline{\mathrm{supp}(K)})$ of the support closure. Following [5], we recall the upper box-counting (or Minkowski-Bouligand) dimension of $\mathrm{bd}(\overline{\mathrm{supp}(K)})$ as a subset of $\mathbb{R}^2$:

$$\rho := \dim_B(\mathrm{bd}(\overline{\mathrm{supp}(K)})) = \limsup_{\delta \to 0} \frac{\log N_\delta(\mathrm{bd}(\overline{\mathrm{supp}(K)}))}{-\log \delta},$$

where $N_\delta(\mathrm{bd}(\overline{\mathrm{supp}(K)}))$ is the number of cells of a $(\delta \times \delta)$-mesh that intersect $\mathrm{bd}(\overline{\mathrm{supp}(K)})$ (see [?]).

Theorem 2. *Suppose that* $p \in]1, +\infty[$, $g \in L^\infty(\Omega)$, *and*

$$\rho \in [0, 2[.$$

Let u *and* $\check{u}_n$ *denote the functions corresponding to the solutions of (1) and (7), respectively.*

Then for any $\epsilon > 0$ *there exists* $N(\epsilon) \in \mathbb{N}$ *such that for any* $n \geq N(\epsilon)$

$$\|u - \check{u}_n\|_{C(0,T;L^p(\Omega))} \leq C \left(\|g - g_n\|_{L^p(\Omega)} + n^{-((2-\rho)/p - \epsilon)} \right) + O(\tau), \quad (8)$$

where the positive constant C *is independent of* n.

$C(0, T; L^p(\Omega))$ is the class of functions on $[0, T] \times \Omega$ which are uniformly continuous corresponding to the time variable and in the space $L^p(\Omega)$ (for the space variable). Theorem 2 shows that $\check{u}_n$ converges to u in $L^p(\Omega)$ when $n \to \infty$ and $\tau \to 0$. The rate of convergence depends particularly on the fractality of the boundary of the graphon K.

4 Networks on Weighted Graphs

Let $K : \Omega^2 \to [0, 1]$, be a symmetric measurable function which will be used to assign weights to the edges of the graphs considered bellow, we allow only positive weights. We define the quotient of K and $\mathcal{Q}_n$ as a weighted graph with

n nodes $K/\mathcal{Q}_n = \Big([n], [n] \times [n], \hat{K}_n\Big)$. Weights $(\hat{K}_n)_{ij}$ obtained by averaging K over the sets in $\mathcal{Q}_n$

$$(\hat{K}_n)_{ij} = n^2 \int_{\Omega_i^{(n)} \times \Omega_j^{(n)}} K(x,y) dx dy. \tag{9}$$

We consider the totally discrete counterpart of 1 on $K/\mathcal{Q}_n$

$$\begin{cases} \dfrac{u_i^h - u_i^{h-1}}{\tau_{h-1}} = \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} (\hat{K}_n)_{ij} \left| u_j^{h-1} - u_i^{h-1} \right|^{p-2} (u_j^{h-1} - u_i^{h-1}), \\ u_i(0) = g_i^0, \quad i \in [n]. \end{cases} \tag{10}$$

Hence, $\check{u}_n(x,t)$ satisfies the following problem :

$$\Big\{ \check{u}_{n_t}(x,t) = -\boldsymbol{\Delta}_p^{\hat{K}_n^w}(\bar{u}_n(x,t)), \check{u}_n^0(x) = g_n^0(x), \tag{11}$$

where K_n^w and g_n^0 are constant piecewise interpolations of $(\hat{K}_n)_{ij}$ and g_i.

Theorem 3. *Suppose that* $p \in]1, +\infty[$, $K : \Omega^2 \to [0,1]$ *is a symmetric measurable function, and* $g \in L^\infty(\Omega)$. *Let* u *and* $\check{u}_n$ *be the solutions of* 1 *and* 11, *respectively. Then*

$$\left\| u - \check{u}_n \right\|_{C(0,T;L^p(\Omega))} \longrightarrow 0, n \to \infty, \tau \to 0. \tag{12}$$

To quantify the rate of convergence in (12), we need to add some supplementary assumptions on the kernel K and the initial data g.

Definition 2. *The total variation of a function* K *is defined by duality : For* $K \in L^1_{loc}(\Omega^2)$ *it is given by*

$$J(K) = \sup \left\{ - \int_{\Omega^2} K \operatorname{div}(\phi) \; dx dy \right\}, \tag{13}$$

where

$$\phi \in S := \{\phi \in C_c^\infty(\Omega^2; \mathbb{R}^N), \left|\phi(x,y)\right| \leq 1 \forall (x,y) \in \Omega^2\}.$$

A function is said to have bounded variation whenever $J(K) < +\infty$. *We call* $\mathrm{BV}(\Omega^2)$ *the set of functions with bounded variation* $K \in L^1(\Omega^2)$ *such that* $J(K) < +\infty$.

Theorem 4. *Suppose that* $p \in]1, +\infty[$, $K : \Omega^2 \to [0,1]$ *is a symmetric and measurable function in* $\mathrm{BV}(\Omega^2)$, *and* $g \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$. *Let* u *and* $\check{u}_n$ *be the solutions of (1) and (11) respectively. Then*

$$\left\| u - \check{u}_n \right\|_{C(0,T;L^p(\Omega))} \leq O(n^{-\frac{1}{p}}) + O(\tau). \tag{14}$$

5 Conclusion and Perspectives

In this work, we deal with the nonlocal p-Laplacian problem on dynamical networks on simple and weighted graphs. We show that the approximation of solutions of the discrete problems on simple and weighted graph sequences converge to those of the continuous problem. We give also a rate of convergence estimate. Specifically, for simple graph sequences, we show how the accuracy of the approximation depends on the regularity of the boundary of support of the graph limit in the same vein as [5] who did it for a nonlocal nonlinear heat equation. In addition, for weighted graphs, we give a precise error estimate under the mild assumption that both the kernel K and the initial data g are also in Lipschitz spaces, which in particular contain functions of bounded variation.

We look in detail to the one-dimensional case, that is $\Omega = [0, 1]$, our results also hold when we deal with approximations in a multidimensional domain, since the extension to larger dimension spaces is straightforward.

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