A Single-Step 2D Thinning Scheme with Deletion of *P*-Simple Points

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Abstract. Thinning is a frequently applied technique for producing skeletons from digital binary pictures in a topology-preserving way. Bertrand proposed a two-step thinning scheme that is based on *P*-simple points. In this paper, we give two sufficient conditions for topology-preserving reductions working on the three possible 2D regular grids. The new conditions combined with parallel thinning strategies and geometrical constraints yield a single-step thinning scheme that deletes solely *P*-simple points.

Keywords: Shape analysis · Digital topology · Regular 2D grids
Topology preservation · *P*-simple points

1 Introduction

Various applications of image processing and pattern recognition are based on *skeleton*s [11]. *Thinning* is an iterative layer-by-layer reduction: the border points of an object that satisfy certain topological and geometric constraints are deleted until only the skeleton of that object is left [4,8].

A *binary picture* on a grid is a mapping that assigns a color of *black* or *white* to each grid element called *point* [8]. A regular partitioning of the 2D Euclidean space is formed by a tessellation of regular polygons (i.e., polygons having equal angles, and sides are all off the same length). There are exactly three polygons that can form such regular tessellations: the equilateral triangle, the square, and the regular hexagon [10]. Although 2D digital pictures sampled on the square grid are generally assumed, triangular and hexagonal grids also attract remarkable interest [8,10].

A *reduction* transforms a binary picture only by changing some black points to white ones which is referred to as *deletion* [8]. Reductions play a key role in some *topological algorithm*s, e.g., thinning [4,7,8].

Topology preservation is a major concern of reductions [7,8]. In [7], Kong proposed some sufficient conditions for topology-preserving 2D reductions on the square grid, then the authors of this paper adapted his results to the hexagonal and triangular grids [5].

Bertrand introduced the notion of a *P-simple point* [1], and he proved that simultaneous deletion of P-simple points is topology-preserving. His concept can be used not only as a verification method but also as a methodology to design parallel thinning algorithms.

In this paper we prove that a reduction deletes only P-simple points if and only if it satisfies our previously published sufficient conditions for topology preservation [5] for the three possible regular 2D grids. Two new sufficient conditions for topology-preserving reductions are derived from our former conditions. Those new conditions provide a single-step (topology-preserving) thinning scheme with deletion of P-simple points.

2   Basic Notions and Results

In this paper, we use the fundamental concepts of digital topology as reviewed by Kong and Rosenfeld [8]. Despite the fact that there are other approaches based on cellular/cubical complexes [9], we consider here the 'conventional paradigm' of digital topology.

Let us denote by $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{H}$ the triangular, the square, and the hexagonal grids, respectively, and throughout this paper, if we will use the notation $\mathcal{V}$, we will mean that $\mathcal{V}$ belongs to $\{\mathcal{T}, \mathcal{S}, \mathcal{H}\}$. The elements of the considered grids (i.e., regular polygons) are called *points*. Two points are 1-*adjacent* if they share an edge and they are 2-*adjacent* if they share an edge or a vertex, see Fig. 1. Let us denote by $N_j^{\mathcal{V}}(p)$ the set of points being j-adjacent to a point p in the grid $\mathcal{V}$, and let $N_j^{*\mathcal{V}}(p) = N_j^{\mathcal{V}}(p) \setminus \{p\}$ $(j = 1, 2)$.
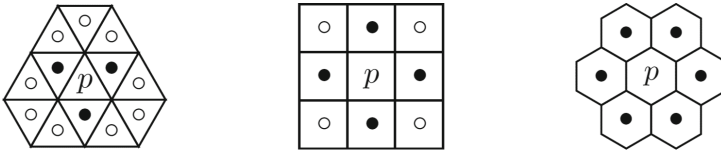
Fig. 1. The considered adjacency relations on the three possible regular 2D grids. Points that are 1-adjacent to the central point p are marked "•", while points being 2-adjacent but not 1-adjacent to p are depicted by "○".

A sequence of distinct points $\langle p_0, p_1, \ldots, p_m \rangle$ is called a j-*path* from p_0 to p_m in a non-empty set of points $X \subseteq \mathcal{V}$ if each point of the sequence is in X and p_i is j-adjacent to p_{i-1} for each $i = 1, 2, \ldots, m$. Two points are said to be j-*connected* in a set X if there is a j-path in X between them. A set of points X is j-*connected* in the set of points $Y \supseteq X$ if any two points in X are j-connected in Y. A j-*component* of a set of points X is a maximal (with respect to inclusion) j-connected subset of X.

Let $(k, \bar{k})$ be an ordered pair of adjacency relations. Throughout this paper, it is assumed that $(k, \bar{k})$ belongs to $\{(1, 2), (2, 1)\}$. A $(k, \bar{k})$ *binary digital picture*

is a quadruple $(\mathcal{V}, k, \bar{k}, B)$ [8], where set $\mathcal{V}$ contains all points of the considered grid, $B \subseteq \mathcal{V}$ denotes the set of *black point*s, and each point in $\mathcal{V} \setminus B$ is said to be a *white point*. A *black component* or *object* is a k-component of B, while a *white component* is a $\bar{k}$-component of $\mathcal{V} \setminus B$.

A black point p is said to be a *border point* if p is $\bar{k}$-adjacent to at least one white point (i.e., $N_{\bar{k}}^{*\mathcal{V}}(p) \setminus B \neq \emptyset$). A border-point p is called an *isolated point* if all points in $N_k^{*\mathcal{V}}(p)$ are white (i.e., $\{p\}$ is a singleton object).

A reduction in a 2D picture is *topology-preserving* if each object in the input picture contains exactly one object in the output picture, and each white component in the output picture contains exactly one white component in the input picture [8].

A black point is said to be *simple* for a set of black points if its deletion is a topology-preserving reduction [7,8]. In [5], the authors gave both formal and easily visualized characterizations of simple points in all the considered five types of pictures on the regular 2D grids (i.e., two for $\mathcal{T}$, two for $\mathcal{S}$, and one for $\mathcal{H}$). Note that the simpleness of p is a local property (i.e., it can be decided by examining $N_2^{*\mathcal{V}}(p)$), and only non-isolated border points may be simple [5].

In [5], the authors formulated the following unified sufficient conditions for topology-preserving reductions:

Theorem 1. [5] *For any picture $(\mathcal{V}, k, \bar{k}, B)$, a reduction is topology-preserving if all of the following conditions hold.*

1. *Only simple points for B are deleted.*
2. *For any two $\bar{k}$-adjacent black points, $p, q \in B$ that are deleted, p is simple for $B \backslash \{q\}$.*
3. *If $(k, \bar{k}) = (2, 1)$, no object is deleted completely, such that is composed by two or more mutually 2-adjacent points, and is not formed by two 1-adjacent points.*

In order to construct topology-preserving reductions and provide a verification method, Bertrand proposed the notion of a *P-simple point*:

Definition 1. [1] *Let $Q \subset B$ be a set of black points in a picture. A point $q \in Q$ is called a P-simple point for Q if q is simple for $B \backslash R$ for any $R \subseteq Q \backslash \{q\}$.*

Bertrand gave a local characterization of P-simple points in $(26, 6)$-pictures on the 3D cubic grid [1], and Bertrand and Couprie reported a similar characterization in $(2, 1)$-pictures on the square grid $\mathcal{S}$ [3]. Later, the authors of this paper presented both formal and easily visualized characterizations of P-simple points in all the considered five types of 2D pictures [6].

Bertrand's approach provides the following sufficient condition for topology preservation:

Theorem 2. [1] *A reduction that deletes a subset composed solely of P-simple points is topology-preserving.*

3   Linking *P*-Simple Points and Our Sufficient Conditions

In this section, we show that the two kinds of conditions stated by Theorems 1 and 2 are equivalent.

Since the simpleness of a point is a local property, we can state the following proposition:

Proposition 1. *Let $Q \subset B$ be a set of points in a picture $(\mathcal{V}, k, \bar{k}, B)$. A point $q \in Q$ is P-simple for Q if for any $R \subseteq N_2^{*\mathcal{V}}(q) \cap Q$, q is simple for $B \backslash R$.*

Theorem 3. *A reduction satisfies all conditions of Theorem 1 if and only if it deletes only P-simple points.*

Proof. In order to prove the '*if*' part, let us assume that a reduction deletes the set of *P*-simple points $Q \subset B$ (for Q).

– Let $q \in Q$ be a point that is deleted by the reduction. Since $\emptyset \subset Q$, by Definition 1, q is simple for $B \backslash \emptyset = B$. Hence Condition 1 of Theorem 1 holds.
– Let $q_1, q_2 \in Q$ be two $\bar{k}$-adjacent points that are deleted by the reduction in question. As $\{q_2\} \subseteq Q \backslash \{q_1\}$, by Definition 1, q_1 is simple for $B \backslash \{q_2\}$, which means that Condition 2 of Theorem 1 is also fulfilled.
– Let us assume that Condition 3 of Theorem 1 is violated. Hence there is an object Q (according to Condition 3 of Theorem 1) that is completely deleted by the reduction. Consider a point $q \in Q$. Since the investigated reduction deletes only *P*-simple points, q is simple for $B \backslash (Q \setminus \{q\})$ by Definition 1. Since q is an isolated (non-simple black) point for $B \backslash (Q \setminus \{q\})$, we arrived at a contradiction.

Then we establish the '*only if*' part for $(1, 2)$-pictures on $\mathcal{S}$. Let us assume that a reduction satisfies all conditions of Theorem 1 and it deletes the set of points $Q \subset B$ from a picture $(\mathcal{S}, 1, 2, B)$.

In [5], the authors gave easily visualized characterizations of simple points in all the considered five types of pictures by sets of matching templates. The four base matching templates assigned to $(1, 2)$-pictures on $\mathcal{S}$ are depicted in Fig. 2.

Let $p \in Q$. Since Condition 1 of Theorem 1 holds, p is simple for B. Without loss of generality, we may investigate only the four templates shown in Fig. 2.

By Proposition 1, the following cases are to be investigated:

(a) Assume that p is matched by the template in Fig. 2a. In this case, only black point q is to be investigated. Since p is a non-simple (isolated black) point for $B \backslash \{q\}$, by Condition 2 of Theorem 1, $q \notin Q$.
(b) Assume that p is matched by the template in Fig. 2b. It can be readily seen that p is not simple for $B \backslash \{s\}$. Hence, by Condition 2 of Theorem 1, $s \notin Q$. Let us investigate the remaining two black points q and r.
   - Assume that $q \in Q$ and $r \notin Q$. Since p is matched by the template in Fig. 2a in $B \backslash \{q\}$, p remains simple after the deletion of q.
   - Assume that $r \in Q$ and $q \notin Q$. Since p is matched by the template in Fig. 2a in $B \backslash \{r\}$, p remains simple after the deletion of r.
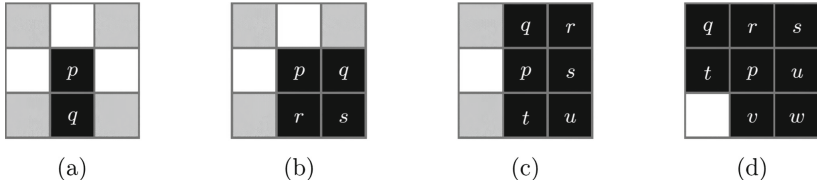
Fig. 2. The four base matching templates for characterizing a simple point p in $(1, 2)$-pictures on $\mathcal{S}$. Notations: each black template position matches a black point; each white element matches a white point; each position depicted in gray matches either a white point or a black point. All the rotated and reflected versions of these four templates also match simple points.

- Assume that $q \in Q$ and $r \in Q$. Since q is not simple for $B\backslash\{r\}$, and r is not simple for $B\backslash\{q\}$, by Condition 2 of Theorem 1, we arrived at a contradiction.

(c) Assume that p is matched by the template in Fig. 2c. It can be readily seen that p is not simple for $B\backslash\{r\}$, $B\backslash\{s\}$, and $B\backslash\{u\}$. Hence, by Condition 2 of Theorem 1, the three points r, s, and u are not in Q. Let us investigate the remaining two black points q and t.

- Assume that $q \in Q$ and $t \notin Q$. Since p is matched by the template in Fig. 2b in $B\backslash\{q\}$, p remains simple after the deletion of q.
- Assume that $t \in Q$ and $q \notin Q$. Since p is matched by the template in Fig. 2b in $B\backslash\{t\}$, p remains simple after the deletion of t.
- Assume that $q \in Q$ and $t \in Q$. Since p is matched by the template in Fig. 2a in $B\backslash\{q, t\}$, p remains simple after the deletion of $\{q, t\}$.

(d) Assume that p is matched by the template in Fig. 2d. It can be readily seen that p is not simple for $B\backslash\{q\}$, $B\backslash\{r\}$, $B\backslash\{s\}$, $B\backslash\{u\}$, and $B\backslash\{w\}$. Hence, by Condition 2 of Theorem 1, the five points q, r, s, u, and w are not in Q. Let us investigate the remaining two black points t and v.

- Assume that $t \in Q$ and $v \notin Q$. Since p is matched by the template in Fig. 2c in $B\backslash\{t\}$, p remains simple after the deletion of t.
- Assume that $v \in Q$ and $t \notin Q$. Since p is matched by the template in Fig. 2c in $B\backslash\{v\}$, p remains simple after the deletion of v.
- Assume that $t \in Q$ and $v \in Q$. Since p is matched by the template in Fig. 2b in $B\backslash\{t, v\}$, p remains simple after the deletion of $\{t, v\}$.

Since p remains simple after the deletion of each subset of Q, p is a P-simple point for Q. Hence only P-simple points are deleted.

The proofs of the '*only if*' part for the remaining four kinds of pictures can be carried out analogously with the help of matching templates given in [5]. □

4   New Sufficient Conditions

Theorem 1 (i.e., our unified sufficient conditions for topology-preserving reductions) provides a method of verifying that a thinning algorithm always preserves

topology. The following new conditions allow us to directly construct deletion rules for topological algorithms.

Theorem 4. *For any picture* $(\mathcal{V}, k, \bar{k}, B)$, *a reduction* **R** *satisfies all conditions of Theorem 1 if the following conditions hold for each point* $p \in B$ *deleted by* **R***:*

1. *p is simple for B.*
2. *For any $q \in N_{\bar{k}}^{*\mathcal{V}}(p) \cap B$ that is simple for B, p is simple for $B \backslash \{q\}$.*
3. *If $(k, \bar{k}) = (2, 1)$, p does not belong to an object that is composed by two or more mutually 2-adjacent points, and is not formed by two 1-adjacent points.*

Proof. It can be readily seen that if **R** satisfies Condition i of this theorem, Condition i of Theorem 1 is also satisfied for each $i \in \{1, 2, 3\}$.   □

Let us consider the addressing schemes depicted in Fig. 3, which refer every point in the three possible regular 2D grids by a pair of coordinates. The *lexicographical order relation* '$\prec$' between two distinct points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ is defined as follows: $p \prec q \Leftrightarrow (p_y < q_y) \vee ((p_y = q_y) \wedge (p_x < q_x))$. Let Q be a finite subset of $\mathcal{V}$. Then, point $p \in Q$ is said to be the *smallest element* of Q if for any $q \in Q \backslash \{p\}$, $p \prec q$.
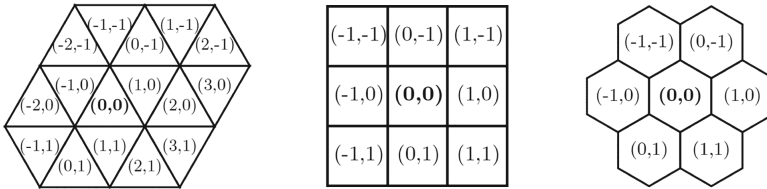
Fig. 3. Feasible addressing schemes for the three possible regular 2D grids.

Theorem 5. *For any picture* $(\mathcal{V}, k, \bar{k}, B)$, *a reduction* **R** *satisfies all conditions of Theorem 1 if the following conditions hold for each point* $p \in B$ *deleted by* **R***:*

1. *p is simple for B.*
2. *For any $q \in N_{\bar{k}}^{*\mathcal{V}}(p) \cap B$ that is simple for B and $p \prec q$, p is simple for $B \backslash \{q\}$.*
3. *If $(k, \bar{k}) = (2, 1)$, p is not the smallest element of an object that is composed by two or more mutually 2-adjacent points, and is not formed by two 1-adjacent points.*

Proof. It can be readily seen that if **R** satisfies Condition i of this theorem, Condition i of Theorem 1 is also satisfied for each $i \in \{1, 2, 3\}$.   □

The new sufficient conditions (i.e., Theorems 4 and 5) directly specify the following five pairs of reductions $(\mathbf{R}_{\text{sym}}^{\mathcal{T},1,2}, \mathbf{R}_{\text{asym}}^{\mathcal{T},1,2})$, $(\mathbf{R}_{\text{sym}}^{\mathcal{T},2,1}, \mathbf{R}_{\text{asym}}^{\mathcal{T},2,1})$, $(\mathbf{R}_{\text{sym}}^{\mathcal{S},1,2}, \mathbf{R}_{\text{asym}}^{\mathcal{S},1,2})$, $(\mathbf{R}_{\text{sym}}^{\mathcal{S},2,1}, \mathbf{R}_{\text{asym}}^{\mathcal{S},2,1})$, and $(\mathbf{R}_{\text{sym}}^{\mathcal{H},1,2}, \mathbf{R}_{\text{asym}}^{\mathcal{H},1,2}) = (\mathbf{R}_{\text{sym}}^{\mathcal{H},2,1}, \mathbf{R}_{\text{asym}}^{\mathcal{H},2,1})$, where $\mathbf{R}_{\text{sym}}^{\mathcal{V},k,\bar{k}}$ deletes all points from a $(k, \bar{k})$ picture on $\mathcal{V}$ that satisfy all conditions of Theorem 4, and $\mathbf{R}_{\text{asym}}^{\mathcal{V},k,\bar{k}}$ deletes all points from a $(k, \bar{k})$ picture on $\mathcal{V}$ that satisfy all conditions of Theorem 5.

The following proposition is an easy consequence of Theorems 3, 4, and 5:

Proposition 2. *Reductions* $\mathbf{R}_{sym}^{\mathcal{V},k,\bar{k}}$ *and* $\mathbf{R}_{asym}^{\mathcal{V},k,\bar{k}}$ *delete only P-simple points.*

Figure 4 is to illustrate the different behaviors of $\mathbf{R}_{\text{sym}}^{\mathcal{S},2,1}$ and $\mathbf{R}_{\text{asym}}^{\mathcal{S},2,1}$. It is important to emphasize the disparity between Conditions 2 and 3 of Theorems 4 and 5. Theorem 4 ensures *symmetric* conditions, since both $\bar{k}$-adjacent black points p and q are preserved if p is not simple after the deletion of q (in that case, q is not simple after the deletion of p, as well), and all 'small' objects composed by three or more mutually k-adjacent points remain unchanged. The conditions of Theorem 5 are *asymmetric*, since only the smallest elements of pairs of points and objects considered in Conditions 2 and 3 are preserved.
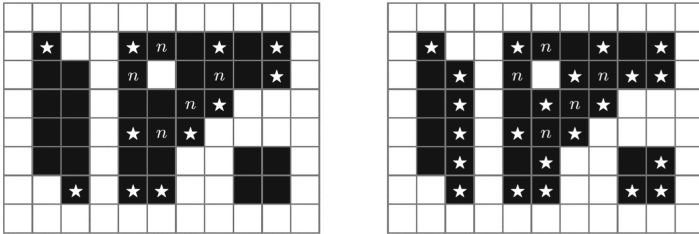
Fig. 4. Comparison of reductions $\mathbf{R}_{\text{sym}}^{\mathcal{S},2,1}$ (left) and $\mathbf{R}_{\text{asym}}^{\mathcal{S},2,1}$ (right) for a $(2, 1)$ picture on the square grid $\mathcal{S}$. Non-simple points are marked 'n', and deleted (simple and P-simple) points are marked white stars. (Unmarked black points are simple but non-deleted points.)

5   A Single-Step 2D Thinning Scheme

In [2], Bertrand proposed a two-step (topology-preserving) thinning scheme that is based on P-simple points. One phase/reduction of the iterative thinning process is performed as follows:

1. A set of points $Q \subset B$ is (somehow) chosen and labeled.
2. All P-simple points in Q are deleted (simultaneously).

Note that Step 2 deals with *tricolor* pictures (say: the value '0' corresponds to white points, the value '1' is assigned to the elements in $B \backslash Q$, and value '2' corresponds to points in Q). Hence that two-step scheme is both space- and time-consuming.

Proposition 2 provides a single-step (topology-preserving) thinning scheme with deletion of *P*-simple points. The deletion rule of a reduction of the iterative thinning process can be directly constructed by combining the reduction $\mathbf{R}_{\mathrm{sym}}^{\mathcal{V},k,\bar{k}}$ (or $\mathbf{R}_{\mathrm{asym}}^{\mathcal{V},k,\bar{k}}$) with different thinning strategies (i.e., *fully parallel*, *subiteration-based*, and *subfield-based* [4]) and various geometric constraints (say *endpoints* [4]). The generated deletion rule is a common *Boolean function* that is to be evaluated for the neighborhood of the investigated points in binary (*two-level*) pictures. That Boolean function can be stored in a pre-calculated *look-up-table*, hence, the proposed single-step scheme can be implemented efficiently.

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