Ordered Level Planarity, Geodesic Planarity and Bi-Monotonicity

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Abstract. We introduce and study the problem ORDERED LEVEL PLANARITY which asks for a planar drawing of a graph such that vertices are placed at prescribed positions in the plane and such that every edge is realized as a y-monotone curve. This can be interpreted as a variant of LEVEL PLANARITY in which the vertices on each level appear in a prescribed total order. We establish a complexity dichotomy with respect to both the maximum degree and the level-width, that is, the maximum number of vertices that share a level. Our study of ORDERED LEVEL PLANARITY is motivated by connections to several other graph drawing problems.

GEODESIC PLANARITY asks for a planar drawing of a graph such that vertices are placed at prescribed positions in the plane and such that every edge e is realized as a polygonal path p composed of line segments with two adjacent directions from a given set S of directions symmetric with respect to the origin. Our results on ORDERED LEVEL PLANARITY imply $\mathcal{NP}$-hardness for any S with $|S| \geq 4$ even if the given graph is a matching. Katz, Krug, Rutter and Wolff claimed that for matchings MANHATTAN GEODESIC PLANARITY, the case where S contains precisely the horizontal and vertical directions, can be solved in polynomial time [GD 2009]. Our results imply that this is incorrect unless $\mathcal{P} = \mathcal{NP}$. Our reduction extends to settle the complexity of the BI-MONOTONICITY problem, which was proposed by Fulek, Pelsmajer, Schaefer and Štefankovič.

ORDERED LEVEL PLANARITY turns out to be a special case of T-LEVEL PLANARITY, CLUSTERED LEVEL PLANARITY and CONSTRAINED LEVEL PLANARITY. Thus, our results strengthen previous hardness results. In particular, our reduction to CLUSTERED LEVEL PLANARITY generates instances with only two non-trivial clusters. This answers a question posed by Angelini, Da Lozzo, Di Battista, Frati and Roselli.

Due to space constraints, some proofs in this manuscript are only sketched or omitted entirely. Full proofs of all claims can be found in the appendix of the preprint [17].

1   Introduction

In this paper we introduce ORDERED LEVEL PLANARITY and study its complexity. We establish connections to several other graph drawing problems, which we survey in this first section. We proceed from general problems to more and more constrained ones.

Upward Planarity: An *upward* planar drawing of a directed graph is a plane drawing where every edge $e = (u, v)$ is realized as a y-monotone curve that goes upward from u to v. Such drawings provide a natural way of visualizing a partial order on a set of items. The problem UPWARD PLANARITY of testing whether a directed graph has an upward planar drawing is $\mathcal{NP}$-complete [10]. However, if the y-coordinate of each vertex is prescribed, the problem can be solved in polynomial time [15]. This is captured by the notion of level graphs.

Level Planarity: A *level graph* $\mathcal{G} = (G, \gamma)$ is a directed graph $G = (V, E)$ together with a *level assignment* $\gamma : V \to \{0, \dots, h\}$ where γ is a surjective map with $\gamma(u) < \gamma(v)$ for every edge $(u, v) \in E$. Value h is the *height* of $\mathcal{G}$. The vertex set $V_i = \{v \mid \gamma(v) = i\}$ is called the i-th *level* of $\mathcal{G}$ and $\lambda_i = |V_i|$ is its *width*. The *level-width* λ of $\mathcal{G}$ is the maximum width of any level in $\mathcal{G}$. A *level* planar drawing of $\mathcal{G}$ is an upward planar drawing of G where the y-coordinate of each vertex v is $\gamma(v)$. The horizontal line with y-coordinate i is denoted by L_i. The problem LEVEL PLANARITY asks whether a given level graph has a level planar drawing. The study of the complexity of LEVEL PLANARITY has a long history [7,9,13–15], culminating in a linear-time approach [15]. LEVEL PLANARITY has been extended to drawings of level graphs on surfaces different from the plane such as standing cylinder, a rolling cylinder or a torus [1,3,4].

An important special case are *proper* level graphs, that is, level graphs in which $\gamma(v) = \gamma(u) + 1$ for every edge $(u, v) \in E$. Instances of LEVEL PLANARITY can be assumed to be proper without loss of generality by subdividing long edges [7,15]. However, in variations of LEVEL PLANARITY where we impose additional constraints, the assumption that instances are proper can have a strong impact on the complexity of the respective problems [2].

Level Planarity with Various Constraints: CLUSTERED LEVEL PLANARITY is a combination of CLUSTER PLANARITY and LEVEL PLANARITY. The task is to find a level planar drawing while simultaneously visualizing a given cluster hierarchy according to the rules of CLUSTER PLANARITY. The problem is $\mathcal{NP}$-complete in general [2], but efficiently solvable for proper instances [2,8].

T-LEVEL PLANARITY is a consecutivity-constrained version of LEVEL PLANARITY: every level V_i is equipped with a tree T_i whose set of leaves is V_i. For every inner node u of T_i the leaves of the subtree rooted at u have to appear consecutively along L_i. The problem is $\mathcal{NP}$-complete in general [2], but efficiently solvable for proper instances [2,18]. The precise definitions of both problems and a longer discussion about the related work can be found in [17].

Very recently, Brückner and Rutter [6] explored a variant of LEVEL PLANARITY in which the left-to-right order of the vertices on each level has to

be a linear extension of a given partial order. They refer to this problem as CONSTRAINED LEVEL PLANARITY and they provide an efficient algorithm for single-source graphs and show $\mathcal{NP}$-completeness of the general case.

A Common Special Case - Ordered Level Planarity: We introduce a natural variant of LEVEL PLANARITY that specifies a total order for the vertices on each level. An *ordered* level graph $\mathcal{G}$ is a triple $(G = (V, E), \gamma, \chi)$ where (G, γ) is a level graph and $\chi : V \to \{0, \ldots, \lambda - 1\}$ is a *level ordering* for G. We require that χ restricted to domain V_i bijectively maps to $\{0, \ldots, \lambda_i - 1\}$. An *ordered level planar drawing* of an ordered level graph $\mathcal{G}$ is a level planar drawing of (G, γ) where for every $v \in V$ the x-coordinate of v is $\chi(v)$. Thus, the position of every vertex is fixed. The problem ORDERED LEVEL PLANARITY asks whether a given ordered level graph has an ordered level planar drawing.

In the above definitions, the x- and y-coordinates assigned via χ and γ merely act as a convenient way to encode total and partial orders respectively. In terms of realizability, the problems are equivalent to generalized versions where χ and γ map to the reals. In other words, the fixed vertex positions can be any points in the plane. All reductions and algorithms in this paper carry over to these generalized versions, if we pay the cost for presorting the vertices according to their coordinates. ORDERED LEVEL PLANARITY is also equivalent to a relaxed version where we only require that the vertices of each level V_i appear along L_i according to the given total order without insisting on specific coordinates. We make use of this equivalence in many of our figures for the sake of visual clarity.

Geodesic Planarity: Let $S \subset \mathbb{Q}^2$ be a finite set of directions symmetric with respect to the origin, i.e. for each direction $s \in S$, the reverse direction $-s$ is also contained in S. A plane drawing of a graph is *geodesic* with respect to S if every edge is realized as a polygonal path p composed of line segments with two adjacent directions from S. Two directions of S are *adjacent* if they appear consecutively in the projection of S to the unit circle. Such a path p is a geodesic with respect to some polygonal norm that corresponds to S. An instance of the decision problem GEODESIC PLANARITY is a 4-tuple $\mathcal{G} = (G = (V, E), x, y, S)$ where G is a graph, x and y map from V to the reals and S is a set of directions as stated above. The task is to decide whether $\mathcal{G}$ has a *geodesic drawing*, that is, G has a geodesic drawing with respect to S in which every vertex $v \in V$ is placed at $(x(v), y(v))$.

Katz et al. [16] study MANHATTAN GEODESIC PLANARITY, which is the special case of GEODESIC PLANARITY where the set S consists of the two horizontal and the two vertical directions. Geodesic drawings with respect to this set of direction are also referred to as orthogeodesic drawings [11,12]. Katz et al. [16] show that a variant of MANHATTAN GEODESIC PLANARITY in which the drawings are restricted to the integer grid is $\mathcal{NP}$-hard even if G is a perfect matching. The proof is by reduction from 3-PARTITION and makes use of the fact the number of edges that can pass between two vertices on a grid line is bounded. In contrast, they claim that the standard version of MANHATTAN GEODESIC PLANARITY is polynomial-time solvable for perfect matchings [16, Theorem 5]. To this end, they sketch a plane sweep algorithm that maintains

a linear order among the edges that cross the sweep line. When a new edge is encountered it is inserted as low as possible subject to the constraints implied by the prescribed vertex positions. When we asked the authors for more details, they informed us that they are no longer convinced of the correctness of their approach. Theorem 2 of our paper implies that the approach is indeed incorrect unless $\mathcal{P} = \mathcal{NP}$.

Bi-Monotonicity: Fulek et al. [9] present a Hanani-Tutte theorem for y-*monotone* drawings, that is, upward drawings in which all vertices have distinct y-coordinates. They accompany their result with a simple and efficient algorithm for Y-MONOTONICITY, which is equivalent to LEVEL PLANARITY restricted to instances with level-width $\lambda = 1$. They propose the problem BI-MONOTONICITY and leave its complexity as an open problem. The input of BI-MONOTONICITY is a triple $\mathcal{G} = (G = (V, E), x, y)$ where G is a graph and x and y injectively map from V to the reals. The task is to decide whether $\mathcal{G}$ has a bi-monotone drawing, that is, a plane drawing in which edges are realized as curves that are both y-monotone and x-monotone and in which every vertex $v \in V$ is placed at $(x(v), y(v))$.

Main Results: In Sect. 3 we study the complexity of ORDERED LEVEL PLANARITY. While UPWARD PLANARITY is $\mathcal{NP}$-complete [10] in general but becomes polynomial-time solvable [15] for prescribed y-coordinates, we show that prescribing both x-coordinates and y-coordinates renders the problem $\mathcal{NP}$-complete. We complement our result with efficient approaches for some special cases of ordered level graphs and, thereby, establish a complexity dichotomy with respect to the level-width and the maximum degree.

Theorem 1. ORDERED LEVEL PLANARITY *is* $\mathcal{NP}$*-complete, even for maximum degree* $\Delta = 2$ *and level-width* $\lambda = 2$*. For level-width* $\lambda = 1$ *or* $\Delta^+ = \Delta^- = 1$ *or proper instances* ORDERED LEVEL PLANARITY *can be solved in linear time, where* Δ^+ *and* Δ^- *are the maximum in-degree and out-degree respectively.*

ORDERED LEVEL PLANARITY restricted to instances with $\lambda = 2$ and $\Delta = 2$ is an elementary problem. We expect that it may serve as a suitable basis for future reductions. As a proof of concept, the remainder of this paper is devoted to establishing connections between ORDERED LEVEL PLANARITY and several other graph drawing problems. Theorem 1 serves as our key tool for settling their complexity. In Sect. 2 we study GEODESIC PLANARITY and obtain:

Theorem 2. GEODESIC PLANARITY *is* $\mathcal{NP}$*-hard for any set of directions* S *with* $|S| \geq 4$ *even for perfect matchings in general position.*

Observe the aforementioned discrepancy between Theorem 2 and the claim by Katz et al. [16] that MANHATTAN GEODESIC PLANARITY for perfect matchings is in $\mathcal{P}$. BI-MONOTONICITY is closely related to a special case of MANHATTAN GEODESIC PLANARITY. With a simple corollary we settle the complexity of BI-MONOTONICITY and, thus, answer the open question by Fulek et al. [9].

Theorem 3. BI-MONOTONICITY *is $\mathcal{NP}$-hard even for perfect matchings.*

ORDERED LEVEL PLANARITY is an immediate and very constrained special case of CONSTRAINED PLANARITY. Further, we establish ORDERED LEVEL PLANARITY as a special case of both CLUSTERED LEVEL PLANARITY and T-LEVEL PLANARITY by providing the following reductions.

Theorem 4. ORDERED LEVEL PLANARITY *with maximum degree $\Delta = 2$ and level-width $\lambda = 2$ reduces in linear time to* T-LEVEL PLANARITY *with maximum degree $\Delta' = 2$ and level-width $\lambda' = 4$.*

Theorem 5. ORDERED LEVEL PLANARITY *with maximum degree $\Delta = 2$ and level-width $\lambda = 2$ reduces in quadratic time to* CLUSTERED LEVEL PLANARITY *with maximum degree $\Delta' = 2$, level-width $\lambda' = 2$ and $\kappa' = 3$ clusters.*

Angelini et al. [2] propose the complexity of CLUSTERED LEVEL PLANARITY for clustered level graphs with a flat cluster hierarchy as an open question. Theorem 5 answers this question by showing that $\mathcal{NP}$-hardness holds for instances with only two non-trivial clusters.

2   Geodesic Planarity and Bi-Monotonicity

In this section we establish that deciding whether an instance $\mathcal{G} = (G, x, y, S)$ of GEODESIC PLANARITY has a geodesic drawing is $\mathcal{NP}$-hard even if G is a perfect matching and even if the coordinates assigned via x and y are in *general position*, that is, no two vertices lie on a line with a direction from S. The $\mathcal{NP}$-hardness of BI-MONOTONICITY for perfect matchings follows as a simple corollary. Our results are obtained via a reduction from ORDERED LEVEL PLANARITY.

Lemma 1. *Let $S \subset \mathbb{Q}^2$ with $|S| \geq 4$ be a finite set of directions symmetric with respect to the origin.* ORDERED LEVEL PLANARITY *with maximum degree $\Delta = 2$ and level-width $\lambda = 2$ reduces to* GEODESIC PLANARITY *such that the resulting instances are in general position and consist of a perfect matching and direction set S. The reduction can be carried out using a linear number of arithmetic operations.*

Proof Sketch. In this sketch, we prove our claim only for the classical case that S contains exactly the four horizontal and vertical directions. Our reduction is carried out in two steps. Let $\mathcal{G}_o = (G_o = (V, E), \gamma, \chi)$ be an ORDERED LEVEL PLANARITY instance with maximum degree $\Delta = 2$ and level-width $\lambda = 2$. In Step (i) we turn $\mathcal{G}_o$ into an equivalent GEODESIC PLANARITY instance $\mathcal{G}'_g = (G_o, x', \gamma, S)$. In Step (ii) we transform $\mathcal{G}'_g$ into an equivalent GEODESIC PLANARITY instance $\mathcal{G}_g = (G_g, x, y, S)$ where G_g is a perfect matching and the vertex positions assigned via x and y are in general position.

Step (i): In order to transform $\mathcal{G}_o$ into $\mathcal{G}'_g$ we apply a shearing transformation. We translate the vertices of each level V_i by $3i$ units to the right, see Fig. 1(a)
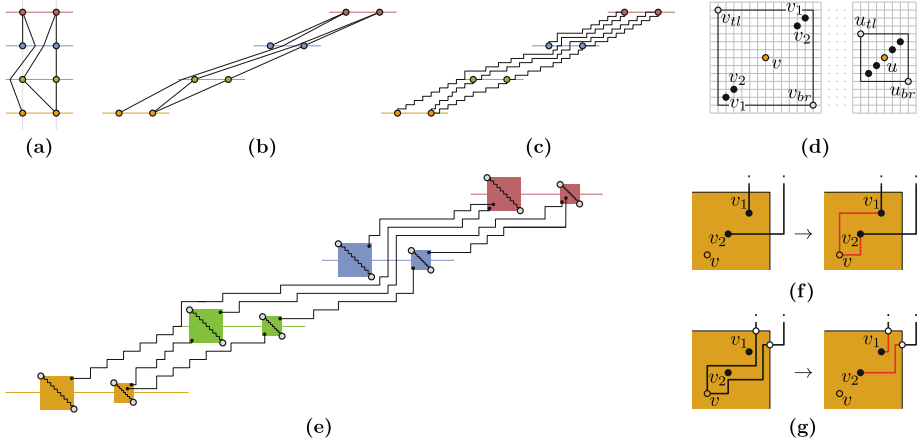
Fig. 1. (a), **(b)** and **(c)**: Illustrations of Step (i). **(d)** The two gadget squares of each level. Grid cells have size $1/48 \times 1/48$. **(e)** Illustration of Step (ii). Turning a drawing of $\mathcal{G}_g$ into a drawing of $\mathcal{G}'_g$ **(f)** and vice versa **(g)**.

and (b). Clearly, every geodesic drawing of $\mathcal{G}'_g$ can be turned into an ordered level planar drawing of $\mathcal{G}_o$. On the other hand, consider an ordered level planar drawing Γ_o of $\mathcal{G}_o$. Without loss of generality we can assume that in Γ_o all edges are realized as polygonal paths in which bend points occur only on the horizontal lines L_i through the levels V_i where $0 \leq i \leq h$. Further, we may assume that all bend points have x-coordinates in the open interval $(-1, 2)$. We shear Γ_o by translating the bend points and vertices of level V_i by $3i$ units to the right for $0 \leq i \leq h$, see Fig. 1(b). In the resulting drawing Γ'_o, the vertex positions match those of $\mathcal{G}'_g$. Furthermore, all edge-segments have a positive slope. Thus, since the maximum degree is $\Delta = 2$ we can replace all edge-segments with L_1-geodesic rectilinear paths that closely trace the segments and we obtain a geodesic drawing Γ'_g of $\mathcal{G}'_g$, see Fig. 1(c).

Step (ii): In order to turn $\mathcal{G}'_g = (G_o = (V, E), x', \gamma, S)$ into the equivalent instance $\mathcal{G}_g = (G_g, x, y, S)$ we transform G_o into a perfect matching. To this end, we split each vertex $v \in V$ by replacing it with a small gadget that fits inside a square r_v centered on the position $p_v = (x'(v), \gamma(v))$ of v, see Fig. 1(e). We call r_v the *square* of v and use p_v^{tr}, p_v^{tl}, p_v^{br} and p_v^{bl} to denote the top-right, top-left, bottom-right and bottom-left corner of r_v, respectively. We use two different sizes to ensure general position. The size of the gadget square is $1/4 \times 1/4$ if $\chi(v) = 0$ and it is $1/8 \times 1/8$ if $\chi(v) = 1$. The gadget contains a degree-1 vertex for every edge incident to v. In the following we explain the gadget construction in detail, for an illustration see Fig. 1(d). Let $\{v, u\}$ be an edge incident to v. We create an edge $\{v_1, u\}$ where v_1 is a new vertex which is placed at $p_v^{tr} - (1/48, 1/48)$ if u is located to the top-right of v and it is placed at $p_v^{bl} + (1/48, 1/48)$ if u is located to the bottom-left of v. Similarly,

if v is incident to a second edge $\{v, u'\}$, we create an edge $\{v_2, u'\}$ where v_2 is placed at $p_v^{tr} - (1/24, 1/24)$ or $p_v^{bl} + (1/24, 1/24)$ depending on the position of u'. Finally, we create a *blocking* edge $\{v_{tl}, v_{br}\}$ where v_{tl} is placed at p_v^{tl} and v_{br} is placed at p_v^{br}. The thereby assigned coordinates are in general position and the construction can be carried out in linear time.

Assume that $\mathcal{G}_g$ has a geodesic drawing Γ_g. By construction, all blocking edges have a top-left and a bottom-right endpoint. On the other hand, all other edges have a bottom-left and a top-right endpoint. As a result, a non-blocking edge $e = \{u, v\}$ can not pass through any gadget square r_w, except the squares r_u or r_v since e would have to cross the blocking edge of r_w. Accordingly, it is straight-forward to obtain a geodesic drawing of Γ_g': We remove the blocking edges, reinsert the vertices of V according to the mappings x' and γ and connect them to the vertices of their respective gadgets in a geodesic fashion. This can always be done without crossings. Figure 1(f) shows one possibility. If the edge from v_2 passes to the left of v_1, we may have to choose a reflected version. Finally, we remove the vertices v_1 and v_2 which now act as subdivision vertices.

On the other hand, let Γ_g' be a geodesic planar drawing of $\mathcal{G}_g'$. Without loss of generality, we can assume that each edge $\{u, v\}$ passes only through the squares of u and v. Furthermore, for each $v \in V$ we can assume that its incident edges intersect the boundary of r_v only to the top-right of $p_v^{tr} - (1/48, 1/48)$ or to the bottom-left of $p_v^{bl} + (1/48, 1/48)$, see Fig. 1(g). Thus, we can simply remove the parts of the edges in the interior of the gadget squares and connect the gadget vertices to the intersection points of the edges with the gadget squares in a geodesic fashion.   □

The bit size of the numbers involved in the calculations of our reduction is linearly bounded in the bit size of the directions of S. Together with Theorem 1 we obtain the proof of Theorem 2. The instances generated by Lemma 1 are in general position. In particular, this means that the mappings x and y are injective. We obtain an immediate reduction to Bi-Monotonicity. The correctness follows from the fact that every L_1-geodesic rectilinear path can be transformed into a bi-monotone curve and vice versa. Thus, we obtain Theorem 3.

3   Ordered Level Planarity

To show $\mathcal{NP}$-hardness of Ordered Level Planarity we reduce from a 3-Satisfiability variant described in this paragraph. A *monotone* 3-Satisfiability formula is a Boolean 3-Satisfiability formula in which each clause is either *positive* or *negative*, that is, each clause contains either exclusively positive or exclusively negative literals respectively. A *planar* 3SAT formula $\varphi = (\mathcal{U}, \mathcal{C})$ is a Boolean 3-Satisfiability formula with a set $\mathcal{U}$ of variables and a set $\mathcal{C}$ of clauses such that its *variable-clause graph* $G_\varphi = (\mathcal{U} \uplus \mathcal{C}, E)$ is planar. The graph G_φ is bipartite, i.e. every edge in E is incident to both a *clause* vertex from $\mathcal{C}$ and a *variable* vertex from $\mathcal{U}$. Furthermore, edge $\{c, u\} \in E$ if and only if a literal of variable $u \in \mathcal{U}$ occurs in $c \in \mathcal{C}$. Planar Monotone

3-SATISFIABILITY is a special case of 3-SATISFIABILITY where we are given a planar and monotone 3-SATISFIABILITY formula φ and a *monotone rectilinear representation* $\mathcal{R}$ of the variable-clause graph of φ. The representation $\mathcal{R}$ is a contact representation on an integer grid in which the variables are represented by horizontal line segments arranged on a line ℓ. The clauses are represented by E-shapes turned by 90° such that all positive clauses are placed above ℓ and all negative clauses are placed below ℓ, see Fig. 2a. PLANAR MONOTONE 3-SATISFIABILITY is $\mathcal{NP}$-complete [5]. We are now equipped to prove the core lemma of this section.

Lemma 2. PLANAR MONOTONE 3-SATISFIABILITY *reduces in polynomial time to* ORDERED LEVEL PLANARITY. *The resulting instances have maximum degree* $\Delta = 2$ *and all vertices on levels with width at least 3 have out-degree at most 1 and in-degree at most 1.*
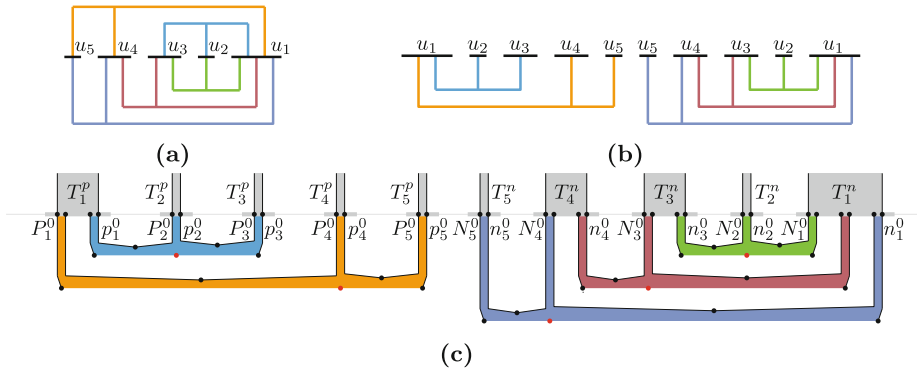

Fig. 2. (a) Representation $\mathcal{R}$ of φ with negative clauses $(\overline{u}_1 \vee \overline{u}_4 \vee \overline{u}_5)$, $(\overline{u}_1 \vee \overline{u}_3 \vee \overline{u}_4)$ and $(\overline{u}_1 \vee \overline{u}_2 \vee \overline{u}_3)$ and positive clauses $(u_1 \vee u_4 \vee u_5)$ and $(u_1 \vee u_2 \vee u_3)$ and **(b)** its modified version $\mathcal{R}'$ in Lemma 2. **(c)** Tier $\mathcal{T}_0$.

Proof Sketch. We perform a polynomial-time reduction from PLANAR MONO-TONE 3-SATISFIABILITY. Let $\varphi = (\mathcal{U}, \mathcal{C})$ be a planar and monotone 3-SATISFIABILITY formula with $\mathcal{C} = \{c_1, \ldots, c_{|\mathcal{C}|}\}$. Let G_φ the variable-clause graph of φ. Let $\mathcal{R}$ be a monotone rectilinear representation of G_φ. We construct an ordered level graph $\mathcal{G} = (G, \gamma, \chi)$ such that $\mathcal{G}$ has an ordered level planar drawing if and only if φ is satisfiable. In this proof sketch we omit some technical details such as precise level assignments and level orderings.

Overview: The ordered level graph $\mathcal{G}$ has $l_3 + 1$ levels which are partitioned into four *tiers* $\mathcal{T}_0 = \{0, \ldots, l_0\}$, $\mathcal{T}_1 = \{l_0 + 1, \ldots, l_1\}$, $\mathcal{T}_2 = \{l_1 + 1, \ldots, l_2\}$ and $\mathcal{T}_3 = \{l_2 + 1, \ldots, l_3\}$. Each clause $c_i \in \mathcal{C}$ is associated with a *clause edge* $e_i = (c_i^s, c_i^t)$ starting with c_i^s in tier $\mathcal{T}_0$ and ending with c_i^t in tier $\mathcal{T}_2$. The clause edges have to be drawn in a system of tunnels that encodes the 3-SATISFIABILITY formula φ. In $\mathcal{T}_0$ the layout of the tunnels corresponds directly

to the rectilinear representation $\mathcal{R}$, see Fig. 2c. For each E-shape there are three tunnels corresponding to the three literals of the associated clause. The bottom vertex c_i^s of each clause edge e_i is placed such that e_i has to be drawn inside one of the three tunnels of the E-shape corresponding to c_i. This corresponds to the fact that in a satisfying truth assignment every clause has at least one satisfied literal. In tier $\mathcal{T}_1$ we merge all the tunnels corresponding to the same literal. We create variable gadgets that ensure that for each variable u edges of clauses containing u can be drawn in the tunnel associated with either the negative or the positive literal of u but not both. This corresponds to the fact that every variable is set to either true or false. Tiers $\mathcal{T}_2$ and $\mathcal{T}_3$ have a technical purpose.

We proceed by describing the different tiers in detail. Recall that in terms of realizability, ORDERED LEVEL PLANARITY is equivalent to the generalized version where γ and χ map to the reals. For the sake of convenience we will begin by designing $\mathcal{G}$ in this generalized setting. It is easy to transform $\mathcal{G}$ such that it satisfies the standard definition in a polynomial-time post processing step.
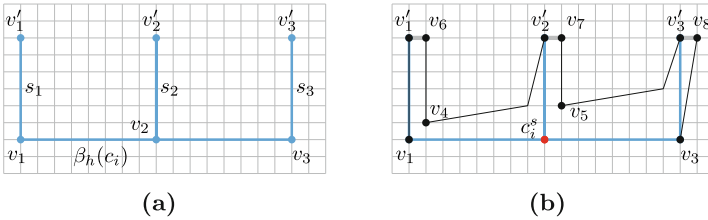

Fig. 3. **(a)** The E-shape and **(b)** the clause gadget of clause c_i. The thick gray lines represent the gates of c_i.

Tier 0 and 2, Clause Gadgets: The clause edges $e_i = (c_i^s, c_i^t)$ end in tier $\mathcal{T}_2$. It is composed of $l_2 - l_1 = |\mathcal{C}|$ levels each of which contains precisely one vertex. We assign $\gamma(c_i^t) = l_1 + i$. Observe that this imposes no constraint on the order in which the edges enter $\mathcal{T}_2$.

Tier $\mathcal{T}_0$ consists of a system of tunnels that resembles the monotone rectilinear representation $\mathcal{R}$ of $G_\varphi = (\mathcal{U} \uplus \mathcal{C}, E)$, see Fig. 2c. Intuitively it is constructed as follows: We take the top part of $\mathcal{R}$, rotate it by $180°$ and place it to the left of the bottom part such that the variables' line segments align, see Fig. 2b. We call the resulting representation $\mathcal{R}'$. For each E-shape in $\mathcal{R}'$ we create a *clause gadget*, which is a subgraph composed of 11 vertices that are placed on a grid close to the E-shape, see Fig. 3. The red vertex at the bottom is the lower vertex c_i^s of the clause edge e_i of the clause c_i corresponding to the E-shape. Without loss of generality we assume the grid to be fine enough such that the resulting ordered level graph can be drawn as in Fig. 2c without crossings. Further, we assume that the y-coordinates of every pair of horizontal segments belonging to distinct E-shapes differ by at least 3. This ensures that all vertices on levels with width at least 3 have out-degree at most 1 and in-degree at most 1 as stated in the lemma.
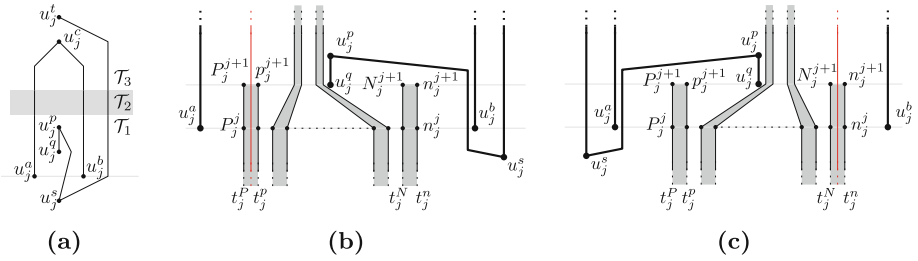
Fig. 4. (a) The variable gadget of u_j in **(b)** positive and **(c)** negative state.

The clause gadget (without the clause edge) has a *unique* ordered level planar drawing in the sense that for every level V_i the left-to-right sequence of vertices and edges intersected by the horizontal line L_i through V_i is identical in every ordered level planar drawing. This is due to the fact that the order of the top-most vertices v_1', v_6, v_2', v_7, v_3' and v_8 is fixed. We call the line segments $v_1'v_6$, $v_2'v_7$ and $v_3'v_8$ the *gates* of c_i. Note that the clause edge e_i has to intersect one of the gates of c_i. This corresponds to the fact the at least one literal of every clause has to be satisfied.

The subgraph G_0 induced by $\mathcal{T}_0$ (without the clause edges) has a unique ordered level planar drawing. In tier $\mathcal{T}_1$ we bundle all gates that belong to one literal together by creating two long paths for each literal. These two paths form the *tunnel* of the corresponding literal. All clause edges intersecting a gate of some literal have to be drawn inside the literal's tunnel, see Fig. 2c. To this end, for $j = 1, \ldots, |\mathcal{U}|$ we use N_j^0 (n_j^0) to refer to the left-most (right-most) vertex of a negative clause gadget placed on a line segment of $\mathcal{R}'$ representing $u_j \in \mathcal{U}$. The vertices N_j^0 and n_j^0 are the first vertices of the paths forming the *negative* tunnel T_j^n of the negative literal of variable u_j. Analogously, we use P_j^0 (p_j^0) to refer to the left-most (right-most) vertex of a positive clause gadget placed on a line segment of $\mathcal{R}'$ representing u_j. The vertices P_j^0 and p_j^0 are the first vertices of the paths forming the *positive* tunnel T_j^p of the positive literal of variable u_j. If for some j the variable u_j is not contained both in negative and positive clauses, we artificially add two vertices N_j^0 and n_j^0 or P_j^0 and p_j^0 on the corresponding line segments in order to avoid having to treat special cases in the remainder of the construction.

Tier 1 and 3, Variable Gadgets: Recall that every clause edge has to pass through a gate that is associated with some literal of the clause, and, thus, every edge is drawn in the tunnel of some literal. We need to ensure that it is not possible to use tunnels associated with the positive, as well as the negative literal of some variable simultaneously. To this end, we create a *variable gadget* with vertices in tier $\mathcal{T}_1$ and tier T_3 for each variable. The variable gadget of variable u_j is illustrated in Fig. 4a. The variable gadgets are nested in the sense that they start in $\mathcal{T}_1$ in the order $u_1, u_2, ..., u_{|\mathcal{U}|}$, from bottom to top and they end in the reverse order in $\mathcal{T}_3$, see Fig. 5. We force all tunnels with index at least j to

be drawn between the vertices u_j^a and u_j^b. This is done by subdividing the tunnel edges on this level, see Fig. 4b. The *long edge* (u_j^s, u_j^t) has to be drawn to the left or right of u_j^c in $\mathcal{T}_3$. Accordingly, it is drawn to the left of u_j^a or to the right of u_j^b in $\mathcal{T}_1$. Thus, it is drawn either to the right (Fig. 4b) of all the tunnels or to the left (Fig. 4c) of all the tunnels. As a consequence, the *blocking edge* (u_j^s, u_j^p) is also drawn either to the right or the left of all the tunnels. Together with the edge (u_j^q, u_j^p) it prevents clause edges from being drawn either in the positive tunnel T_j^p or negative tunnel T_j^n of variable u_j which end at level $\gamma(u_j^q)$ because they can not reach their endpoints in $\mathcal{T}_2$ without crossings. We say T_j^p or T_j^n are *blocked* respectively.

The construction of the ordered level graph $\mathcal{G}$ can be carried out in polynomial time. Note that maximum degree is $\Delta = 2$ and that all vertices on levels with width at least 3 have out-degree at most 1 and in-degree at most 1 as claimed in the lemma.

Correctness: It remains to show that $\mathcal{G}$ has an ordered level planar drawing if and only if φ is satisfiable. Assume that $\mathcal{G}$ has an ordered level planar drawing Γ. We create a satisfying truth assignment for φ. If T_j^n is blocked we set u_j to true, otherwise we set u_j to false for $j \in 1, \ldots, |\mathcal{U}|$. Recall that the subgraph G_0 induced by the vertices in tier $\mathcal{T}_0$ has a unique ordered level planar drawing. Consider a clause c_i and let f, g, j be the indices of the variables whose literals are contained in c_i. Clause edge $e_i = (e_i^s, e_i^t)$ has to pass level l_0 through one of the gates of c_i. More precisely, it has to be drawn between either N_f^0 and n_f^0,
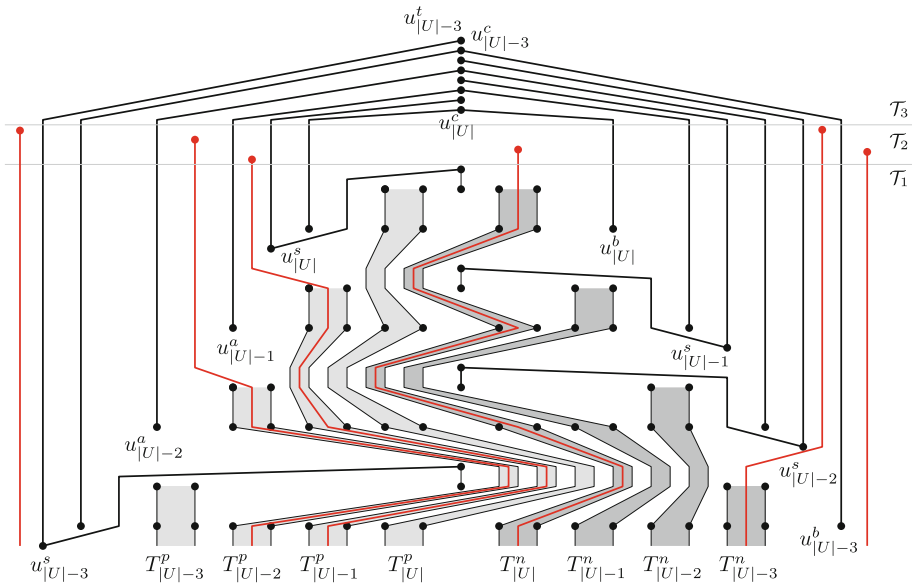

Fig. 5. Nesting structure of the variable gadgets.

N_g^0 and n_g^0 or N_j^0 and n_j^0 if c_i is negative or between either P_f^0 and p_f^0, P_g^0 and p_g^0 or P_j^0 and p_j^0 if c_i is positive, see Fig. 2c. First, assume that c_i is negative and assume without loss of generality that it traverses l_0 between N_j^0 and n_j^0. In this case clause edge e_i has to be drawn in T_j^n. Recall that this is only possible if T_j^n is not blocked, which is the case if u_j is false, see Fig. 4c. Analogously, if c_i is positive and e_i traverses w.l.o.g. between p_j^P and p_j^p, then u_j is true, Fig. 4b. Thus, we have established that one literal of each clause in $\mathcal{C}$ evaluates to true for our truth assignment and, hence, formula φ is satisfiable.

Now assume that φ is satisfiable and consider a satisfying truth assignment. We create an ordered level planar drawing Γ of $\mathcal{G}$. It is clear how to create the unique subdrawing of G_0. The variable gadgets are drawn in a nested fashion, see Fig. 5. For $j = 1, \ldots, |\mathcal{U}| - 1$ we draw edge (u_j^a, u_j^c) to left of u_{j+1}^a and u_{j+1}^s and edge (u_j^b, u_j^c) to right of u_{j+1}^b and u_{j+1}^s. In other words, the pair $((u_j^a, u_j^c), (u_j^b, u_j^c))$ is drawn between all such pairs with index smaller than j. Recall that the vertices u_j^a, u_j^b, u_j^s, u_j^p and u_j^q are located on higher levels than the according vertices of variables with index smaller than j and that u_j^t and u_j^c are located on lower levels than the according vertices of variables with index smaller than j.

For $j = 1, \ldots, |\mathcal{U}|$ if u_j is positive we draw the long edge (u_j^s, u_j^t) to the right of u_j^b and u_j^c and, accordingly, we have to draw all tunnels left of u_j^s and u_j^q (except for T_j^n, which has to be drawn to the left of u_j^s and end to the right of u_j^q), see Fig. 4b. If u_j is negative we draw the long edge (u_j^s, u_j^t) to the left of u_j^b and u_j^c and, accordingly, we have to draw all tunnels right of u_j^s and u_j^q (except for T_j^p, which has to be drawn to the right of u_j^s and end to the left of u_j^q), see Fig. 4c. We have to draw the blocking edge (u_j^s, u_j^p) to the right of n_j^{j+1} if u_j is positive and to the left of P_j^{j+1} if u_j is negative.

It remains to describe how to draw the clause edges. Let c_i be a clause. There is at least one true literal in c_i. Let k be the index of the corresponding variable. We describe the drawing of clause edge $e_i = (c_i^s, c_i^t)$ from bottom to top. We start by drawing e_i in the tunnel T_k^p (T_k^n) if c_i is positive (negative). After the variable gadget of u_k the edge e_i leaves its tunnel and is drawn to the left (right) of all gadgets of variables with higher index, see Fig. 5.   □

We obtain $\mathcal{NP}$-hardness for instances with maximum degree $\Delta = 2$. In fact, we can restrict our attention to instances level-width $\lambda = 2$. To this end, we split levels with width $\lambda_i > 2$ into $\lambda_i - 1$ levels containing exactly two vertices each.

Lemma 3. *An instance* $\mathcal{G} = (G = (V, E), \gamma, \chi)$ *of* ORDERED LEVEL PLANARITY *with maximum degree* $\Delta \leq 2$ *can be transformed in linear time into an equivalent instance* $\mathcal{G}' = (G' = (V', E'), \gamma', \chi')$ *of* ORDERED LEVEL PLANARITY *with level-width* $\lambda' \leq 2$ *and maximum degree* Δ'. *If in* $\mathcal{G}$ *all vertices on levels with width at least 3 have out-degree at most 1 and in-degree at most 1, then* $\Delta' \leq 2$. *Otherwise,* $\Delta' \leq \Delta + 1$.

The reduction in Lemma 2 requires degree-2 vertices. With $\Delta = 1$, the problem becomes polynomial-time solvable. In fact, even if $\Delta = 2$ one can easily

solve it as long as the maximum in-degree and the maximum out-degree are both bounded by 1. Such instances consists of a set P of y-monotone paths.

We write $p \prec q$, meaning that $p \in P$ must be drawn to the left of $q \in P$, if p and q have vertices v_p and v_q that lie adjacent on a common level. If $\prec$ is acyclic, we can draw $\mathcal{G}$ according to a linear extension of $\prec$, otherwise there exists no solution.

Lemma 4. ORDERED LEVEL PLANARITY *restricted to instances with maximum in-degree* $\Delta^- = 1$ *and maximum out-degree* $\Delta^+ = 1$ *can be solved in linear time.*

For $\lambda = 1$ ORDERED LEVEL PLANARITY is solvable in linear time since LEVEL PLANARITY can be solved in linear time [15]. Proper instances can be solved in linear-time via a sweep through every level. The problem is obviously contained in $\mathcal{NP}$. The results of this section establish Theorem 1.

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