Colored Point-Set Embeddings of Acyclic Graphs

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Abstract. We show that any planar drawing of a forest of three stars whose vertices are constrained to be at fixed vertex locations may require $\Omega(n^{\frac{2}{3}})$ edges each having $\Omega(n^{\frac{1}{3}})$ bends in the worst case. The lower bound holds even when the function that maps vertices to points is not a bijection but it is defined by a 3-coloring. In contrast, a constant number of bends per edge can be obtained for 3-colored paths and for 3-colored caterpillars whose leaves all have the same color. Such results answer to a long standing open problem.

1 Introduction

A pioneering paper by Pach and Wenger [9] studied the problem of computing a planar drawing of a graph G with the constraint that the mapping of the vertices to the points in the plane, that represent the vertices of G, is given as part of the input. Pach and Wenger proved that, for any given mapping, a planar graph with n vertices admits a planar drawing such that the curve complexity, i.e. the number of bends per edge, is $O(n)$. Furthermore, they proved that the bound on the curve complexity is (almost surely) tight as n tends to infinity when G has $O(n)$ independent edges. This implies that the curve complexity of a planar drawing with vertices at fixed locations may be $\Omega(n)$ even for structurally very simple graphs such as paths or matchings, for which the number of independent edges is linear in n.

These results have motivated the study of a relaxed version of the problem where the function that associates vertices of the graph to points of the plane is not a bijection. Namely, an instance of the k-colored point set embeddability problem receives as input an n-vertex planar graph G such that every vertex is

given one of k distinct colors and a set S of n distinct points, such that, each point is given one of the k distinct colors. The number of points of S having a certain color i is the same as the number of vertices of G having color i. The goal is to compute a planar drawing of G with curve complexity independent of n where every vertex of a specific color is represented by a point of the same color. When $k = n$ the k-colored point set embeddability problem coincides with the problem of computing a drawing with vertices at fixed locations and thus the lower bounds by Pach and Wenger hold. Therefore several papers have focused on small values of k (typically $k \leq 3$) to see whether better bounds on the curve complexity could be achieved in this scenario (see, e.g., [1,4–7]).

For $k = 1$, Kaufmann and Wiese [8] proved that every planar graph admits a 1-colored point set embedding onto any point set with curve complexity at most 2. For $k = 2$, outerplanar graphs always admit a 2-colored point set embedding with $O(1)$ curve complexity [2]. However, for $k \geq 2$, there are 2-connected k-colored planar graphs for which a k-colored point set embedding may require $\Omega(n)$ bends on $\Omega(n)$ edges [1]. This result extends the lower bound of Pach and Wenger [9] to a much more relaxed set of constraints on the location of the vertices, but it does so by using 2-connected graphs instead of just (not necessarily connected) planar graphs. For example, the problem of establishing tight bounds on the curve complexity of k-colored forests for small values of $k \geq 3$ is a long standing open problem (see, e.g., [1]). We explicitly address this gap in the literature and consider the k-colored point set embeddability problem for acyclic graphs and $k \geq 3$. Our main results are as follows.

– In Sect. 3, we prove that a planar drawing of a forest of three stars and n vertices may require $\Omega(n^{\frac{2}{3}})$ edges with $\Omega(n^{\frac{1}{3}})$ bends each, even if the mapping of the vertices to the points is defined by using a set of k colors with $k \geq 3$. In contrast, a constant number of bends per edge can always be achieved if the number of stars is at most two (for any number of colors) or the number of colors is at most two (for any number of stars).
– Since the above result implies that 3-colored point set embeddings of 3-colored caterpillars may have a non-constant curve complexity, in Sect. 4 we study subfamilies of 3-colored caterpillars for which constant curve complexity is possible. We prove that every 3-colored path and every 3-colored caterpillar whose leaves all have the same color admit a 3-colored point-set emebdding with constant curve complexity onto any 3-colored point set.
– Finally, still in Sect. 4, we prove that any 4-colored path π such that the vertices of colors 0 and 1 precede all vertices of colors 2 and 3 when moving along π has a 4-colored point set embedding with at most five bends per edge onto any 4-colored point-set.

Concerning the lower bound, it is worth mentioning that the argument by Pach and Wenger [9] does not apply to families of graphs where the number of independent edges is not a function of n. Hence, our lower bound extends the one by Pach and Wenger about the curve complexity of planar drawings with vertices at fixed locations also to those graphs for which the number of

independent edges does not grow with n. For reasons of space some proofs are omitted in this paper and can be found in [3].

2   Preliminaries

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a partition $\{V_0, V_1, \ldots, V_{k-1}\}$ of V. The integers $0, 1, \ldots, k-1$ are called *colors* and G is called a *k-colored graph*. For each vertex $v \in V_i$ we denote by $col(v)$ the color i of v.

Let S be a set of distinct points in the plane. For any point $p \in S$, we denote by $x(p)$ and $y(p)$ the x- and y-coordinates of p, respectively. We denote by $CH(S)$ the convex hull of S. Throughout the paper we always assume that the points of S have different x-coordinates (if not we can rotate the plane so to achieve this condition). A *k-coloring* of S is a partition $\{S_0, S_1, \ldots, S_{k-1}\}$ of S. A set of points S with a k-coloring is called a *k-colored point set*. For each point $p \in S_i$, $col(p)$ denotes the color i of p. A k-colored point set S is *compatible with* a k-colored graph G if $|V_i| = |S_i|$ for every i. If G is planar we say that G has a *topological point-set embedding* on S if there exists a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as simple Jordan arc. We say that G has a *k-colored point-set embedding* on S if there exists a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as a polyline λ. A point shared by any two consecutive segments of λ is called a *bend* of e. The maximum number of bends along an edge is the *curve complexity* of the k-colored point-set embedding. A *k-colored sequence* σ is a sequence of (possibly repeated) colors c_0, $c_1, \ldots, c_{n-1}$ such that $0 \leq c_j \leq k-1$ $(0 \leq j \leq n-1)$. We say that σ is *compatible with* a k-colored graph G if color i occurs $|V_i|$ times in σ. Let S be a k-colored point set. Let $p_0, \ldots, p_{n-1}$ be the points of S with $x(p_0) < \ldots < x(p_{n-1})$. The k-colored sequence $col(p_0), \ldots col(p_{n-1})$ is called the *k-colored sequence induced by S*, and is denoted as $seq(S)$. A set of points S is *one-sided convex* if they are in convex position and the two points with minimum and maximum x-coordinate are consecutive along $CH(S)$. In a k-colored one-sided convex point set, the sequence of colors encountered clockwise along $CH(S)$, starting from the point with minimum x-coordinate, coincides with $seq(S)$.

A *Hamiltonian cycle* of a graph G is a simple cycle that contains all vertices of G. A graph G that admits a Hamiltonian cycle is said to be *Hamiltonian*. A planar graph G is *sub-Hamiltonian* if either G is Hamiltonian or G can be augmented with dummy edges (but not with dummy vertices) to a graph $\mathsf{aug}(G)$ that is Hamiltonian and planar. A *subdivision* of a graph G is a graph obtained from G by replacing each edge by a path with at least one edge. Internal vertices on such a path are called *division vertices*. Every planar graph has a subdivision that is sub-Hamiltonian. Let G be a planar graph and let $\mathsf{sub}(G)$ be a sub-Hamiltonian subdivision of G. The graph $\mathsf{aug}(\mathsf{sub}(G))$ is called a *Hamiltonian augmentation of G* and will be denoted as $\mathsf{Ham}(G)$. Let $\mathcal{C}$ be the Hamiltonian cycle of a Hamiltonian augmentation $\mathsf{Ham}(G)$ of G. Let e be an edge of $\mathcal{C}$,

let $\mathcal{P} = \mathcal{C} \setminus e$ be the Hamiltonian path obtained by removing e from $\mathcal{C}$, and let $v_0, v_1, \ldots, v_{n'-1}$ be the vertices of G in the order they appear along $\mathcal{P}$. Finally, let $\sigma = c_0, c_1, \ldots, c_{n'-1}$ be a k-colored sequence. $\mathcal{P}$ is a *k-colored Hamiltonian path consistent with σ* if $\mathrm{col}(v_i) = c_i$ $(0 \leq i \leq n' - 1)$. $\mathcal{C}$ is a *k-colored Hamiltonian cycle consistent with σ* if there exists an edge $e \in \mathcal{C}$ such that $\mathcal{P} = \mathcal{C} \setminus e$ is a k-colored Hamiltonian path consistent with σ. $\mathsf{Ham}(G)$ is called a *k-colored Hamiltonian augmentation of G consistent with σ*. The following theorem has been proved in [2] (see also [1,6]).

Theorem 1. [2] *Let G be a k-colored planar graph and S be a k-colored point set consistent with G. If G has a k-colored Hamiltonian augmentation consistent with $seq(S)$ and at most d division vertices per edge then G admits a k-colored point-set embedding on S with at most $2d + 1$ bends per edge.*

The next lemma can be easily derived from Theorem 1.

Lemma 1. *Let G be a k-colored graph, and S be a k-colored one-sided convex point set compatible with G. If G has a topological k-colored point-set embedding on S such that each edge crosses $CH(S)$ at most b times, then G admits a k-colored point-set embedding on S with at most $2b + 1$ bends per edge.*

Let $G = (V, E)$ be a planar graph. A *topological book embedding* of G is a planar drawing such that all vertices of G are represented as points of a horizontal line ℓ, called the *spine*. Each of the half-planes defined by ℓ is a *page*. Each edge of a topological book embedding is either in the top page, or completely in the bottom page, or it can be on both pages, in which case it crosses the spine. Each crossing between an edge and the spine is called a *spine crossing*. It is also assumed that in a topological book embedding every edge consists of one or more circular arcs, such that no two consecutive arcs are in the same page[1]. Let G be a k-colored graph and let σ be a k-colored sequence compatible with G. A topological book embedding of G is consistent with σ if the sequence of vertex colors along the spine coincides with σ. Let S be a k-colored point set compatible with a k-colored planar graph G and let $seq(S)$ be the k-colored sequence induced by S. The following lemma can be proved similarly to Lemma 1.

Lemma 2. *If G admits a topological book embedding consistent with $seq(S)$ and having at most h spine crossing per edge, then G admits a point-set embedding on S with curve complexity at most $2h + 1$.*

3   Point-Set Embeddings of Stars

In this section we establish that a 3-colored point-set embedding of a forest of three stars may require $\Omega(n^{\frac{1}{3}})$ bends along $\Omega(n^{\frac{2}{3}})$ edges by exploiting a previous

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[1] The more general concept of *h-page topological book embedding* exists, where each arc can be drawn on one among h different pages. For simplicity we use the term topological book embedding to mean 2-page topological book embedding.

result about biconnected outerplanar graphs. We start by recalling the result in [2]. An *alternating point set* S_n is a 3-colored one-sided convex point set such that: (i) S_n has n points for each color 0, 1, and 2, and (ii) when going along the convex hull $CH(S_n)$ of S_n in clockwise order, the sequence of colors encountered is $0, 1, 2, 0, 1, 2, \ldots$. Each set of consecutive points colored $0, 1, 2$ is called a *triplet*.
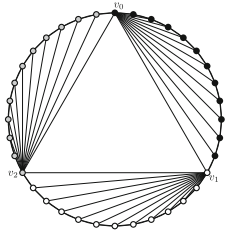

A 3-*fan*, denoted as G_n, is a 3-colored outerplanar graph with $3n$ vertices ($n \geq 2$) and defined as follows. G_n consists of a simple cycle formed by n vertices of color 0, followed (in the counterclockwise order) by n vertices of color 1, followed by n vertices of color 2. The vertex of color i adjacent in the cycle to a vertex of color $i - 1$ (indices taken modulo 3) is denoted as v_i. Also, in G_n every vertex colored i is adjacent to v_i ($i = 0, 1, 2$) and vertices v_0, v_1, v_2 form a 3-cycle of G_n. See, e.g. Fig. 1. The following theorem has been proved in [2].

Fig. 1. A 3-fan G_n for $n = 12$.

Theorem 2. [2] *Let h be a positive integer and let G_n be a 3-fan for $n \geq 79h^3$, and let S_n be an alternating point set compatible with G_n. In every 3-colored point-set embedding of G_n on S_n there is one edge with more than h bends.*

The forest of stars that we use to establish our lower bound is called a 3-*sky* and is denoted by F_n. It consists of three stars T_0, T_1, T_2 such that: (i) each T_i ($i = 0, 1, 2$) has n vertices ($n \geq 2$); (ii) all the vertices of each T_i ($i = 0, 1, 2$) have the same color i.

Let Γ_n be a point-set embedding of F_n on S_n. An *uncrossed triplet* of Γ_n is a triplet p_i, p_{i+1}, p_{i+2} of points of S_n such that, when moving along $CH(S_n)$ in clockwise order, no edge of Γ_n crosses $CH(S_n)$ between p_i and p_{i+1} and between p_{i+1} and p_{i+2}. A triplet is *crossed k times* if the total number of times that $CH(S_n)$ is crossed by some edges between p_i and p_{i+1} and between p_{i+1} and p_{i+2} is k. A *leaf triplet* of Γ_n is a triplet of S_n whose points represent leaves of F_n. Analogously, a *root triplet* is a triplet of S_n whose points represent the three roots of F_n. The following lemma establishes the first relationship between the curve complexity of some special types of 3-colored point-set embeddings of F_n and those of a 3-fan G_n.

Lemma 3. *Let F_n be a 3-sky, S_n be an alternating point set compatible with F_n, and Γ_n be a 3-colored topological point-set embedding of F_n on S_n. If Γ_n has an uncrossed leaf triplet and each edge of Γ_n crosses $CH(S_n)$ at most b times, then the 3-fan G_n has a 3-colored topological point-set embedding on S_n such that each edge crosses $CH(S_n)$ at most $3b + 2$ times.*

Proof. We show how to use Γ_n to construct a topological point-set embedding of the 3-fan G_n on S_n with at most $3b + 2$ crossings of $CH(S_n)$ per edge.

Let p_j, p_{j+1}, p_{j+2} be an uncrossed leaf triplet. Every point of the triplet represents a leaf of a different star (because they have different color). Denote by q_i the point of Γ_n representing the root of T_i ($i = 0, 1, 2$) and denote by e_i the edge connecting q_i to p_{j+i}. The idea is to connect the three points q_0, q_1, q_2
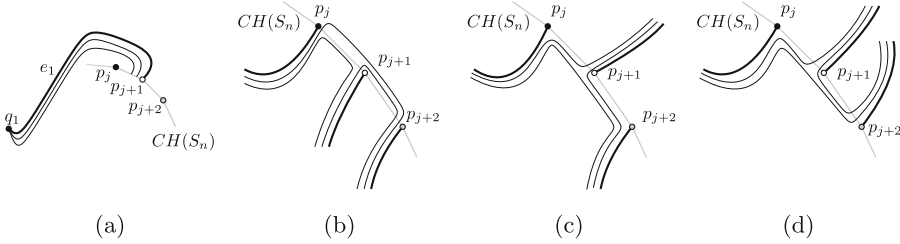
Fig. 2. Insertion of a cycle connecting q_0, q_1 and q_2. (a) Drawing of the curves following the edge e_1. (b), (c), and (d) Connection of the six curves to form a cycle.

with a 3-cycle that does not cross any existing edges. For each edge e_i we draw two curves that from q_i run very close to e_i until they reach $CH(S_n)$. The two curves are drawn on the same side of e_i such that they are consecutive in the circular order of the edges around q_i (see Fig. 2(a) for an illustration). These two curves do not intersect any existing edges and cross $CH(S_n)$ the same number of times as e_i. The six drawn curves are now suitably connected to realize a cycle C connecting q_0, q_1, q_2. Depending on which side the various curves reach $CH(S_n)$, the connections are different. However in all cases we can connect two curves to form a single edge by crossing $CH(S_n)$ at most two additional times and without violating planarity (see Fig. 2(b), (c) and (d)). Thus, we have added to Γ_n three edges e_i' connecting q_i to q_{i+1} (indices taken modulo 3), each crossing $CH(S_n)$ at most $2b + 2$ times. Also, since the two curves that follow an edge e_i are both drawn on the same side of e_i, the cycle C does not have any vertices inside. Notice that, depending on the case with respect to the connection of the curves, q_0, q_1, and q_2 appear along C either in the clockwise or in the counterclockwise order. W.l.o.g. we assume that the clockwise order is q_0, q_1, and q_2.

The obtained drawing is not yet a topological point-set embedding of G_n because the cycle C' connecting all the vertices is missing. We first add edges connecting leaves of the same color. Let $e_i' = \bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{n-2} = e_{i-1}'$ be the edges incident to q_i in the circular order around q_i (this is the counterclockwise order under our assumption that q_0, q_1, and q_2 are located in the clockwise order along C). We add an edge between the leaf of $\bar{e}_j$ and the leaf of $\bar{e}_{j+1}$ (for $j = 0, 1, \ldots, n-3$) as follows. Starting from the leaf of $\bar{e}_j$, we draw a curve following the edge $\bar{e}_j$ until we arrive very close to q_i and then we follow $\bar{e}_{j+1}$ until we reach the leaf of $\bar{e}_{j+1}$. The added edges do not cross any existing edges and cross $CH(S_n)$ a number of times equal to the number of times that $\bar{e}_j$ crosses $CH(S_n)$ plus the number of times that $\bar{e}_{j+1}$ crosses $CH(S_n)$, so at most $2b$.

It remains to add the edges of C' connecting vertices of different colors. There are three such edges and they connect vertex v_i ($i = 0, 1, 2$) of G_n to a vertex of color $i + 1$ (indices taken modulo 3). We add an edge connecting q_i to a leaf of color $i + 1$ as follows. Let e_{i+1}'' be the edge incident to q_{i+1} that follows e_i' in the clockwise order around q_{i+1} (this is an edge connecting q_{i+1} to a leaf of color $i + 1$). Starting from q_i we draw a curve following the edge e_i' until we

arrive very close to q_{i+1} and then we follow e''_{i+1} until we reach the leaf of e''_{i+1}. The constructed curve connects q_i to a leaf of color $i+1$ and does not cross any existing edge. It crosses $CH(S_n)$ at most the number of times that e'_{i+1} crosses $CH(S_n)$ (that is $2b + 2$) plus the number of times that e''_{i+1} crosses $CH(S_n)$ (that is b). Thus the total number of crossing of $CH(S_n)$ is at most $3b + 2$.   □

The next two lemmas explain how to obtain a 3-colored topological book embedding that satisfies Lemma 3.

Lemma 4. *Let F_n be a 3-sky, S_n be an alternating point set compatible with F_n, and Γ_n be a 3-colored topological point-set embedding of F_n on S_n with a root triplet. If Γ_n has a leaf triplet τ that is crossed c times $(c < n)$ and each edge crosses $CH(S_n)$ at most b times, then there exists a 3-sky $F_{n'}$ which is a subgraph of F_n and an alternating point set $S_{n'}$ which is a subset of S_n such that: (i) $n' \geq n - c$; (ii) there exists a 3-colored topological point-set embedding $\Gamma_{n'}$ of $F_{n'}$ on $S_{n'}$ such that each edge crosses $CH(S_{n'})$ at most $b+1$ times; (iii) τ is an uncrossed leaf triplet of $\Gamma_{n'}$.*

Lemma 5. *Let F_n be a 3-sky, S_n be an alternating point set compatible with F_n, and Γ_n be a 3-colored topological point-set embedding of F_n on S_n. If each edge of Γ_n crosses $CH(S_n)$ at most b times, then there exists a 3-sky $F_{n'}$ which is a subgraph of F_n and an alternating point set $S_{n'}$ which is a subset of S_n such that: (i) $n' \geq \frac{n}{3} - 3$; (ii) there exists a 3-colored topological point-set embedding $\Gamma_{n'}$ of $F_{n'}$ on $S_{n'}$ such that each edge crosses $CH(S_{n'})$ at most $b+2$ times; (iii) $\Gamma_{n'}$ has a root triplet.*

Lemma 6. *Let h be a positive integer, F_n be a 3-sky for $n = 520710h^3$, and S_n be an alternating point set compatible with F_n. In every 3-colored point-set embedding of F_n on S_n there exist at least h^2 edges with more than h bends.*

Proof (sketch). Let F_{n_i}, $i = 1, 2, \ldots, h^2$, be a 3-sky for $n_i = 520689h^3 + 21h \cdot i$ and let S_{n_i} be an alternating point set compatible with F_{n_i}. We prove by induction on i that in every 3-colored point-set embedding of F_{n_i} on S_{n_i} there exist i edges with more than h bends. Notice that for $i = h^2$, we have $n_i = n$.

Base case: $i = 1$: We have to prove that in any 3-colored point-set embedding of F_{n_1} on S_{n_1} with $n_1 = 520689h^3 + 21h$, there exists one edge with more than h bends. Suppose as a contradiction that there exists a 3-colored point-set embedding Γ_{n_1} of F_{n_1} on S_{n_1} with curve complexity h. Γ_{n_1} is also a 3-colored topological point-set embedding of F_{n_1} on S_{n_1} such that each edge crosses $CH(S_{n_1})$ at most $2h$ times (each edge consists of at most $h + 1$ segments). By Lemma 5 there exists a 3-colored point-set embedding $\Gamma_{n'}$ of a 3-sky $F_{n'}$ on an alternating point set $S_{n'}$ such that: (i) $n' \geq \frac{n_1}{3}$; (ii) each edge of $\Gamma_{n'}$ crosses $CH(S_{n'})$ at most $2h + 2$ times; (iii) $\Gamma_{n'}$ has a root triplet.

Since each edge of $\Gamma_{n'}$ crosses $CH(S_{n'})$ at most $2h + 2$ times and there are $3(n' - 1) \geq n_1 - 3$ edges in total, there are at most $(2h + 2)(n_1 - 3)$ crossings of $CH(S_{n_1})$ in total. The number of leaf triplets in $\Gamma_{n'}$ is $n' - 1 \geq \frac{n_1}{3} - 1$. It follows that there is at least one leaf triplet τ crossed at most $\frac{3(2h+2)(n_1-3)}{(n_1-3)} = 6h+6 \leq 7h$

times. By Lemma 4 there exists a 3-colored point-set embedding $\Gamma_{n''}$ of a 3-sky $F_{n''}$ on an alternating point set $S_{n''}$ such that: (i) $n'' \geq n' - 7h$; (ii) each edge of $\Gamma_{n''}$ crosses $S_{n''}$ at most $2h+3$ times; (iii) τ is uncrossed. By Lemma 3, the 3-fan $G_{n''}$ has a 3-colored topological point-set embedding on $S_{n''}$ such that each edge crosses $CH(S_{n''})$ at most $6h + 11$ times and by Lemma 1 a 3-colored point set embedding with curve complexity at most $12h + 23$. On the other hand, since $n_1 = 520689h^3 + 21h$, we have that $n'' \geq n' - 7h \geq \frac{n_1}{3} - 7h = \frac{520689h^3 + 21h}{3} - 7h = \frac{520689}{3}h^3 \geq \frac{520689}{3}h^3 = 79(13h)^3$ and by Theorem 2, in every 3-colored point-set embedding of $G_{n''}$ on $S_{n''}$ at least one edge that has more than $13h$ bends – a contradiction.

Inductive step: $i > 1$. We have to prove that in any 3-colored point-set embedding of F_{n_i} on S_{n_i} with $n_i = 520689h^3 + 21h \cdot i$, there exist i edges with more than h bends.

We first prove that there exists at least one edge with more than h bends. Suppose as a contradiction that there exists a 3-colored point-set embedding Γ_{n_i} of F_{n_i} on S_{n_i} with curve complexity h. With the same reasoning as in the base case, there would exist a 3-colored point set embedding with curve complexity at most $12h+23$ of a 3-fan $G_{n''}$, with $n'' \geq \frac{n_i}{3} - 7h$. Since $n_i = 520689h^3 + 21h \cdot i$, we have that $n'' \geq \frac{n_i}{3} - 7h = \frac{520689h^3 + 21h \cdot i}{3} - 7h = \frac{520689}{3}h^3 + 7h(i-1) \geq \frac{520689}{3}h^3 = 79(13h)^3$ and by Theorem 2, in every 3-colored point-set embedding of $G_{n''}$ on $S_{n''}$ at least one edge has more than $13h$ bends – again a contradiction.

This proves that there is at least one edge e crossed more than h times. We now remove this edge and the whole triplet that contains the point representing the leaf of e. We then arbitrarily remove $21h - 1$ triplets. The resulting drawing is a 3-colored point-set embedding $\Gamma_{n'''}$ of $F_{n'''}$ on $S_{n'''}$ for $n''' = n_{i-1}$. By induction, it contains $i-1$ edges each having more than h bends. It follows that Γ_{n_i} has i edges each having more than h bends. Since for $i = h^2$ we have $n_i = n$, the statement follows. □

Theorem 3. *For sufficiently large n, there exists a 3-colored forest F_n consisting of three monochromatic stars with n vertices and a 3-colored point set S_n in convex position compatible with F_n such that any 3-colored point-set embedding of F_n on S_n has $\Omega(n^{\frac{2}{3}})$ edges having $\Omega(n^{\frac{1}{3}})$ bends.*

We conclude this section with some results deriving from Theorem 3 and/or related to it. Firstly, Theorem 3 extends the result of Theorem 2 since it implies that a 3-colored point set embedding of G_n may require $\Omega(n^{\frac{2}{3}})$ edges with $\Omega(n^{\frac{1}{3}})$ bends each. Moreover, the result of Theorem 3 implies an analogous result for a k-colored forest of at least three stars for every $k \geq 3$. In particular, when $k = n$ we have the following result that extends the one by Pach and Wenger [9].

Corollary 1. *Let F be a forest of three n-vertex stars. Every planar drawing of F with vertices at fixed vertex locations has $\Omega(n^{\frac{2}{3}})$ edges with $\Omega(n^{\frac{1}{3}})$ bends each.*

One may wonder whether the lower bound of Theorem 3 also holds when the number of colors or the number of stars is less than three. However, it is immediate to see that this is not the case, i.e., the following theorem holds.

Theorem 4. *Let F be a k-colored forest of h stars and S be a set of points compatible with F. If $\max\{k,h\} = 2$ then F has a k-colored point-set embedding on S with curve complexity at most* 2.

Since a caterpillar can be regarded as a set of stars whose roots are connected in a path, the lower bound of Theorem 3 also holds for caterpillars. This answers an open problem in [1] about the curve complexity of k-colored point-set embeddings of trees for $k \geq 3$. Note that $O(1)$ curve complexity for 2-colored outerplanar graphs has been proved in [2].

Corollary 2. *For sufficiently large n, a 3-colored point-set embedding of a 3-colored caterpillar may require $\Omega(n^{\frac{2}{3}})$ edges having $\Omega(n^{\frac{1}{3}})$ bends.*

4   Point-Set Embeddings of Paths and Caterpillars

In the light of Corollary 2, one may ask whether there exist subclasses of 3-colored caterpillars for which constant curve complexity can be guaranteed. In this section we first prove that this is the case for 3-colored paths and then we extend the result to 3-colored caterpillars whose leaves all have the same color.

Based on Lemma 2, we prove that a 3-colored path P has a topological book embedding consistent with $seq(S)$ and having a constant number of spine crossings. Namely, we first remove the vertices and points of one color from P and S, obtaining a 2-colored path P' and a compatible 2-colored point set S'. Next, we construct a topological book embedding $\gamma_{P'}$ of P' consistent with $seq(S')$ with at most two spine crossings per edge and with suitable properties. Then we use such properties to reinsert the third color and obtain a topological book embedding of P consistent with $seq(S)$.

P' and $\sigma' = seq(S')$ can be regarded as two binary strings of the same size where one color is represented by bit 0 and the other one by bit 1. P' and σ' are *balanced* if the number of 0's (1's, resp.) in P' equals the number of 0's (1's, resp.) in σ'. P' and σ' are a *minimally balanced pair* if there does not exist a prefix of P' and a corresponding prefix of σ' that are balanced.

Lemma 7. *Let P and σ be a minimally balanced pair of length $k > 1$. Let $b_j(P)$ denote the j-th bit of P and $b_j(\sigma)$ denote the j-th bit of σ. Then $b_1(P) \neq b_k(P)$, $b_k(P) = b_1(\sigma)$, and $b_1(P) = b_k(\sigma)$.*

Let Γ be a topological book embedding, ℓ be the spine of Γ, and p be a point of ℓ (possibly representing a vertex). We say that p is visible from above (below) if the vertical ray with origin at p and lying in the top (bottom) page does not intersect any edge of Γ. We say that the segment $\overline{pq}$ is *visible from above (below)* if each point r in the segment is visible from above (below). Let u and v be two vertices of Γ that are consecutive along the spine ℓ, we say that segment $\overline{uv}$ is *accessible* if it contains a segment that is visible from below. A vertex v of Γ is *hook visible* if there exists a segment $\overline{pq}$ of the spine such that $\overline{pq}$ is visible from below and for any point r of $\overline{pq}$ we can add an edge in the top page of Γ
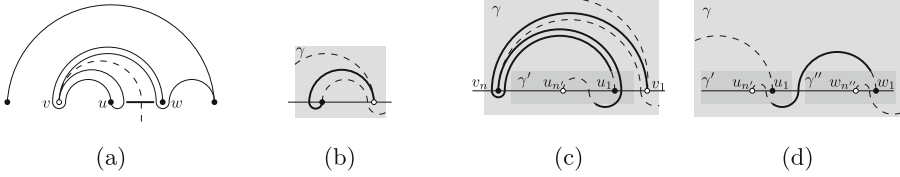
Fig. 3. (a) Illustration of the hook visibility property. The bold segment is the access interval. (b)-(d) Proof of Lemma 8: (b) Base cas; (c) Case 1; (d) Case 2.

connecting v with r without crossing any other edges of Γ (see Fig. 3(a)); $\overline{pq}$ is the *access interval for vertex* v. If the access interval is to the right (left) of v we say that v is *hook visible from the right (left)*.

Lemma 8. *Let P be a 2-colored path and σ be a 2-colored sequence compatible with P. Path P admits a topological book embedding γ consistent with σ and with the following properties: (a) Every edge of γ crosses the spine at least once and at most twice. (b) For any two vertices u and v that are consecutive along the spine ℓ of γ, segment $\overline{uv}$ is accessible from below. (c) Every spine crossing is visible from below. (d) The first vertex v_1 of P is visible from above; the last vertex v_n of P is hook visible from the right; to the right of its access interval there is only one vertex and no spine crossing.*

Proof. We prove the statement by induction on the length n of P (and of σ). If $n = 1$ the statement trivially holds. If $n = 2$ we draw the unique edge of P with one spine crossing immediately to the left of the leftmost vertex in the drawing (see Fig. 3(b)). Also in this case the statement holds. Suppose that $n > 2$ and that the statement holds for every $k < n$. We distinguish between two cases.

Case 1: P and σ are a minimally balanced pair. By Lemma 7 the first vertex of P has the same color as the last element of σ, the last element of P has the same color as the first element of σ and these two colors are different. This means that by removing the first and the last elements from both P and σ, we obtain a new 2-colored path P' of length $n - 2$ and a new 2-colored sequence σ' compatible with P'. By induction, P' admits a topological book embedding γ' consistent with σ' and satisfying properties (a)–(d). To create a topological book embedding of P consistent with σ, we add a point p_1 before all the points of γ', whose color is the same as the last vertex v_n of P, and a point p_2 after all points of γ', whose color is the same as the first vertex v_1 of P. Vertex v_1 is mapped to p_2 and vertex v_n is mapped to p_1. We connect v_1 to the first vertex u_1 of P' with an edge incident to p_2 from above, crossing the spine once immediately before p_1 and once immediately after p_1 and arriving to u_1 from above (by property (d), u_1 is visible from above). We then connect the last vertex $u_{n'}$ of P' to v_n. Since $u_{n'}$ is hook visible by property (d), we connect it to v_n with an edge that starting from $u_{n'}$ reaches the access interval of $u_{n'}$, crosses the spine between the last vertex of γ' and p_2 and reaches p_1 from above. As shown in Fig. 3(c) the two edges (v_1, u_1) and $(u_{n'}, v_n)$ can be added without creating any crossing. Property (a)

holds by construction. About properties (b) and (c), we added two arcs in the bottom page. The first one connects a point immediately before v_n and a point immediately after it, so the segment of the spine between v_n and its following vertex is accessible from below; also, the addition of this arc does not change the accessibility of the spine crossing of γ'. The second arc added in the bottom page connects a point q in the access interval of $u_{n'}$ with a point immediately after u_1; by property (d) of γ' there is no vertex or spine crossings between q and u_1. Thus the segments connecting u_1 to its preceding and to its following vertices are visible from below and property (b) holds; furthermore the addition of this arc does not change the accessibility to existing spine crossings. Since the new created spine crossings are visible from below, property (c) also holds. It is immediate to see that also property (d) holds; see for example Fig. 3(c).

Case 2: P and σ are not a minimally balanced pair. In this case there exists a prefix (i.e. a subpath) P' of P and a corresponding prefix σ' of σ that are balanced. P' is 2-colored path and σ' is a 2-colored sequence compatible with P' and their length is less than n. By induction, P' admits a topological book embedding γ' consistent with σ' and statisfying properties (a)–(d). On the other hand, $P'' = P \setminus P'$ is also a 2-colored path and $\sigma'' = \sigma \setminus \sigma'$ is a 2-colored sequence consistent with P''. Thus, P'' also admits a topological book embedding γ'' consistent with σ'' and statisfying properties (a)–(d). Since the last vertex u_n of P' is hook visible in γ' and the first vertex w_1 of P'' is visible from above in γ'', the two vertices can be connected with an edge that crosses the spine twice (see Fig. 3(d)), thus creating a topological book embedding γ of P consistent with σ. Property (a) holds by construction. The only arc added in the bottom page connects a point q in the access interval of u_n and a point q' immediately after the first vertex u_1 of P'. By property (d) of γ' there is no vertex or spine crossing between q and u_1 and between u_1 and q', thus properties (b) and (c) hold for γ. Property (d) holds because it holds for γ' and γ''.   □

Lemma 9. *A 3-colored path admits a topological book embedding with at most two spine crossings per edge consistent with any compatible 3-colored sequence.*

Proof (sketch). Let c_2 be a color distinct from the colors of the end-vertices of P. Let $v_1, v_2, \ldots, v_k$ be a maximal subpath of P colored c_2. Let u_1 and u_2 be the vertices along P before v_1 and after v_k, respectively. We replace the subpath $u_1, v_1, v_2, \ldots, v_k, u_2$ with an edge (u_1, u_2). We do the same for every maximal subpath colored c_2. Let P' be the resulting 2-colored path and σ' be the 2-colored sequence obtained from σ by removing all elements of color c_2.

By Lemma 8, P' admits a topological book embedding γ' consistent with σ' that satisfies properties (a), (b), (c) and (d). We add to γ' a set Q of points colored c_2 to represent the removed vertices that will be added back. These points must be placed so that the sequence of colors along the spine coincides with σ. By property (b) of γ' all these points can be placed so that they are accessible from below. We now have to replace some edges of P' with paths of vertices colored c_2. Let (u_1, u_2) be an edge that has to be replaced by a path $u_1, v_1, v_2, \ldots, v_k, u_2$. For each vertex v_i to be added ($i = 1, 2, \ldots, k$) we add an

image point to the drawing. The image points are added as follows. By property (a), the edge (u_1, u_2) crosess the spine at least once. Let χ be the point where (u_1, u_2) crosses the spine for the first time when going from u_1 to u_2. By property (c) χ is visible from below. This means there is a segment s of ℓ with χ as an endpoint that is visibile from below. We place $k-1$ image points $p_1, p_2, \ldots, p_{k-1}$ inside this segment, while χ is the k-th image point p_k (it is the leftmost if s is to the left of χ, while it is the rightmost if s is to the right of χ). The first arc of the edge (u_1, u_2) is replaced by an arc connecting u_1 to p_1. Each image point p_i is connected to the p_{i+1} ($i = 1, 2, k-1$) by means of an arc in the top page. Finally, the last image point p_k is already connected to u_2 by means of the remaining part of the original edge (u_1, u_2). Notice that the edge (u_1, p_1) does not cross the spine, and the same is true for any edge (p_i, p_{i+1}), while the edge (p_k, u_2) crosses the spine at most once (the original edge had at most two spine crossing one of which was at $\chi = p_k$). We have replaced the edge (u_1, u_2) with a path $\pi = \langle u_1, p_1, p_2, \ldots, p_k, u_2 \rangle$ with $k+1$ edges, as needed. However, the points representing the intermediate vertices of this path are not the points of the set Q. The idea then is to "connect" the image points to the points of Q. To this aim, we add matching edges in the bottom page between the image points and the points of Q. Since both the points of Q and the image points are visible from below, these matching edges do not cross any other existing edge. Moreover, by using a simple brackets matching algorithm, we can add the matching edges so that they do not cross each other. Finally the matching edges can be used to create the actual path that represent $u_1, v_1, v_2, \ldots, v_k, u_2$.   □

The following theorem is a consequence of Lemmas 2 and 9.

Theorem 5. *Every 3-colored path admits a 3-colored point-set embedding with curve complexity at most 5 on any compatible 3-colored point set.*

Theorem 5 can be extended to a subclass of 3-colored caterpillars.

Theorem 6. *Every 3-colored caterpillar with monochromatic leaves admits a 3-colored point-set embedding with curve complexity at most 5 on any compatible 3-colored point set.*

The above results motivate the study of 4-colored graphs, in particular a natural question is whether 4-colored paths admit point-set embedding on any set of points with constant curve complexity.

Theorem 7. *Let P be a 4-colored path with n vertices and let S be a 4-colored point set compatible with P. If the first $h \geq 2$ vertices along P only have two colors and the remaining $n - h$ only have the other two colors, then P has a 4-colored point-set embedding on S with curve complexity at most 5.*

5   Open Problems

Motivated by the results of this paper we suggest the following open problems: (i) Investigate whether the lower bound of Theorem 3 is tight. We recall that an

upper bound of $O(n)$ holds for all n-colored planar graphs [9]. (ii) Study whether constant curve complexity can always by guaranteed for 4-colored paths. (iii) Characterize the 3-colored caterpillars that admit a 3-colored point-set embedding with constant curve complexity on any given set of points.

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