

Chapter 18

Mathematical Problem Solving in Choice-Affluent Environments

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Abstract This chapter presents a proposal for an exploratory confluence model of mathematical problem solving in different instructional contexts. The proposed model aims at bridging the knowledge of how problem solving occurs and the knowledge of how to enhance problem solving. The model relies on the premise that a key solution idea to a problem is constructed as a result of shifts of attention stipulated by the solver's individual resources, interaction with peers, or with a source of knowledge about the solution. The exposition converges to the conclusion that successful problem solving is likely to occur in choice-affluent learning environments, in which the solvers are empowered to make informed choices of a challenge to cope with, problem-solving schemata, a mode of interaction, an extent of collaboration, and an agent to learn from. The theoretical argument is supported by an example from an empirical study.

Keywords Mathematical problem solving · Shifts of attention
Choice-affluent environments

18.1 Introduction

The centrality of problem solving in doing and studying mathematics is broadly recognized in the mathematics education research community (e.g., Halmos 1980; Mason 2016a; Schroeder and Lester 1989). Research on problem solving keeps growing, and many approaches to translating the developed problem-solving

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frameworks and accumulated research results into recommendations for practice have been articulated (e.g., Schoenfeld 1983, 1985; Felmer et al. 2016). In particular, the professional literature suggests various specifications of “good” mathematical problems (e.g., Lappan and Phillips 1998), characterizations of problem-solving classrooms (e.g., Engle and Conant 2002; Lampert 1990; Schoenfeld et al. 2014), and sets of principles for teaching for and through problem solving (Cai 2010; Heller and Hungate 1985; Koichu et al. 2007a; Lester 2013; Lester and Cai 2016; Schoenfeld 1983). In many cases, the recommendations are presented as generalized reflections on successful classroom practices, experiments, or series of experiments (e.g., Koichu et al. 2007a, b; Lester 2013; Schoenfeld 1992). In some cases, the recommendations are based also on theories of problem-solving architecture (e.g., Ambrus and Barczy-Veres 2016; Clark et al. 2006) or decision making (e.g., Schoenfeld 2013).

It is indicative, however, that recent reflections of the state of the art tend to emphasize lacunas and open questions rather than the accomplishments of problem-solving research as a servant of mathematics instruction (Mason 2016a, b; Lester 2013; Schoenfeld 2013; Vinner 2014). Vinner (2014), for instance, questions the feasibility and even the relevance of problem solving for exam-oriented school education. Mason (2016a) reminds us that a variety of factors should be taken into account in order to make problem-solving instruction feasible in school and university settings. In his words, “all aspects of the human psyche, cognition, affect, behavior, attention, will and metacognition or witnessing must be involved” (p. 110). He then characterizes a research approach attempting to isolate particular features of problem solving as simplistic and unlikely to bring the desired change. In the same volume (Felmer et al. 2016), Mason (2016b) suggests that the crucial yet not sufficiently understood issue for adopting a problem-solving approach to mathematics teaching is the *when-issue*, that is, the issue of “*when* to introduce exploratory tasks, *when* to intervene, and in what way” (p. 263).

Lester (2013) acknowledges that research on mathematical problem solving has provided some valuable information about problem-solving instruction, but argues that the progress is slow and, generally speaking, insufficient. As one of the explanations of “this unfortunate state of affairs” (p. 251), Lester (2013) reiterates the claim that he and Charles made 25 years ago (Lester and Charles 1992): Research on mathematical problem solving remains largely atheoretical. Lester (2013) then argues that the comprehensive framework for research on problem-solving instruction proposed by Lester and Charles (1992) is still worth pursuing. Likewise, Schoenfeld (2013) reflects on the gains and limitations of a problem-solving framework that he authored 30 years ago (Schoenfeld 1985). He then suggests that the current challenge is to advance from a *framework* for examining problem solving to a *model* that would specify the theoretical architecture of this activity, i.e., would say “what matters” in problem solving (Schoenfeld 2013, p. 17), explain “how decision making occurs within that architecture” (p. 17) and theorize “how ideas grow and can be shared in interaction” (p. 20).

Stimulated by the aforementioned calls, the goal of this chapter is to present a particular proposal for an exploratory model of problem solving that would bridge

our knowledge of how problem solving occurs with the knowledge of how to support and enhance problem solving in instructional settings.¹ The proposed model is confluence, namely, it consolidates ideas from several conceptual and theoretical frameworks. The consolidation is pursued by means of a strategy that is referred to as *networking theories by iterative unpacking*. In brief, this strategy consists of sequencing theoretical developments so that at each step of theorizing one theory serves as an overarching conceptual framework, in which another theory, either existing or emerging, is embedded in order to elaborate on the chosen element(s) of the overarching theory. Mason's theory of shifts of attention (Mason 1989, 2008, 2010) serves as the overarching conceptual framework of the proposed model. Throughout the chapter, the model is illustrated by consideration of a single geometry problem, which is analyzed theoretically and then based on empirical evidence. Thoughts about possible implications of the model are shared in the concluding section.

18.2 The Proposed Model at a Glance

The proposed model is schematically presented in Fig. 18.1. The model is referred to as the shifts and choices model (SCM) in the rest of the chapter.

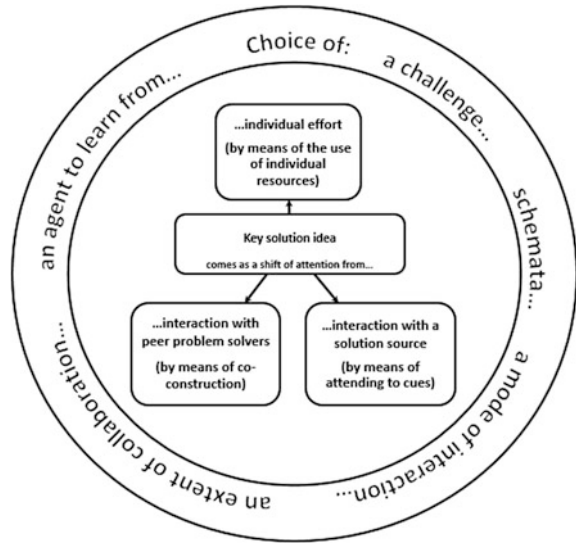
The inner part of Fig. 18.1 concerns the process that might be termed, with reference to Pólya (1945/1973) or Schoenfeld (1992), as a heuristic search embedded in the planning phase of problem solving. The main query associated with this part of the SCM is, simply stated: Where can a solution to a challenging problem come from? A more elaborated formulation of the query is as follows: Through which activities and resources does a problem solver construct a pathway of shifts of attention towards an invention of a key solution idea to a mathematical problem?

The outer part of Fig. 18.1 concerns a configuration of choices available to a problem solver. This part of the SCM deals with the following query: What choices is a problem solver empowered to make when constructing or co-constructing a chain of shifts of attention towards invention of a key solution idea to a mathematical problem that he or she perceives as a challenge? Among endless conscious and unconscious choices that individuals face when solving problems on their own or with others, the model takes into account the following: a choice of a challenge to be dealt with, a choice of schemata for dealing with a challenge, a choice of the mode of interaction, a choice of an extent of collaboration, and a choice of an agent to learn from.

As Fig. 18.1 shows, a *key solution idea* notion is in the core of the SCM. It is a solver-centered notion. Namely, a key solution idea is a *heuristic idea* that is

¹In a way, this paper synthesizes and develops ideas that have been presented separately in Palatnik and Koichu (2014, 2015) and Koichu (2015a, b, 2017).

Fig. 18.1 Schematic presentation of the proposed model (SCM)



invented by the solver and evokes the conviction that it can lead to a *full solution* to the problem. A full solution is referred to as a solution that, to the solver’s knowledge, would be acceptable in a situation in which it is communicated. Furthermore, Raman (2003) explains the heuristic idea notion as follows²: it is “an idea based on informal understandings, e.g., grounded in empirical data or represented by a picture, which may be suggestive but does not necessarily lead directly to a formal proof” (p. 322). Note that not any heuristic idea is a key solution idea. An in-depth discussion of an *idea* notion is beyond the scope of this chapter. It is sufficient to mention here that the Oxford Dictionary defines *idea* as “a thought or suggestion as to a possible course of action” (<https://en.oxforddictionaries.com/definition/idea>). Accordingly, a heuristic idea notion can operationally be treated as a piece of content-level mathematical discourse (see Sfard 2007) suggestive as to a possible way of solving the problem. An elaborated example is presented below.

The SCM relies on three premises:

1. Even when a problem is solved in collaboration, it has a *situational solver*: an individual who invents and eventually shares its key solution idea.
2. A key solution idea can be invented by a situational solver as a shift of attention in a sequence of his or her shifts of attention when coping with the problem.
3. Generally speaking, a solver’s pathway of the shifts of attention is stipulated by choices the solver is empowered to make and by enacting the following types of resources:

²See also Koichu et al. (2007a) and Liljedahl et al. (2016) for detailed discussions of approaches to conceptualizing heuristics.

- (i) individual resources,
- (ii) interaction with peer solvers who do not know the solution and struggle in their own ways with the problem or attempt to solve it together, and
- (iii) interaction with a source of knowledge about the solution or its parts, such as a textbook, an internet resource, a teacher, or a classmate who has already found the solution but is not yet disclosing it.

These three possibilities are intended to embrace all frequent situations of problem solving in instructional settings. Needless to say, the possibilities can be employed separately or can complement each other in problem solving.

18.3 Elaboration on the Elements of the Proposed Model

Discussion of the elements of the SCM in this section is supported by consideration of the two-circle problem (Fig. 18.2). The reader is invited to approach it before continuing reading.

18.3.1 *Invention of a Key Solution Idea as a Shift of Attention*

Mason's theory of shifts of attention had initially been formulated as a conceptual tool to dismantle constructing abstractions (Mason 1989) and was then extended to the phenomena of mathematical thinking and learning (Mason 2008, 2010). Palatnik and Koichu (2014, 2015) adapted the theory as a tool for analyzing insight problem solving. To characterize attention shifts, Mason (2008) considers *what* attended to by an individual and *how* it is attended to. To address the "how" question, he distinguishes five different *ways of attending* or *structures of attention*.

According to Mason (2008), *holding the wholes* is the structure of attention where a person is gazing at the whole without focusing on particulars. This is what probably happens when a reader flashes a glance at Fig. 18.2. *Discerning details* is a structure of attention in which one's attention is caught by a detail that becomes distinguished from the rest of the elements of the attended object. For example, one's attention can be caught by the segments EF and GH or by triangle MEF in Fig. 18.2. Mason (2008) suggests that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships* between the discerned elements is a development from discerned details that often occurs automatically: It refers to specific connections between specific elements. Say, for instance, that when gazing at the central part of Fig. 18.2, one notices that segments EF and GH look equal and can be considered sides of a quadrilateral $EFHG$. The *perceiving properties* structure of attention is different from the *recognizing relationships* structure in a subtle but essential way. In the words of Mason (2008): "When you

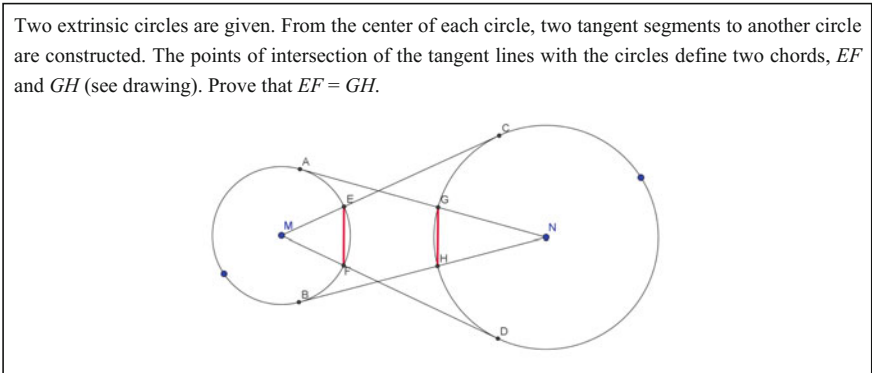


Fig. 18.2 The two-circle problem (translated from Sharygin and Gordin 2001, No. 3463)

are aware of a possible relationship and you are looking for elements to fit it, you are perceiving a property” (p. 38). In our example, one can draw the segments EG and FH . The perceived property would be “ $EFHG$ is a rectangle.” Finally, *reasoning on the basis of perceived properties* is a structure of attention in which selected properties are attended to as the only basis for further reasoning. For example, one might consider what needs to be proved for sides EG and FH in order to prove that $EFHG$ is a rectangle (see Fig. 18.3).

Palatnik and Koichu (2014, 2015) added a “why” question to Mason’s “what” and “how” questions about attention: Why do individuals make shifts from one object of attention to another in the way that they do? One way of addressing this question is related to the obstacles embedded for the solver in attending to a particular object and to continuous evaluation of potential gains and losses of the choice to keep attending to the object or shift the attention to another one.

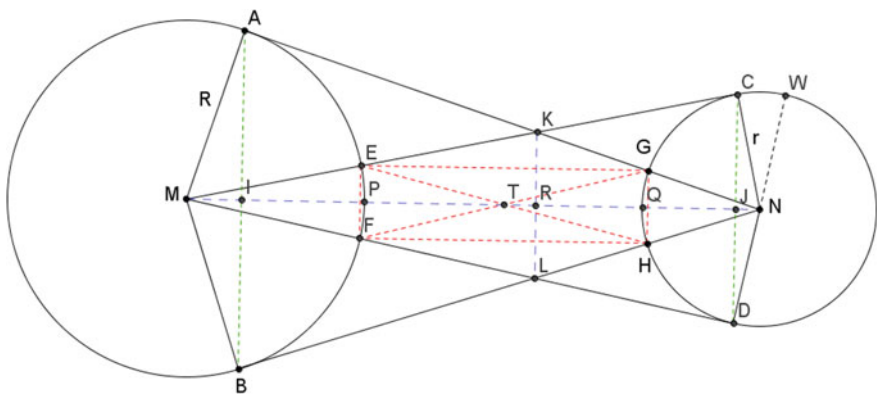


Fig. 18.3 Auxiliary construction for the two-circle problem

For example, one can try proving that $EFHG$ is a rectangle by applying the available schemata associated with the rectangle notion, such as: it is enough to prove that $EFHG$ is a parallelogram with a right angle, so let's try proving first that $EF = FH$. To prove this conjecture, it is enough to prove that $\triangle EKG \cong \triangle FLH$ and so on. (The reader is invited to check that this reasoning line is not particularly productive). At some point one can decide that enough attention has been given to $EFHG$ and consider another object. A shift is likely to be mediated by mathematical resources within the reach for the solver. Our imaginary solver might think: "What else can be done? How can the congruency of two segments be proved? Maybe, it is worth including the segments into some pair of triangles and prove that they are congruent. Are EFM and GHN congruent? Apparently not. Should I stop considering EF and GH for a while and focus on their halves, EP and GQ ?" Such a shift may seem trivial (especially if one knows the solution), but in fact it is not.

In a while, the solver might consider EP and GQ to be the sides of the right-angle triangles MEP and NGQ . The solver can then perceive the following property: $\triangle MEP \sim \triangle MCN$ and $\triangle NGQ \sim \triangle NAM$. There is a gap, however, between perceiving a property and choosing it as the only basis for further reasoning. Indeed, the solver should somehow realize that the triangle similarity can help in proving that two segments are equal. If our imaginary solver arrives this far, she has a good chance of inventing a key solution idea to the problem. One such idea consists of the observation that EP and GQ can be expressed through the same elements, R , r and MN , based on the aforementioned triangle similarities. This idea can be developed into a full solution to the problem: $EP = \frac{r \cdot R}{MN}$ from the first similarity and $GQ = \frac{r \cdot R}{MN}$ from the second similarity, consequently, $EP = GQ$, QED.

The presented imaginary scenario suggests how the process of inventing a key solution idea can be seen in terms of the solver's shifts of attention. Generally speaking, the solver attends to the objects embedded in the problem formulation and mentally manipulates them by applying available schemata. The process is goal directed, but particular shifts can be sporadic. However, it should be noted that the presented scenario is neither complete nor compelling. More should be done in order to realistically characterize one's problem solving process as a chain of shifts of attention. In particular, the specificity of problem solving in socially different educational contexts should be considered. This is done in the next three subsections.

18.3.2 Shifts of Attention in Individual Problem Solving

The lion's share of the data corpus that underlies the development of the foremost problem-solving frameworks (e.g., Schoenfeld 1985; Carlson and Bloom 2005) consists of cases of individual problem solving. Carlson and Bloom (2005) consider four phases in individual problem solving by an expert mathematician: orientation, planning, executing, and checking. The model also includes a sub-cycle,

“conjecture-test-evaluate,” and operates with various problem-solving attributes (this notion is due to Schoenfeld 1985), such as conceptual knowledge, heuristics, metacognition, control, and affect. Generally speaking, Carlson and Bloom’s framework offers a kit of conceptual tools that can be used for producing thick descriptions of individual problem solving. These tools enter the SCM as tools for addressing “how” and “why” questions about the shifts of attention.

For example, when our imaginary problem solver individually coped with the two-circle problem, she first directed her attention to proving that a particular quadrilateral is a rectangle and then shifted her attention to proving the similarity of two pairs of triangles. The pre- and post-stages of the shift can be described as two “conjecture-test-evaluate” sub-cycles within the planning phase. The shift itself can be characterized in terms of her mathematical, heuristic, and affective resources.

18.3.3 Shifts of Attention When Interaction with Peers Is Available

While studying problem-solving behaviors in small groups of students, Clark et al. (2014) extended Carlson and Bloom’s (2005) framework by introducing two new categories/codes. They termed them *questioning* and *group synergy*. The former category was introduced in order to give room in the data analysis to various questions (for assistance, for clarification, for status, and for direction) that the participants had asked. The latter category appeared to be necessary in order

to capture the combination and confluence of two or more group members’ problem-solving moves that could only occur when solving problems as a member of a group. ... A key characteristic of this group synergy code is that it leads to increased group interaction and activity, sometimes in unanticipated and very productive ways. (Clark et al. 2014, p. 10–11)

Indeed, when a possibility to collaborate with peers is available to problem solvers, their shifts of attention can be stipulated by inputs of the group members, especially when the inputs are shared in some common problem-solving space in a non-tiresome way. Peer interaction can increase one’s chances to produce a key solution idea, but can also be overwhelming or distracting. In particular, when nobody in a group knows how to solve the problem, the other members’ inputs of potential value are frequently undistinguishable for the solver from inputs of no value.

Schwartz et al. (2000) deeply explored the cognitive gains of two children who failed to solve a problem individually, but who improved when working in peer interaction. They distinguished between the *two-wrongs-make-a-right* and *two-wrongs-make-a-wrong* phenomena. The mechanisms of co-construction behind the two-wrongs-make-a-right phenomenon were the mechanisms of disagreement, hypothesis testing, and inferring new knowledge through challenging and conceding. These mechanisms might be involved in those cases of collaborative

problem solving in which group synergy led the participants in Clark et al.'s (2014) study to “very productive ways” (p. 11) of solving the given problems. To further explore the phenomenon of group synergy, it seems necessary to me to acknowledge that the above mechanisms can become active on condition that at least sometimes solvers shift their attention from an object that they are exploring to an object attended to by a peer.

Consider as an example once more the two-circle problem, but this time based on real data. The data were collected from a two-year experiment conducted in a class of 17 regular (i.e., not identified as gifted) 10th grade students. During the experiment, many difficult problems were offered to the students to solve over the course of 5–7 days for each problem in an environment combining classroom work and work from home. The work from home was supported by an online discussion forum at Google+. The forum devoted to the two-circle problem was active for 4 days and contained 230 entries. Three different solutions were finally produced, including the one presented in Sect. 18.3.1. An excerpt from the beginning of the forum is presented in Fig. 18.4.

Evidently, the forum participants are still far from any productive heuristic idea. Some of them are at the orientation stage and, generally speaking, are occupied by creating initial drawings. Maya and Shira begin to develop the direction that “ $EFHG$ is a rectangle,” which, as we know, is a dead end. The excerpt is suggestive about the following phenomena: The students independently choose different objects of attention (e.g., a kite-looking quadrilateral, a pair of triangles) and then share what they do. Sometimes the attempts to get somebody’s attention are successful, and sometimes they are not. The students occasionally choose to explore the same object together. The excerpt is also suggestive about the aforementioned mechanisms of productive peer interaction.

How can such interactions influence an individual pathway of shifts of attention? Let me address this question with particular focus on one student, Maya. According to the teacher, Maya has neither been an active student in a classroom nor a successful student in terms of mathematics exams. However, Maya was one of the most active participants in the two-circle problem forum. She initiated six out of 34 discussion threads, replied to 17 threads, uploaded two drawings and a scanned hand-written solution. Based on her reflective questionnaire, we know that she devoted about 5 h to the problem during 4 days and that for about 3 h she worked outside the forum. A part of the Maya’s devious pathway of shifts of attention is presented on Fig. 18.5.

The key solution idea of Maya was similar to the idea presented in Sect. 18.3.1, but she considered different pairs of similar triangles, $\triangle KNL \sim \triangle GNH$ and $\triangle KML \sim \triangle EMF$ (see Fig. 18.3). To prove their similarity, Maya first proved that $KL \parallel EF \parallel GH$. This was the major challenge for her. She then concluded the proof by consideration of proportions stemming from these similarities, in conjunction to a proportion based on the “bridging” pair of similar triangles, $\triangle MAK \sim \triangle NCK$.

Maya was a situational problem solver in the group. It is of note that Maya’s interactions with the peers were mostly around the objects of attention chosen by her. She switched her attention to the objects suggested by the other students only

Teacher (Day 1, 04:37): Good week, everybody! Today we begin to work on the third task of the project [posts the two-circle problem]. Good luck! [one “like”]

Shira (Day 1, 22:14): Look guys if you understand anything [in the drawing on the right side]. But this is, in general, the direction that I took :-) [two “likes”]

Zila (Day 1, 22:16): The truth is that I also thought about this way as the beginning, but how can you develop it?

...

Zila (Day 1, 23:49): How the quadrilateral (looks like a kite at Shira’ drawing), you said that it is a parallelogram?

Shira (Day 2, 07:22): I think it does not matter what the drawing looks like. I just did not draw exactly :-). But this is, in general, the direction, and you’re invited to continue :-)

Maya (Day 2, 19:11): I began working on the exercise only now and your auxiliary construction is exactly what I drew, even without looking! ^ _____ ^ [this sign roughly means “I am proud of myself”]. I thought to use the segment between the centers and the distance from the center of the circle, I’ve not really begun, just a sort of conjecture :) [one “like”]

Zila (Day 2, 19:12): [Maya], did you reach the conclusion that this is a parallelogram (there is a good chance that it is!)

Maya (Day 2, 19:27): So far I’ve drawn a sketch, and I’ll begin thinking in a moment.

Shira (Day 2, 21:20): The teacher, Maya, and I talked by phone for about 15 min about the exercise and thought and thought and got something. We first understood that the key points are those points that define the given arcs. So, the auxiliary construction should be from them or they should be included in two triangles so that it will be possible to prove their congruence. [one “like”]

Teacher (Day 2, 21:26): Excellent. It is worth adding to your previous ideas. . .

Maya (Day 2, 21:40): I invented this idea with Shira! In general, the idea is to get to the rectangle $EFHG$. I thought about how to prove the congruence of triangles $EKG = FLN$ [six “likes”].

Zila (Day 2, 21:42): Maya, they are congruent.

Maya (Day 2, 21:52): How???

Zila (Day 2, 21:53): According to the calculation of the angles.

Zila (Day 2, 21:53): Or, in fact, they are not.

Maya (Day 2, 22:01): Hhhhhhhh... I am sitting already for two hours on this exercise. . . . If it would be that simple, I’d already be successful.

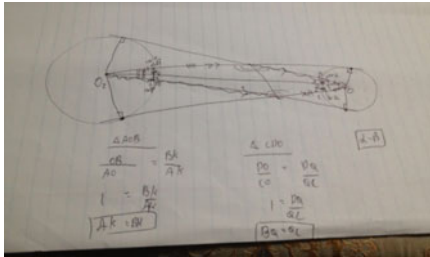


Fig. 18.4 An excerpt from the beginning of the two-circle problem forum at Google+

MN and the radii (with Shira) \rightarrow Points defining arcs GH and $EF \rightarrow EFHG \rightarrow EKG = FLN \rightarrow KL \rightarrow$ Possible similarity of GHN and CMD , ANB and $EMF \rightarrow$ Similarity of MAK and KCN , LDN and $MLB \rightarrow MAN$ (Meirav’s idea, which appears to be a dead-end for Maya) \rightarrow return to $EFHG \rightarrow$ Return to Arcs GH and $EF \rightarrow MN$, chords GH and $EF \rightarrow KL$, similarity of NHG and EMF , KNL and $MKL \rightarrow ??? \rightarrow$ solution

Fig. 18.5 Maya’s pathway of shifts of attention in solving the two-circle problem

occasionally. Interestingly, Maya never explicitly acknowledged that the ideas published on the forum had helped her. However, some steps in her solution can be traced back to the ideas suggested and explored by the other participants.³

18.3.4 Shifts of Attention When Interaction with a Solution Source Is Available

The option to interact with a source of knowledge about a key solution idea to a problem can drastically change a pathway of one's shifts of attention, up to the point that the entire process can stop being a problem-solving process and become a solution-comprehending process. The proposed model seeks to encompass only the situations in which a solution source can be present as a provider of cues to the solution or as a convenient storage of potentially useful facts, but not as a source of *telling* the solution. Such situations are common, for instance, when a teacher orchestrates a classroom problem-solving discussion by favoring some of the students' ideas over the others. In this way, the problem is usually solved before the bell rings. A danger in this situation, however, is that solvers may be deprived of inventing the solution themselves or being misled by a deceptive feeling, such as "we solved the problem with the teacher, so next time I will be able to do so alone."

When a source of knowledge about the solution is present but does not give the solution, the solver may attempt to extract it from the source (e.g., see questions for assistance and questions for direction in Clark et al. 2014). In some cases, one's shifts of attention may occur as a straightforward result of such attempts. In other cases, a shift may occur as a result of a conflict that emerges when more knowledgeable and less knowledgeable interlocutors assign different meanings to the same assertions (cf. Sfard 2007 for commognitive conflict).

For example, the assertion "triangle similarity is a good idea" can either pass unnoticed in the group discourse or be a trigger for solvers to shift their object of attention. The effect of the assertion would depend on who it has come from, a regular member of the group or a teacher or a peer who acts as if she has already solved the problem. In one case, the assertion can be perceived as "it is possible that similarity helps"; in another, "I've tried it and it helped"; and in yet another, "this is the direction approved by the authority." In any case, the perceived meaning of the assertion does not necessarily match the intended meaning. In line with Sfard (2007), one can suggest that a conflict of meanings can either hinder the communication or help the solver to progress.

Let me illustrate this suggestion by an additional excerpt from the two-circle problem forum. This time I focus on an episode from the fourth day of the forum, when Maya announced that she solved the problem. As one can see (Fig. 18.6),

³Unfortunately, there is no room in this chapter for presenting the entire story and the method of its SCM-driven analysis. It will be done elsewhere (Koichu and Harris, in preparation).

Maya: I succceeeded!!!!!!! And the teacher was right—it is so simple that I'd like to die. When I saw that I got nothing from [consideration of] this rectangle [$EFHG$ on Fig. X.3], I stopped and changed the direction. I went to proportions and solved the problem in several theorems, but anyway the long way towards the rectangle helped me to prove that KL is parallel to GH and to EF , and this is what helped me with the proportions. I'll now organize everything and upload the solution ☺.

Meirav: But I've tried a lot of times to find some parallelograms and it did not work out. A hint?

Maya: Look at Thaleses...

Meirav: Thanks!!

Alina: How did you prove that KL is parallel to GH and EF ?

Zila: She used the theorems about tangent lines and triangle similarity. Is it right?

Maya: Oh, it took years....

Fig. 18.6 Interactions with Maya as a source of knowledge about the solution

when Maya announced her success, she was perceived by the classmates as a source of knowledge about the solution. This fact essentially shaped the interaction. The students did not ask Maya to share the full solution—they knew that this would be against the rules of the forum—but they sought direction. Meirav asked for a hint. Alina inquired about a particular solution step. Zila assumed the role of translator of Maya's ideas. Interestingly, the situation is not as festive as it probably looks. First, when Maya actually published her solution, it appeared to have a logical flaw. She succeeded in producing a mathematically valid proof only after polishing her reasoning in interaction with two forum participants. Second, the hint provided by Maya in response to Meirav's request sounds as if the theorem of Thales might help, but this was not the case. Neither Maya nor other students used this theorem. Apparently, "Thaleses" was an informal tag for the idea "use proportions." Third, Zila's suggestion that Maya used "the theorems about tangent line and triangle similarity" was not helpful at best and misleading at worst. Maya indeed used these theorems, but not in proving that $KL \parallel EF \parallel GH$.

The point is that interaction with a source of knowledge about the solution affected the students' pathways of shifts of attention. It seems that an interplay of different meanings assigned to the same statements (e.g., "Thaleses") was an indispensable characteristics of such an interaction.

18.3.5 Shifts of Attention and Choices That Problem Solvers Are Empowered to Make

The argument presented in the previous subsections can be condensed into the following sentence: One's pathway of shifts of attention when solving a mathematical problem is essentially stipulated by choices that he or she is empowered to make. At a glance, this sentence echoes a description of a problem-solving process offered by Poincaré (1908/1948): A problem-solving process consists of a multi-stage pathway of conscious and unconscious steps towards the minimalistic

choice of a “proper” combination of ideas out of a huge number of possible combinations. Let us recall however that the Poincaré description concerns only the choices of solution steps and applies to mathematicians. Accordingly, it concerns problem solving having very specific characteristics: The choice of a problem is made by its solver, the solver has immensely rich mathematical resources and is confident in his or her mathematical ability, and has virtually unlimited time and motivation for pursuing the problem. Few of these characteristics hold for mathematical problem solving in instructional settings, but let me argue that Poincaré’s idea of choice can usefully be stretched.

Let us come back to the case of Maya once more. This time I wish to summarize the choices available to Maya when solving the Two-Circle Problem. As the presented excerpts suggest, Maya acted in a situation in which she could choose her solution moves. Moreover, she was empowered to make many additional choices such as whether to attempt to solve the given problem, follow problem-solving ideas of her peers, or merely ignore the challenge, as some of her classmates did; whether to work independently or with her peers; who to communicate with and when; which ideas to respond to and when; which and whose ideas to include into her own reasoning line; which ideas to share and how; and how much time to devote to the problem. It seems that only two choices were not up to Maya: Which problem to solve and when. In the described episode, these choices have been made for her.

The presented situation was particularly rich with opportunities for the students to choose. It is possible to recall or imagine instructional situations in which a configuration of choices available to the problem solvers would be different and include more or less choices. In fact, any instructional situation involving problem solving can be thought of in terms of choices that mathematics teachers empower their students to make, either intentionally or not.

18.4 Pedagogical Uncertainty and Choice-Affluent Environments

The diversity of choices involved in problem solving in instructional settings is immense, as are the diversity of individual pathways of shifts of attention. Being aware of this, we must acknowledge a fundamental role of pedagogical uncertainty: We can never know in advance which pathways of the shifts of attention the students construct when solving problems; thus, we will probably never be able to formulate universal recommendations as to how to organize a problem-solving classroom so that it would fit the individual needs and traits of each student. In the other words, we can probably never produce a satisfactory answer to Mason’s (2016b) “when” question, which he posits as *the* question of problem-solving instruction: “*when* to introduce exploratory tasks, *when* to intervene, and in what way” (p. 263). In yet other words, there are probably no configuration or configurations of teaching decisions that would be optimal for enhancing problem solving for all.

There is an alternative, however, to the perennial search for such configurations of decisions. Its roots can be traced to seminal work of Dewey (1938/1963), who substantiated the idea that students must be involved in choosing what they learn and how. I term this alternative as *constructing choice-affluent learning environments*. By a choice-affluent learning environment, I mean an environment in which students can at different times choose the most appropriate (1) challenge to pursue, from solving a difficult problem to comprehending a worked-out example; (2) mathematical tools and schemata for dealing with the challenge; (3) extent of collaboration, from being actively involved in exploratory discourse with peers of their choice to being independent solvers; (4) a mode of interactions, that is, whether to talk, listen, or be temporarily disengaged from the collective discourse, as well as whether to be a proposer of an idea, a responder to the ideas by the others, or a silent observer; and (5) agent to learn from, that is, the opportunity choose whose and which ideas are worthwhile of their attention.

How feasible are choice-affluent environments in school reality? One example of such an environment, an asynchronous problem-solving forum characterized by exploratory discourse, was presented above. An additional example involving engagement of students in long-term mathematics research projects has been presented elsewhere (Palatnik and Koichu 2015; Koichu 2017). Furthermore, it can be argued that even a lesson in a classroom that has a time constraint can be a choice-affluent environment.

For example, one characteristic of what Liljedahl (2016) termed the *thinking classroom* is the use of vertical surfaces (e.g., whiteboards or blackboards) as media that substitute for notebooks or working sheets that traditionally lie on the student desks. In such a classroom, students are given time to solve mathematical problems in a small group while standing and writing on the vertical surfaces instead of sitting and writing in their notebooks. Accordingly, students all have access not only to the content of the whiteboard of their own small groups, but can also see what is written on the other groups' whiteboards. I had a chance⁴ to observe the following phenomenon in such a lesson: When a small group felt that they were in progress, they paid little attention to the work of the other groups. But when the students felt that they were stuck, some of them looked over the other groups' work without directly interacting with the members of these groups with the hope of getting a useful cue or evaluating their progress in comparison with the other groups' progress. In this way, they got what can be called *non-intrusive assistance*. Furthermore, the students engaged themselves in interactions exactly when they needed them and not when the teacher decided for them that they needed them. In terms of the definition of a choice-affluent environment, the students in Peter Liljedahl's class were empowered to make choices (2), (3), and (5) above.

⁴I thank Peter Liljedahl for the opportunity to attend such a class during my visit to Simon Fraser University in 2016.

18.5 Concluding Remarks

Developing a model of mathematical problem solving that would be applicable to different educational contexts is motivated by several causes. First, with few exceptions, the existing problem-solving frameworks utilize different conceptual tools for exploring problem solving in socially different educational contexts. Second, the foremost frameworks (Carlson and Bloom 2005; Schoenfeld 1985) are comprehensive within the problem-solving contexts within which they have emerged, but it is sometimes difficult to apply them to additional contexts. Third, the central issue of connecting our knowledge of how problem solving occurs and how to enhance this activity in instructional settings is still underdeveloped (e.g., Schoenfeld 2013).

In this chapter, a particular way of constructing a model of mathematical problem solving is presented. The proposed model capitalizes on the Mason's (1989, 2008, 2010) theory of shifts of attention (which was initially developed for other reasons) and consideration of choices that problem solvers are empowered to make. Hence, the model has been named the shifts and choices model, or the SCM. In fact, the model is a confluence and embeds theoretical tools from the existing problem-solving frameworks and theories. As mentioned, the model is only exploratory. Its use as a research tool is stipulated by the availability of research methodologies for identifying shifts of attention in socially different problem-solving contexts. In part, such methodologies are available from past research, but they should be further developed. The use of the model as a pedagogical tool depends on further unpacking the mechanisms underlying the choices that problem solvers make in different instructional situations and on research on how a teacher's decisions affect these choices. Little research has been conducted in this direction so far.⁵

Furthermore, the choice-affluent learning environment notion is introduced in this chapter. It is important to note that I do not argue for the claim that the more choices that are left to the students, the better. I rather argue for being aware of the fundamental role of pedagogical uncertainty related to Mason's (2016b) "when" question, for being aware of configurations of choices that we, mathematics teachers, inevitably create for our students and for being aware of the complexity and sensitivity of the student pathways of shifts of attention when they are engaged in problem solving.

I choose to end this chapter by mentioning some of the questions that I have had a tendency to ask myself as a mathematics educator ever since I have begun looking at my own lessons through the lenses provided by the SCM: What student choices do I tend to support? What student choices am I aware of? How do my students choose what they choose? How can I support the desirable choices without

⁵A study by Flowerday and Schraw (2000) on teachers' beliefs about instructional choices is an exception and a useful step towards understanding how teachers construct choices for their students.

choosing for the students? Which choices should I leave to the students? To what extent were my lessons during the last week choice affluent? My best hope in relation to this chapter is that some of the readers would find some of these questions worth thinking about.

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