Chapter 19
Quantities, Numbers, Number Names and the Real Number Line

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19.1 Introduction

The starting point of this paper is a quotation from Davydov's *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula,* whose original edition in Russian dates back to 1972 and was translated into English in 1990.

> When we designed a mathematics course, we proceeded from the fact that the students' creation of a detailed and thorough conception of a *real number,* underlying which is the concept of *quantity,* is currently the end purpose of this entire instructional subject from grade 1 to grade 10. Numbers (natural and real) are a particular aspect of this more general mathematical entity. (Davydov 1990, p. 167)

> In our course the teacher, relying on the knowledge previously acquired by the children, introduces number as a particular case of the representation of a *general relationship* of quantities, where one of them is taken as a measure and is computing the other. (Davydov 1990, p. 169)

19.2 Two Conceptions of Quantity: Counting and Measure

Number and operations have two aspects: conceptual (what numbers are) and nominal (how we name and denote numbers). Conceptually, numbers arise from a sense of *quantity* of some experiential species of objects – count (of a set or collection), distance, area, volume, time, rate, etc. And in fact before children enter school, they have already acquired a sense of quantity, of rough comparison of size, as well as of

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counting. Number is not intrinsically attached to a quantity; rather it arises from *measuring* one quantity by another, taken to be the 'unit': How 'much' (or many) of the unit is needed to constitute the given quantity? This is the *measurement framework* in which fractions are often introduced, via part-whole relations, the whole playing the role of the unit, which is a choice to be made and has to be specified. The discrete (counting) context in which whole numbers are often developed is distinguished by the use of the single-object set as the unit, so that the very concept of the unit, and its possible variability, is not necessarily subject to conscious consideration. This choice is so natural, and often taken for granted, that the concept of a *chosen unit of measurement* need not enter explicit discussion. If number is first developed exclusively in this discrete context, then fractions, introduced later, might appear to be, conceptually, a new and more complex species of number quite separate from whole numbers. This might make it difficult to see how the two kinds of numbers eventually, coherently, inhabit the same real number line. Indeed, this integration entails seeing the placement of whole numbers on the number line from the point of view (not of discrete counting, but) of continuous linear measure.

This distinction is further reinforced by the fact that fractions have their own notational representation, distinct from the base-ten place value of whole numbers. The operations on numbers likewise have conceptual models, but notational representations of number are needed in order to construct *computational algorithms*. To calculate, say, a sum of two numbers is not to ask about what the sum *means*. Instead, given two numbers A and B in notation system S, a calculation is a construction of a representation of A + B in same notation system S. That is why '2 + 11', though a logically correct answer to 'What is '5 + 8?', is *not* the correct answer, 13, to the question: '*calculate* 5 + 8'. At the same time, important as the notation is, its emphasis without links to the conceptual foundations can make it seem that quantities are the same as their number names, which is false, and potentially misleading.

Two possible pathways exist for the development of whole numbers:

Counting

Using the discrete context of finite sets, introduce whole numbers as cardinals, and addition as the cardinal of a disjoint union, and the experience of enumerating and comparing sets. (This rests on a discrete model of quantity.)

Measure

One uses the general context of *quantity* of *various* species of experiential objects and addition as disjoint union or concatenation (composition and decomposition). This allows discussion of comparison of quantities ('which one is more?'), and implicitly that the larger quantity *equals* the smaller plus some complementary quantity. This can be done before any numerical values have been attached to the quantities, with the relations expressed symbolically.

Then number is introduced by choice of a unit, and the number attached to a quantity is how much of the unit is needed to constitute the given quantity. Whole numbers then are represented in the form of quantities that are measured exactly by a set of copies of the unit.

The measure pathway was articulated in detail by Davydov (1975). My first purpose here is to discuss the measure pathway and cite some possible virtues that merit our attention. In particular, I will note that it makes available from the beginning the *continuous* number line as a coherent geometric environment in which all numbers of school mathematics eventually reside.

My second purpose is to discuss our base-ten place value notation for whole numbers (and finite decimals) and their operations, emphasising its extraordinary power and its impact on the progress of mathematics and science. I will also describe a particular instructional model[1] for the introduction of place value. This model can be seen to provide an activity context for not only conceptual understanding of place value, but also one that models the 'intellectual need' (Harel 2003/2007) to *invent* some version of number notation based on hierarchical grouping.

19.3 Implications for the Development of the Real Number Line

19.3.1 Two Narratives

I propose here some affordances of developing number in the measure context. Most importantly, this approach offers a productive context for developing the real number line across the grades. Relying exclusively on the discrete model of counting leads to what I will call the 'construction narrative' of the number line, in which the new kinds of numbers, their notations and their operations are added incrementally without sufficient interconnection. In this narrative, whole numbers and their verbal names and symbolic base-ten representations predominate. New kinds of numbers are added – fractions, negative numbers, a few irrational numbers and eventually infinite decimals. This process of bringing in these new types of number can lead to 'immigration stress' and difficulties of assimilation of the new numbers into one coherent context. In particular, the real number line as a coherent connected number universe with uniformly smooth arithmetic operations is not as explicit as it could be.

In the 'measure narrative', the number line, at least as a geometric continuum, is featured as the environment of linear measurement. A premise of this trajectory is that the mathematical resources that children bring include not only discrete counting, but also a sense of measurement of continuous quantity. A possible metaphor for geometric number line is an (indefinitely long) string, flexible but inelastic. Then linear quantities would be 'measured' by a segment of string. This would permit comparison of size even before such measures acquire numerical names. An example of an activity drawing on this metaphor is to engage students in considering how

[1] This is based on work by Deborah Ball with teacher candidates, representing work done with primary grade children.

far two toy cars travel from a starting point by examining where each car stops along a strip of tape on the floor. In order to compare measures of two things that are remote, one adopts a standard *unit* of measure, against which both quantities can be compared. And then whole number quantities appear as iterated composites of that unit.

To situate numbers on the number line, the '*oriented unit*' is specified on the geometric line by the choice of an ordered pair of points, called 0 and 1, the unit of linear measure then being the segment, [0, 1], between them. The direction from 0 to 1 then also specifies a positive orientation to the number line (which has an intrinsic linear order defined by the fact that, given any three points, one lies in the interval between the other two), whereupon the whole numbers (and eventually all real numbers) can be located on the number line by juxtaposing replicas of [0, 1] in the positive direction.

Of course the counting approach to whole numbers can be interpreted in measure terms, since cardinal is one particular context of measurement. However, counting is only one such (discontinuous) context, and the unit (a set with one member) must be made explicit to extend to the general concept of unit. Other units in the discrete context are made visible when one later encounters (skip) counting in groups. More general continuous measurement environments for whole numbers are robustly represented with materials such as Cuisenaire rods. Eventually, whole numbers (as cardinals) are so well conceptually assimilated that they seem to become (abstract) entities in their own right.

Fractions are often developed from a measure perspective, with fractions, from the start, being conceived as part-whole relationships, and applied to a wide variety of species of quantities: round food; lengths of ribbon; containers of sugar, or of milk; sets of objects; periods of time; etc. In contrast with whole numbers, it is less common to name a fraction without adding the word 'of'. Moreover, we do not hesitate to compare the size of whole numbers, while, with fractions, we are more prone to first ask, 'fractions of what?' – attending to specification of the unit (or whole).

19.3.2 Operations and the Real Number Line

Addition and subtraction appear to be conceptually similar in both the counting and measure regimes, addition corresponding to combination (composition and decomposition of quantities) and subtraction to taking away or comparison.

Multiplication is more subtle and more complex. One model is repeated addition of some fixed quantity, as if applying the counting regime to fixed-size groups of unit quantities. One difficulty with this model is that it obscures the commutativity of multiplication. This is sometimes repaired by use of rectangular arrays, eventually evolving into area models. The difficulty of the area model, from a measure perspective, is that numbers and their products then have different units of measure (e.g. length and area), so that it is problematic to assign meaning to an expression like $a \bullet b + c$. One resolution of this is to use a continuous version of repeated addition, which is scaling (magnification and shrinking). This has the advantage of maintaining the species of quantities involved. These are complex conceptual issues, which I do not pursue here.

Suffice it to say here that, from the point of view of quantity (measurement), we can combine (simplify) additive expressions only when they are quantities of the same species (we do not add apples and oranges, unless combined into some larger category, like ‘fruit’), expressed with a common unit and then the sum or difference is a quantity expressed with that same unit. When dealing with fractions, a quantity like 3/5 is understood to be three one-fifths, where the latter corresponds to a rescaling of the unit. In adding fractions, finding a ‘common denominator’ is then a process of measuring two quantities with a common unit in order to make simplification of the sum possible. Similarly, in multi-digit addition, the alignment of the base-ten representations of the summands assures that the addition in each column is adding digits with the same base-ten units attached.

On the other hand, for multiplication and division, the units of measurement are not restricted but simply parallel the operation, leading to compound units, like: kilometres/hour, foot • pounds, pounds per square inch and class • hours.

Once numbers are named and denoted (in base-ten or with fraction notation), then we develop algorithms for the operations in that notational system. The power of the base-ten system is that addition, subtraction and multiplication can be performed on *any* pair of whole numbers, knowing only how to perform single-digit operations (‘basic facts’) plus how to keep track of positional notation. This puts extraordinary computational power instructionally within reach of young children, a major historical development.

Once fractions and integers have been developed, one has the rational numbers, which are densely distributed on the number line: between any two points there is a rational number. The example of irrational numbers, like $\sqrt{2}$, shows that many points remain to be named. Informal arguments of approximation can indicate how all points can eventually be specified by possibly infinite decimal representations. Moreover, informal assurance can be given that the operations extend by continuity to all real numbers, preserving the *basic rules of arithmetic*. This synthesis of the real number line sets the stage for higher mathematics, for example calculus.

19.4 The Davydov Curriculum

Davydov, a Vygotskian psychologist and educator, and his colleagues in the Soviet Union developed, in the 1960s and 1970s, a curriculum based on the measure approach (1990).

In order to develop the concept of number, the Davydov curriculum delayed the introduction of number in school instruction until late in the first grade. Early lessons concentrate on ‘pre-numerical’ material: properties of objects such as colour, shape and size and then quantities such as length, volume, area, mass and amount of discrete objects (i.e. collections of things, but without yet using number to enumerate 'how many').

According to Davydov, the fundamental problem solved by the invention of number is the task of taking a given quantity (length, volume, mass, area, amount of discrete objects) and reproducing it at a different time or place. Moxhay (2008) describes the following activity that illustrates this.

On one table is a strip of paper tape. The task is to go to another table (in a different room) and cut off, from the supply of paper tape, a piece that is exactly the same length as the original one. But one is not allowed to carry the original paper strip over to the other table. In Davydov's experiments, children sometimes just walked over to the second table and cut off a piece of paper of a random size, hoping that it would be the same length as the original one. In such cases, conditions of the task seemed to the children to make a correct solution impossible (except by luck).

Davydov and his colleagues explained that a solution might involve taking a third object, such as string, and cutting it to be just precisely the length of the paper strip and then carrying this intermediary object (the string) to the other table, where it can be used to lay off a new paper strip of the required length. In this case, the intermediary is equal in length to the object to be reproduced. The curricular approach showed children how to take a given third object, say a piece of wood, and, if it is longer than the paper strip, mark it to show the length of the paper strip. This solution was equivalent to the first one, with the children performing just a different set of operations. But if the only available intermediary object was *smaller* than the paper strip – for example, a wooden block – this was an interesting case, for then the children could learn that they could use the block as a *unit*, as an intermediary that could be placed repeatedly (each time marking the paper with a pencil) and then *counting up* how many times the unit has been laid down. The unit could then be carried to the other table (together with the number), where it would be laid down on the paper tape the number of times that is necessary to reproduce, by cutting, a paper strip of the required length. Note that, only with this last method – selection of a unit and counting how many of it are needed – that number names make an appearance.

Although this is a particular task, solved by a particular discovery on the part of the children, it is said to lead 'genetically' to the solution of all analogous tasks. If the children, working as a collective, grasp the meaning of the construction they have made, then they should (again, collectively, at least at first) be able to attack all analogous problems. Davydov argues that children thus recreate, in brief, the invention of number as a human tool that enables *any* quantity to be reproduced at a different place or time. It is worth noting that this task would lose its force in the discrete context of counting, in which the portability of the unit is much simpler to achieve, but therefore is also invisible and tacit.

Davydov argued that it was important for children to reflect on, become conscious of, the ideas developed through this activity. He develops this as a collective process, with the teacher guiding the children to ask one another questions like, 'How did you do this? Why did you do this? Does your method work? Is that the best method for solving the task?'

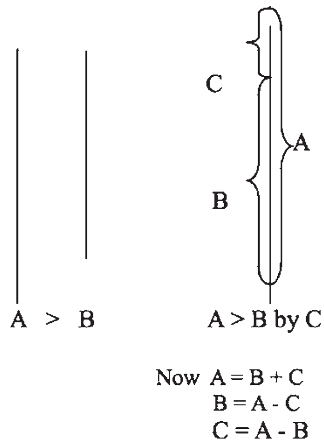

Fig. 19.1 Exercises from Davydov's curriculum

19.4.1 Algebra in the Context of Davydov

In the Davydov curriculum (see Schmittau 2005), children were to study scalar quantities such as the length, area, volume and weight of real objects, which they can experience visually and tactilely, thus gaining a first access to the real number continuum. Early in the first grade, for example, children were shown that they could make two unequal volumes equal by adding to the smaller or subtracting from the larger the difference between the original quantities. They determined that if volume A is greater than volume B, then A = B + C, where C is the quantity complementary to B in A. The children would be led to schematise their result with a 'length' model and symbolise it with equations and inequalities (Fig. 19.1).

The following problem, occurring approximately half-way through the first grade curriculum, provides another example of the role of the schematic in problem solving: *N* apples were in a bowl on the table. *R* people entered the room and each took an apple. How many apples remained? Children first analyse the structure of the problem, identifying it as a part-whole structure, with *N* as the whole and *R* as a part. They schematise the quantitative relations expressed in the problem as follows:

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  N
 / \
R   ?
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Beyond the visibly algebraic form of these equations and relations, introduced quite early, there are further noteworthy features, having to do with the very nature of the '=' sign. When equations are introduced numerically, the first exercises often have the format 8 + 4 = _, with the result that students gain the habit of reading '='

as 'calculate what is on the left, and put the answer on the right'. Thus, they will validate the equation 8 + 4 = 12, but question the truth of 12 = 8 + 4. Moreover, they may fill the blank in 8 + 4 = _ + 7 with 12. I expect that these common confusions would be mitigated with the balancing of the quantities approach of the Davydov curriculum. Of course, other curricula have ways of accomplishing this as well.

19.4.2 Place Value

> The greatest calamity in the history of science was the failure of Archimedes to invent positional notation. Carl Friedrich Gauss (as quoted by Bell 1937, p. 256)

Davydov emphasised the notion of quantity as being primary, the concept of number being later derived as a measure of one quantity by another (the unit). There then arises the task of providing names and notations for numbers. Although the notion of quantity is in some sense cognitively primordial, the naming of numbers, in contrast, is a cultural construct, and it has been accomplished historically in many different ways (see, for example, ICMI Study 13 2006). But the naming of numbers is much more than a cultural convention. It is itself a piece of conceptual technology with huge bearing on the progress of science. Our current Hindu-Arabic system of (base-ten) place value notation, now universally used in science, was solidified relatively late in history. It puts within reach of even young children a quantitative power not reached even by the mathematical genius of ancient Greece. (See the above quotation from Gauss.)

Howe (2011) offers a critique of elementary curriculum in the USA, 'Place value […] is treated as a vocabulary issue: ones place, tens place, hundreds place. It is described procedurally rather than conceptually'. How can one produce in young children and their teachers a robust conceptual understanding of place value? I describe here a method developed by Deborah Ball, one that is now an integral part of the teacher education programme at my university. Teacher candidates experience this sequence for several purposes, among them to appreciate the structure and meaning of a numeration system, in this case, the base-ten system. This approach fits here since its design echoes the instructional approach of Davydov (1975, 1990), Brousseau (1986) and others, who like to introduce a concept using a mathematical problem context whose solution necessitates discovery of that concept.

In this case, *the problem is to collectively count a large collection*. The size of the count is sufficient to require some structural organisation for record-keeping and to make this common across the individual counters so as to be able to coherently combine the different records. It is this need that precipitates the idea of grouping, which leads to a hierarchal structure akin to place value.

The setting here is a methods class for some 25 elementary teacher interns. (The activity is a compressed approximation of what would be done with primary grade children over much longer period of time.) About half of the interns sit in a circle on the floor with the teacher, the others observing and taking notes. On the floor, the teacher pours out a container of over 2000 wooden sticks. She first invites the interns to guess/estimate how many sticks there are. After a wide range of guesses, she asks, ‘How could we find out?’ and it is suggested that they count them. So the counting begins, each intern gathering individual sticks from the pile and lining them up. However, their individual collections quickly become so numerous that they feel a need to somehow consolidate. After some discussion the idea of *grouping* the sticks emerges (see, this volume, Sect. 9.2.2). Note that this arises, not as a mathematical suggestion but as a practical necessity, given the large size of the counting task. And with rubber bands that are available, they begin to form what they call ‘bundles’ of sticks. But then the question arises, ‘How many sticks should be in a bundle?’ Several choices are considered (e.g. 2, 5, 10, 25, 60). The small values are judged not to achieve enough efficiency to be worthwhile and the larger to be possibly unwieldy. It is nonetheless clear that *this is a choice to be made*; it is not mathematically forced. (This opens the space to later contemplate place value in bases other than 10.) More importantly, *this choice should be the same for each person*. Otherwise, there would be no coherent way to count the amalgamated collections at the end. The teacher eventually encourages as consensus making bundles of ten sticks each.

Then the counting continues, and the interns make a bundle as soon as ten loose sticks are available to do so. At any given moment, an intern’s collection has the form of a certain number of bundles, together with at most nine loose sticks. However, the big pile is so numerous that the interns confront the same problem again, this time with their bundles instead of individual sticks. A discussion similar to the earlier one then ensues about grouping the bundles, to form ‘bundles of bundles’, or ‘super-bundles’, as they came to be called. Again the question arose: ‘how many bundles should there be in a super-bundle?’ It was noted that this choice could, in principle, be independent of the first. But it was decided that there would be some mathematical merit in again choosing ten for the number of bundles in a super-bundle. And these could again be bound together with rubber bands. At this point, each intern’s collection consists of a modest number of super-bundles, at most nine bundles and at most nine loose sticks.

Finally, when the big pile was exhausted, the collections of the different interns were brought together. Then the many loose sticks were bundled until at most nine loose sticks remained. In turn, then, the bundles were super-bundled until at most nine bundles remained. Finally, there being over 20 super-bundles, it was decided to make 2 ‘mega-bundles’, each composed of 10 super-bundles. In the end, then, the original pile had been organised into two mega-bundles, four super-bundles, seven bundles and six loose sticks. Thus, the cardinal of this huge collection of could be specified by a list of just four small numbers, (2, 4, 7, 6), specifying the numbers of mega-bundles, super-bundles, bundles and loose sticks, respectively. By construction, the number of sticks in a bundle is 10, in a super-bundle $10^2 = 100$ and in a mega-bundle, $10^3 = 1000$. Thus the very concise ‘coding’ (2, 4, 7, 6) tells us that the total number of sticks is 2000 + 400 + 70 + 6 = 2476 (in base-ten notation) (Fig. 19.2).

Fig. 19.2 Making bundles

This activity, with modest scaffolding, *simulated the invention of the place value system of recording numbers.* Moreover, it dramatically and physically presented the compressive power of the system: four small digits suffice to specify this perceptually very large quantity. In the course of the activity, the teacher could pose a number of questions, about representing particular numbers with the sticks and their bundles and also about how to identify numbers represented by various configurations of bundled sticks, modelling the sorts of interactions that would be carried out with children.

Attention was further drawn to the fact that the bundled sticks remained an authentic representation of quantity, since they could be unbundled to reproduce the original collection. This was put in contrast with other physical representations of number, such as Dienes blocks (see this volume, Sect. 9.3.1.2) for which the ten-rod could not be decomposed into to ten little cubes; rather, this would require a trade.

These physical models of base ten provide concrete models for the arithmetic operations. The correspondence with the symbolic base-ten notation can then be extended to provide concrete meaning to the algorithms for arithmetic computation.

19.5 Conclusion

I have argued that the measure-based introduction to number, as developed for example by Davydov, supports a possibly more coherent development of the real number line. Moreover, I suggest that it allows a smooth transition from whole numbers to fractions and it provides an early introduction to algebraic thinking. Finally, I have described an instructional activity, developed by Ball, that simulates the conceptual development of place value.

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References

Bell, E. T. (1937). *Men of mathematics*. New York: Simon & Schuster.
Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques, 7*(2), 33–115.
Davydov, V. V. (1975). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe (Ed.), *Children's capacity for learning mathematics* (pp. 55–107). Chicago: University of Chicago.
Davydov, V. V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula*, *Soviet studies in mathematics education* (vol. 2, pp. 1–223). Reston: National Council of Teachers of Mathematics. (Original published in 1972.)
Harel, G. (2003/2007). The DNR system as a conceptual framework for curriculum development and instruction. In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Foundations for the future of mathematics education* (pp. 263–280). New York: Lawrence Erlbaum Associates.
Howe, R. (2011). The greatest calamity in the history of science. Editorial, *ICMI Newsletter 18* (Ed. J. C. de Silva).
Leung, F.K.S., Graf, K.D., & Lopez-Real, F.J. (Eds). (2006). *Mathematics education in different cultural traditions: A comparative study of East Asia and the West: The 13th ICMI Study*. New York: Springer.
Moxhay, P. (2008). *Assessing the scientific concept of number in primary school children*. Paper presented at ISCAR 2008, San Diego.
Schmittau, J. (2005). The development of algebraic thinking: A Vygotskian perspective. *ZDM Mathematics Education, 37*(1), 16–22.