“What is Mathematics?” and why we should ask, where one should experience and learn that, and how to teach it

Günter M. Ziegler and Andreas Loos

Abstract “What is Mathematics?” [with a question mark!] is the title of a famous book by Courant and Robbins, first published in 1941, which does not answer the question. The question is, however, essential: The public image of the subject (of the science, and of the profession) is not only relevant for the support and funding it can get, but it is also crucial for the talent it manages to attract—and thus ultimately determines what mathematics can achieve, as a science, as a part of human culture, but also as a substantial component of economy and technology. In this lecture we thus

- discuss the image of mathematics (where “image” might be taken literally!),
- sketch a multi-facetted answer to the question “What is Mathematics?,”
- stress the importance of learning “What is Mathematics” in view of Klein’s “double discontinuity” in mathematics teacher education,
- present the “Panorama project” as our response to this challenge,
- stress the importance of *telling stories* in addition to *teaching* mathematics, and finally,
- suggest that the mathematics curricula at schools and at universities should correspondingly have space and time for at least three different subjects called Mathematics.

This paper is a slightly updated reprint of: Günter M. Ziegler and Andreas Loos, *Learning and Teaching* “*What is Mathematics*”, Proc. International Congress of Mathematicians, Seoul 2014, pp. 1201–1215; reprinted with kind permission by Prof. Hyungju Park, the chairman of ICM 2014 Organizing Committee.

G.M. Ziegler (✉)
Institute Für Mathematik, FU Berlin, Arnimallee 2, 14195 Berlin, Germany
e-mail: ziegler@math.fu-berlin.de

A. Loos
Zeit Online, Askanischer Platz 1, 10963 Berlin, Germany
e-mail: andreas.loos@zeit.de

G. Kaiser (ed.), *Proceedings of the 13th International Congress on Mathematical Education*, ICME-13 Monographs, DOI 10.1007/978-3-319-62597-3_5

What Is Mathematics?

Defining mathematics. According to *Wikipedia* in English, in the March 2014 version, the answer to "What is Mathematics?" is

> **Mathematics** is the abstract study of topics such as quantity (numbers),[2] structure,[3] space,[2] and change.[4][5][6] There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics.[7][8]
>
> Mathematicians seek out patterns (Highland & Highland, 1961, 1963) and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature. Through the use of abstraction and logic, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity for as far back as written records exist. The research required to solve mathematical problems can take years or even centuries of sustained inquiry.

None of this is entirely wrong, but it is also not satisfactory. Let us just point out that the fact that there is no agreement about the definition of mathematics, given as part of a definition of mathematics, puts us into logical difficulties that might have made Gödel smile.[1]

The answer given by *Wikipedia* in the current German version, reads (in our translation):

> **Mathematics** […] is a science that developed from the investigation of geometric figures and the computing with numbers. For *mathematics*, there is no commonly accepted definition; today it is usually described as a science that investigates abstract structures that it created itself by logical definitions using logic for their properties and patterns.

This is much worse, as it portrays mathematics as a subject without any contact to, or interest from, a real world.

The borders of mathematics. Is mathematics "stand-alone"? Could it be defined without reference to "neighboring" subjects, such as physics (which does appear in the English *Wikipedia* description)? Indeed, one possibility to characterize mathematics describes the borders/boundaries that separate it from its neighbors. Even humorous versions of such "distinguishing statements" such as

- "Mathematics is the part of physics where the experiments are cheap."
- "Mathematics is the part of philosophy where (some) statements are true—without debate or discussion."

---

[1]According to *Wikipedia*, the same version, the answer to "Who is Mathematics" should be:

> **Mathematics**, also known as **Allah Mathematics**, (born: **Ronald Maurice Bean**[1]) is a hip hop producer and DJ for the Wu-Tang Clan and its solo and affiliate projects.

This is not the mathematics we deal with here.

- "Mathematics is computer science without electricity." (So "Computer science is mathematics with electricity.")

contain a lot of truth and possibly tell us a lot of "characteristics" of our subject. None of these is, of course, completely true or completely false, but they present opportunities for discussion.

What we do in mathematics. We could also try to define mathematics by "what we do in mathematics": This is much more diverse and much more interesting than the *Wikipedia* descriptions! Could/should we describe mathematics not only as a research discipline and as a subject taught and learned at school, but also as a playground for pupils, amateurs, and professionals, as a subject that presents challenges (not only for pupils, but also for professionals as well as for amateurs), as an arena for competitions, as a source of problems, small and large, including some of the hardest problems that science has to offer, at all levels from elementary school to the millennium problems (Csicsery, 2008; Ziegler, 2011)?

What we teach in mathematics classes. Education bureaucrats might (and probably should) believe that the question "What is Mathematics?" is answered by high school curricula. But what answers do these give?

This takes us back to the nineteenth century controversies about what mathematics should be taught at school and at the Universities. In the German version this was a fierce debate. On the one side it saw the classical educational ideal as formulated by Wilhelm von Humboldt (who was involved in the concept for and the foundation 1806 of the Berlin University, now named Humboldt Universität, and to a certain amount shaped the modern concept of a university); here mathematics had a central role, but this was the classical "Greek" mathematics, starting from Euclid's axiomatic development of geometry, the theory of conics, and the algebra of solving polynomial equations, not only as cultural heritage, but also as a training arena for logical thinking and problem solving. On the other side of the fight were the proponents of "Realbildung": *Realgymnasien* and the technical universities that were started at that time tried to teach what was needed in commerce and industry: calculation and accounting, as well as the mathematics that could be useful for mechanical and electrical engineering—second rate education in the view of the classical German Gymnasium.

This nineteenth century debate rests on an unnatural separation into the classical, pure mathematics, and the useful, applied mathematics; a division that should have been overcome a long time ago (perhaps since the times of Archimedes), as it is unnatural as a classification tool and it is also a major obstacle to progress both in theory and in practice. Nevertheless the division into "classical" and "current" material might be useful in discussing curriculum contents—and the question for what purpose it should be taught; see our discussion in the Section "Three Times Mathematics at School?".

The Courant–Robbins answer. The title of the present paper is, of course, borrowed from the famous and very successful book by Richard Courant and

Herbert Robbins. However, this title is a question—what is Courant and Robbins' answer? Indeed, the book does not give an explicit definition of "What is Mathematics," but the reader is supposed to get an idea from the presentation of a diverse collection of mathematical investigations. Mathematics is much bigger and much more diverse than the picture given by the Courant–Robbins exposition. The presentation in this section was also meant to demonstrate that we need a multi-facetted picture of mathematics: One answer is not enough, we need many.

Why Should We Care?

The question "What is Mathematics?" probably does not need to be answered to motivate *why* mathematics should be taught, as long as we agree that mathematics is important.

However, a one-sided answer to the question leads to one-sided concepts of *what* mathematics should be taught.

At the same time a one-dimensional picture of "What is Mathematics" will fail to motivate kids at school to do mathematics, it will fail to motivate enough pupils to study mathematics, or even to think about mathematics studies as a possible career choice, and it will fail to motivate the right students to go into mathematics studies, or into mathematics teaching. If the answer to the question "What is Mathematics", or the implicit answer given by the public/prevailing *image* of the subject, is not attractive, then it will be very difficult to motivate *why* mathematics should be learned—and it will lead to the wrong offers and the wrong choices as to *what* mathematics should be learned.

Indeed, would anyone consider a science that studies "abstract" structures *that it created itself* (see the German *Wikipedia* definition quoted above) interesting? Could it be relevant? If this is what mathematics is, why would or should anyone want to study this, get into this for a career? Could it be interesting and meaningful and satisfying to teach this?

Also in view of the diversity of the students' expectations and talents, we believe that one answer is plainly not enough. Some students might be motivated to learn mathematics because it is beautiful, because it is so logical, because it is sometimes surprising. Or because it is part of our cultural heritage. Others might be motivated, and not deterred, by the fact that mathematics is difficult. Others might be motivated by the fact that mathematics is useful, it is needed—in everyday life, for technology and commerce, etc. But indeed, it is not true that "the same" mathematics is needed in everyday life, for university studies, or in commerce and industry. To other students, the motivation that "it is useful" or "it is needed" will not be sufficient. All these motivations are valid, and good—and it is also totally valid and acceptable that no single one of these possible types of arguments will reach and motivate *all* these students.

Why do so many pupils and students fail in mathematics, both at school and at universities? There are certainly many reasons, but we believe that motivation is a key factor. Mathematics *is* hard. It is abstract (that is, most of it is not directly connected to everyday-life experiences). It is not considered worth-while. But a lot of the insufficient motivation comes from the fact that students and their teachers do not know "What is Mathematics."

Thus a multi-facetted image of mathematics as a coherent subject, all of whose many aspects are well connected, is important for a successful teaching of mathematics to students with diverse (possible) motivations.

This leads, in turn, to two crucial aspects, to be discussed here next: What image do students have of mathematics? And then, what should teachers answer when asked "What is Mathematics"? And where and how and when could they learn that?

The Image of Mathematics

A 2008 study by Mendick, Epstein, and Moreau (2008), which was based on an extensive survey among British students, was summarized as follows:

> Many students and undergraduates seem to think of mathematicians as old, white, middle-class men who are obsessed with their subject, lack social skills and have no personal life outside maths.
>
> The student's views of maths itself included narrow and inaccurate images that are often limited to numbers and basic arithmetic.

The students' image of what mathematicians are like is very relevant and turns out to be a massive problem, as it defines possible (anti-)role models, which are crucial for any decision in the direction of "I want to be a mathematician." If the typical mathematician is viewed as an "old, white, male, middle-class nerd," then why should a gifted 16-year old girl come to think "that's what I want to be when I grow up"? Mathematics as a science, and as a profession, looses (or fails to attract) a lot of talent this way! However, this is not the topic of this presentation.

On the other hand the first and the second diagnosis of the quote from Mendick et al. (2008) belong together: The mathematicians are part of "What is Mathematics"!

And indeed, looking at the second diagnosis, if for the key word "mathematics" the *images* that spring to mind don't go beyond a per se meaningless "$a^2 + b^2 = c^2$" scribbled in chalk on a blackboard—then again, why should mathematics be attractive, as a subject, as a science, or as a profession?

We think that we have to look for, and work on, multi-facetted and attractive representations of mathematics by images. This could be many different, separate images, but this could also be images for "mathematics as a whole."

Four Images for "What Is Mathematics?"

Striking pictorial representations of mathematics as a whole (as well as of other sciences!) and of their change over time can be seen on the covers of the German "Was ist was" books. The history of these books starts with the series of "How and why" Wonder books published by Grosset and Dunlop, New York, since 1961, which was to present interesting subjects (starting with "Dinosaurs," "Weather," and "Electricity") to children and younger teenagers. The series was published in the US and in Great Britain in the 1960s and 1970s, but it was and is much more successful in Germany, where it was published (first in translation, then in volumes written in German) by Ragnar Tessloff since 1961. Volume 18 in the US/UK version and Volume 12 in the German version treats "Mathematics", first published in 1963 (Highland & Highland, 1963), but then republished with the same title but a new author and contents in 2001 (Blum, 2001). While it is worthwhile to study the contents and presentation of mathematics in these volumes, we here focus on the cover illustrations (see Fig. 1), which for the German edition exist in four entirely different versions, the first one being an adaption of the original US cover of (Highland & Highland, 1961).

All four covers represent a view of "What is Mathematics" in a collage mode, where the first one represents mathematics as a mostly historical discipline (starting with the ancient Egyptians), while the others all contain a historical allusion (such as pyramids, Gauß, etc.) alongside with objects of mathematics (such as prime numbers or π, dices to illustrate probability, geometric shapes). One notable object is the oddly "two-colored" Möbius band on the 1983 cover, which was changed to an entirely green version in a later reprint.

One can discuss these covers with respect to their contents and their styles, and in particular in terms of attractiveness to the intended buyers/readers. What is over-emphasized? What is missing? It seems more important to us to

- think of our own images/representations for "What is Mathematics",
- think about how to present a multi-facetted image of "What is Mathematics" when we teach.

Indeed, the topics on the covers of the "Was ist was" volumes of course represent interesting (?) topics and items discussed in the books. But what do they add up to? We should compare this to the image of mathematics as represented by school curricula, or by the university curricula for teacher students.

In the context of mathematics images, let us mention two substantial initiatives to collect and provide images from current mathematics research, and make them available on internet platforms, thus providing fascinating, multi-facetted images of mathematics as a whole discipline:

- Guy Métivier et al.: "Image des Maths. La recherche mathématique en mots et en images" ["Images of Maths. Mathematical research in words and images"], CNRS, France, at `images.math.cnrs.fr` (texts in French)

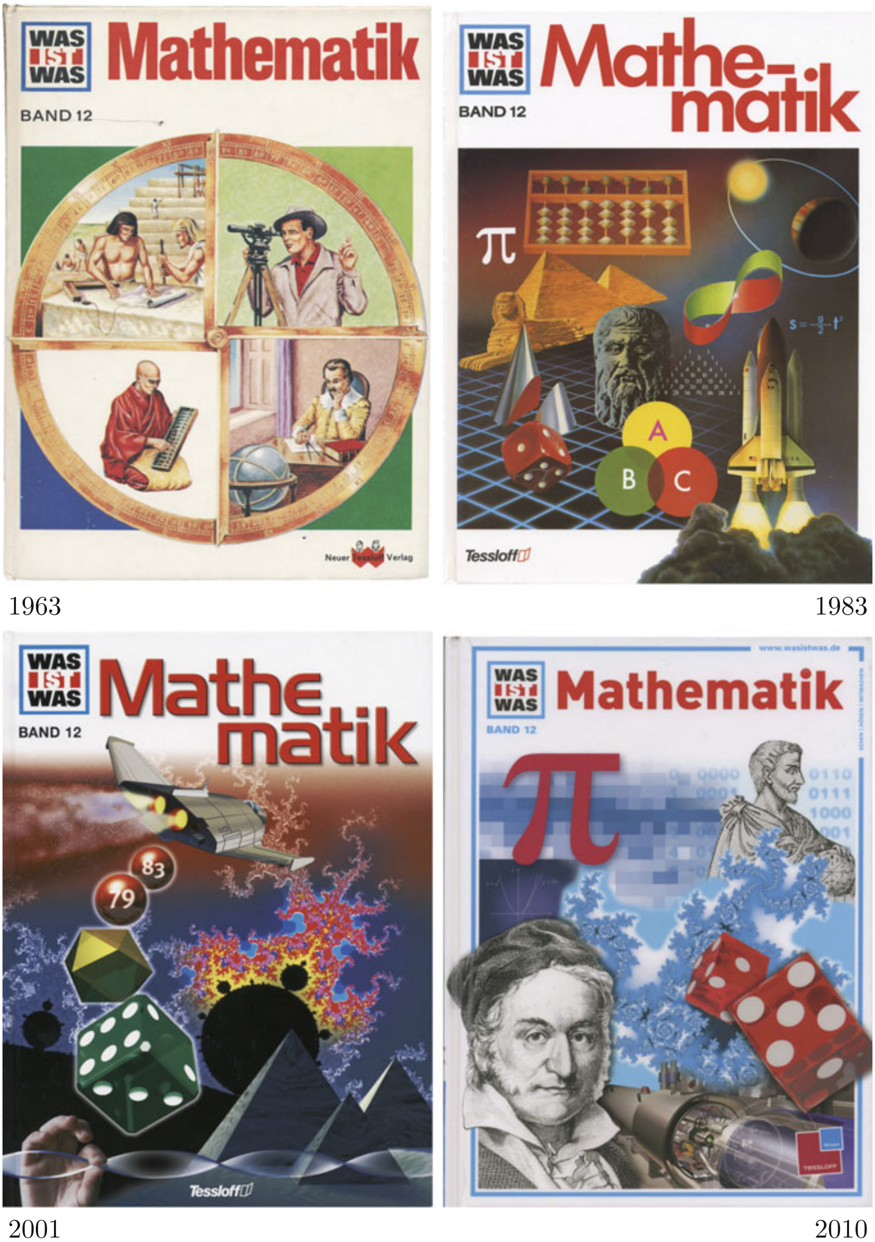

Fig. 1 The four covers of “Was ist was. Band 12: Mathematik” (Highland & Highland, 1963; Blum, 2001)

- Andreas D. Matt, Gert-Martin Greuel et al.: “IMAGINARY. open mathematics,” Mathematisches Forschungsinstitut Oberwolfach, at `imaginary.org` (texts in German, English, and Spanish).

The latter has developed from a very successful travelling exhibition of mathematics images, “IMAGINARY—through the eyes of mathematics,” originally created on occasion of and for the German national science year 2008 “Jahr der Mathematik. Alles was zählt” [“Year of Mathematics 2008. Everything that counts”], see www.jahr-der-mathematik.de, which was highly successful in communicating a current, attractive image of mathematics to the German public—where initiatives such as the IMAGINARY exhibition had a great part in the success.

Teaching “What Is Mathematics” to Teachers

More than 100 years ago, in 1908, Felix Klein analyzed the education of teachers. In the introduction to the first volume of his “Elementary Mathematics from a Higher Standpoint” he wrote (our translation):

> At the beginning of his university studies, the young student is confronted with problems that do not remind him at all of what he has dealt with up to then, and of course, he forgets all these things immediately and thoroughly. When after graduation he becomes a teacher, he has to teach exactly this traditional elementary mathematics, and since he can hardly link it with his university mathematics, he soon readopts the former teaching tradition and his studies at the university become a more or less pleasant reminiscence which has no influence on his teaching (Klein, 1908).

This phenomenon—which Klein calls the *double discontinuity*—can still be observed. In effect, the teacher students “tunnel” through university: They study at university in order to get a degree, but nevertheless they afterwards teach the mathematics that they had learned in school, and possibly with the didactics they remember from their own school education. This problem observed and characterized by Klein gets even worse in a situation (which we currently observe in Germany) where there is a grave shortage of Mathematics teachers, so university students are invited to teach at high school long before graduating from university, so they have much less university education to tunnel at the time when they start to teach in school. It may also strengthen their conviction that University Mathematics is not needed in order to teach.

How to avoid the double discontinuity is, of course, a major challenge for the design of university curricula for mathematics teachers. One important aspect however, is tied to the question of “What is Mathematics?”: A very common highschool image/concept of mathematics, as represented by curricula, is that mathematics consists of the subjects presented by highschool curricula, that is, (elementary) geometry, algebra (in the form of arithmetic, and perhaps polynomials), plus perhaps elementary probability, calculus (differentiation and integration) in one variable—that’s the mathematics highschool students get to see, so they might think that this is all of it! Could their teachers present them a broader picture? The teachers after their highschool experience studied at university, where they probably took courses in calculus/analysis, linear algebra, classical algebra, plus some discrete mathematics, stochastics/probability, and/or numerical analysis/differential equations, perhaps a programming or "computer-oriented mathematics" course. Altogether they have seen a scope of university mathematics where no current research becomes visible, and where most of the contents is from the nineteenth century, at best. The *ideal* is, of course, that every teacher student at university has at least once experienced how "doing research on your own" feels like, but realistically this rarely happens. Indeed, teacher students would have to work and study and struggle a lot to see the fascination of mathematics on their own by doing mathematics; in reality they often do not even seriously start the tour and certainly most of them never see the "glimpse of heaven." So even if the teacher student seriously immerges into all the mathematics on the university curriculum, he/she will not get any broader image of "What is Mathematics?". Thus, even if he/she does *not* tunnel his university studies due to the double discontinuity, he/she will not come back to school with a concept that is much broader than that he/she originally gained from his/her highschool times.

Our experience is that many students (teacher students as well as classical mathematics majors) cannot name a single open problem in mathematics when graduating the university. They have no idea of what "doing mathematics" means—for example, that part of this is a struggle to find and shape the "right" concepts/definitions and in posing/developing the "right" questions and problems.

And, moreover, also the impressions and experiences from university times will get old and outdated some day: a teacher might be active at a school for several decades—while mathematics changes! Whatever is proved in mathematics does stay true, of course, and indeed standards of rigor don't change any more as much as they did in the nineteenth century, say. However, styles of proof do change (see: computer-assisted proofs, computer-checkable proofs, etc.). Also, it would be good if a teacher could name "current research focus topics": These do change over ten or twenty years. Moreover, the relevance of mathematics in "real life" has changed dramatically over the last thirty years.

The Panorama Project

For several years, the present authors have been working on developing a course [and eventually a book (Loos & Ziegler, 2017)] called "Panorama der Mathematik" ["Panorama of Mathematics"]. It primarily addresses mathematics teacher students, and is trying to give them a panoramic view on mathematics: We try to teach an overview of the subject, how mathematics is done, who has been and is doing it, including a sketch of main developments over the last few centuries up to the present—altogether this is supposed to amount to a comprehensive (but not very detailed) outline of "What is Mathematics." This, of course, turns out to be not an easy task, since it often tends to feel like reading/teaching poetry without mastering the language. However, the approach of Panorama is complementing mathematics education in an orthogonal direction to the classic university courses, as we do not *teach* mathematics but *present* (and encourage to *explore*); according to the response we get from students they seem to feel themselves that this is valuable.

Our course has many different components and facets, which we here cast into questions about mathematics. All these questions (even the ones that "sound funny") should and can be taken seriously, and answered as well as possible. For each of them, let us here just provide at most one line with key words for answers:

- When did mathematics start?
  Numbers and geometric figures start in stone age; the science starts with Euclid?
- How large is mathematics? How many Mathematicians are there?
  The Mathematics Genealogy Project had 178854 records as of 12 April 2014.
- How is mathematics done, what is doing research like?
  Collect (auto)biographical evidence! Recent examples: Frenkel (2013)*, Villani* (2012).
- What does mathematics research do today? What are the Grand Challenges?
  The Clay Millennium problems might serve as a starting point.
- What and how many subjects and subdisciplines are there in mathematics?
  See the Mathematics Subject Classification for an overview!
- Why is there no "Mathematical Industry", as there is e.g. Chemical Industry?
  There is! See e.g. Telecommunications, Financial Industry, etc.
- What are the "key concepts" in mathematics? Do they still "drive research"?
  Numbers, shapes, dimensions, infinity, change, abstraction, …; they do.
- What is mathematics "good for"?
  It is a basis for understanding the world, but also for technological progress.
- Where do we *do* mathematics in everyday life?
  Not only where we compute, but also where we read maps, plan trips, etc.
- Where do we *see* mathematics in everyday life?
  There is more maths in every smart phone than anyone learns in school.
- What are the greatest achievements of mathematics through history?
  Make your own list!

An additional question is how to make university mathematics more "sticky" for the tunneling teacher students, how to encourage or how to force them to really connect to the subject as a science. Certainly there is no single, simple, answer for this!

Telling Stories About Mathematics

How can mathematics be made more concrete? How can we help students to connect to the subject? How can mathematics be connected to the so-called real world?

Showing applications of mathematics is a good way (and a quite beaten path). Real applications can be very difficult to *teach* since in most advanced, realistic situation a lot of different mathematical disciplines, theories and types of expertise have to come together. Nevertheless, applications give the opportunity to demonstrate the relevance and importance of mathematics. Here we want to emphasize the difference between *teaching* a topic and *telling* about it. To name a few concrete topics, the mathematics behind weather reports and climate modelling is extremely difficult and complex and advanced, but the "basic ideas" and simplified models can profitably be demonstrated in highschool, and made plausible in highschool level mathematical terms. Also success stories like the formula for the *Google* patent for *PageRank* (Page, 2001), see Langville and Meyer (2006), the race for the solution of larger and larger instances of the Travelling Salesman Problem (Cook, 2011), or the mathematics of chip design lend themselves to "telling the story" and "showing some of the maths" at a highschool level; these are among the topics presented in the first author's recent book (Ziegler, 2013b), where he takes 24 images as the starting points for telling stories—and thus developing a broader multi-facetted picture of mathematics.

Another way to bring maths in contact with non-mathematicians is the human level. Telling stories about how maths is done and by whom is a tricky way, as can be seen from the sometimes harsh reactions on www.mathoverflow.net to postings that try to excavate the truth behind anecdotes and legends. Most mathematicians see mathematics as completely independent from the persons who explored it. History of mathematics has the tendency to become *gossip*, as Gian-Carlo Rota once put it (Rota, 1996). The idea seems to be: As mathematics stands for itself, it has also to be taught that way.

This may be true for higher mathematics. However, for pupils (and therefore, also for teachers), transforming mathematicians into humans can make science more tangible, it can make research interesting as a process (and a job?), and it can be a starting/entry point for real mathematics. Therefore, stories can make mathematics more sticky. Stories cannot replace the classical approaches to teaching mathematics. But they can enhance it.

Stories are the way by which knowledge has been transferred between humans for thousands of years. (Even mathematical work can be seen as a very abstract form of storytelling from a structuralist point of view.) Why don't we try to tell more stories about mathematics, both at university and in school—not legends, not fairy tales, but meta-information on mathematics—in order to transport mathematics itself? See (Ziegler, 2013a) for an attempt by the first author in this direction.

By stories, we do not only mean something like biographies, but also the way of how mathematics is created or discovered: Jack Edmonds' account (Edmonds, 1991) of how he found the blossom shrink algorithm is a great story about how mathematics is actually *done*. Think of Thomas Harriot's problem about stacking cannon balls into a storage space and what Kepler made out of it: the genesis of a mathematical problem. Sometimes scientists even wrap their work into stories by their own: see e.g. Leslie Lamport's *Byzantine Generals* (Lamport, Shostak, & Pease, 1982).

Telling how research is done opens another issue. At school, mathematics is traditionally taught as a closed science. Even touching open questions from research is out of question, for many good and mainly pedagogical reasons. However, this fosters the image of a perfect science where all results are available and all problems are solved—which is of course completely wrong (and moreover also a source for a faulty image of mathematics among undergraduates).

Of course, working with open questions in school is a difficult task. None of the big open questions can be solved with an elementary mathematical toolbox; many of them are not even accessible as questions. So the big fear of discouraging pupils is well justified. On the other hand, why not explore mathematics by showing how questions often pop up on the way? Posing questions in and about mathematics could lead to interesting answers—in particular to the question of "What is Mathematics, Really?"

Three Times Mathematics at School?

So, what is mathematics? With school education in mind, the first author has argued in Ziegler (2012) that we are trying cover three aspects the same time, which one should consider separately and to a certain extent also teach separately:

- Mathematics I: A collection of basic tools, part of everyone's survival kit for modern-day life—this includes everything, but actually not much more than, what was covered by Adam Ries' "Rechenbüchlein" ["Little Book on Computing"] first published in 1522, nearly 500 years ago;
- Mathematics II: A field of knowledge with a long history, which is a part of our culture and an art, but also a very productive basis (indeed a production factor) for all modern key technologies. This is a "story-telling" subject.
- Mathematics III: An introduction to mathematics as a science—an important, highly developed, active, huge research field.

Looking at current highschool instruction, there is still a huge emphasis on Mathematics I, with a rather mechanical instruction on arithmetic, "how to compute correctly," and basic problem solving, plus a rather formal way of teaching Mathematics III as a preparation for possible university studies in mathematics, sciences or engineering. Mathematics II, which should provide a major component of teaching "What is Mathematics," is largely missing. However, this part also could and must provide motivation for studying Mathematics I or III!

What Is Mathematics, Really?

There are many, and many different, valid answers to the Courant-Robbins question "What is Mathematics?"

A more philosophical one is given by Reuben Hersh's book "What is Mathematics, Really?" Hersh (1997), and there are more psychological ones, on the working level. Classics include Jacques Hadamard's "Essay on the Psychology of Invention in the Mathematical Field" and Henri Poincaré's essays on methodology; a more recent approach is Devlin's "Introduction to Mathematical Thinking" Devlin (2012), or Villani's book (2012).

And there have been many attempts to describe mathematics in encyclopedic form over the last few centuries. Probably the most recent one is the gargantuan "Princeton Companion to Mathematics", edited by Gowers et al. (2008), which indeed is a "Princeton Companion to Pure Mathematics."

However, at a time where *ZBMath* counts more than 100,000 papers and books per year, and 29,953 submissions to the `math` and `math-ph` sections of `arXiv.org` in 2016, it is hopeless to give a compact and simple description of what mathematics really is, even if we had only the "current research discipline" in mind. The discussions about the classification of mathematics show how difficult it is to cut the science into slices, and it is even debatable whether there is any meaningful way to separate applied research from pure mathematics.

Probably the most diplomatic way is to acknowledge that there are "many mathematics." Some years ago Tao (2007) gave an open list of mathematics that is/are good for different purposes—from "problem-solving mathematics" and "useful mathematics" to "definitive mathematics", and wrote:

> As the above list demonstrates, the concept of mathematical quality is a high-dimensional one, and lacks an obvious canonical total ordering. I believe this is because mathematics is itself complex and high-dimensional, and evolves in unexpected and adaptive ways; each of the above qualities represents a different way in which we as a community improve our understanding and usage of the subject.

In this sense, many answers to "What is Mathematics?" probably show as much about the persons who give the answers as they manage to characterize the subject.

Acknowledgment The authors' work has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 247029, the DFG Research Center Matheon, and the the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics".

References

Blum, W. (2001). *Was ist was. Band 12: Mathematik*, Tessloff Verlag, Nürnberg. Revised version, with new cover, 2010.

Cook, W. (2011). *In pursuit of the traveling salesman: Mathematics at the limits of computation*. Princeton NJ: Princeton University Press.

Courant, R., & Robbins, H. (1941). *What is mathematics? an elementary approach to ideas and methods* (2nd ed.), Oxford: Oxford University Press. Stewart, I (ed), 1996.

Csicsery, G. (2008). *Hard problems. the road to the world's toughest math contest*, Documentary film, 82 minutes (feature)/45 minutes (classroom version), Washington, DC: Mathematical Association of America.

Devlin, K. J. (2012). *Introduction to mathematical thinking*, published by Keith Devlin, Palo Alto CA.

Edmonds, J. (1991). A glimpse of heaven, In: J. K. Lenstra, A. Schrijver, & A. Rinnooy Kan (eds.) *History of mathematical programming—A collection of personal reminiscences* (pp. 32–54). Amsterdam: CWI and North-Holland.

Frenkel, E. (2013). *Love & math. The heart of hidden reality*. Philadelphia PA: Basic Books/Perseus Books.

Gowers, Timothy, Leader, Imre, & Barrow-Green, June (Eds.). (2008). *The princeton companion to mathematics*. Princeton NJ: Princeton University Press.

Highland, E. H., & Highland, H. J. (1961). *The how and why wonder book of mathematics*. New York: Grosset & Dunlop.

Highland, E. H., & Highland, H. J. (1963). *Was ist was. Band 12: Mathematik*, Neuer Tessloff Verlag, Hamburg, 1963. Revised edition 1969. New cover 1983.

Hersh, R. (1997). *What is mathematics, really?*. Oxford: Oxford University Press.

Klein, F.(1933). *Elementarmathematik vom höheren Standpunkte aus. Teil I: Arithmetik, Algebra, Analysis*, B. G. Teubner, Leipzig, 1908. Vierte Auflage. Heidelberg: Springer.

Lamport, L., Shostak, R., & Pease, M. (1982). The byzantine generals problem. *ACM Transactions on Programming Languages and Systems, 4,* 382–401.

Langville, A. N., & Meyer, C. D. (2006). *Google's pagerank and beyond. The science of search engine rankings*. Princeton and Oxford: Princeton University Press.

Loos, A., & Ziegler, G. M. (2017). *Panorama der Mathematik*. Heidelberg: Springer Spectrum, to appear.

Mendick, H., Epstein, D., & Moreau, M.-P. (2008). *Mathematical images and identities: Education, entertainment, social justice*. London: Institute for Policy Studies in Education, London Metropolitan University.

Page, L. (2001) *Method for node ranking in a linked database*, United States Patent No. US 6,285,999 B1, (submitted: January 9, 1998), http://www.google.com/patents/US6285999

Rota, G.-C. (1996). *Indiscrete thoughts*. Basel: Birkhäuser.

Tao, T. (2007). What is good mathematics? *Bulletin of the American Mathematical Society, 44*(4), 623–634.

Villani, C. (2012). *Théorème vivant*. Paris: Bernard Grasset. (in French).

Ziegler, G. M. (2011). Three competitions. In D. Schleicher & M. Lackmann (Eds.), *Invitation to mathematics. From competition to research* (pp. 195–205). Berlin: Springer.

Ziegler, G. M. (2012). Mathematics school education provides answers—To which questions? *EMS Newsletter* (84), 8–11.

Ziegler, G. M.(2013a). *Do I count? stories from mathematics*, Boca Raton FL: CRC Press/Taylor & Francis. English translation of “Darf ich Zahlen? Geschichten aus der Mathematik”, Piper, München, 2010.
Ziegler, G. M. (2013b). *Mathematik—Das ist doch keine Kunst!*. München: Knaus.