Topic Study Group No. 20: Visualization in the Teaching and Learning of Mathematics

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The TSG 20 aimed to focus on issues in visualization in the teaching and learning of mathematics at all levels. The group welcomed studies that tackle wide range of issues including: • What is the role of visualization within and across mathematical knowledge disciplines? • Are there kinds, qualities, and/or hierarchies of visualization and visual skills? • How do learners from different cultural contexts and of varying levels of ability and disability employ visualization in learning mathematics? • Considering recent advances on embodied cognition in mathematics, what theoretical frameworks could link visual and haptic modalities in an effective manner? • What theories on visualization can take into account the specific cognitive nature of mathematical activity and thinking? • What methodological considerations must be accounted for in investigations that focus on visualization? How should tasks, instruments, and measures be designed that will enable investigators to assess changes in students' understanding and learning? • What aspects of mathematics teacher education programs will help teachers understand the affordances and challenges of using visualization as a learning tool in mathematics? • What visual-based tasks can foster creativity leading to meaningful mathematical knowledge? Further, are visual-driven students more creative than nonvisual and other types of learners? In particular, we note how recent and emerging

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technological tools and digital mathematics media enable learners to perform visual engagement and interaction.

The opening session chaired by Boon Liang Chua included two invited lectures. It commenced with the first plenary, *The explanatory value of mathematical visualisations: a philosophical and pragmatic approach*, by Joachim Frans from Germany. This presentation was about the explanatory value of mathematics visualization that lays the foundation for the use and importance of visualization. The next communication was *Means for learning about students' knowledge: automatic assessment of visual examples* presented by Michal Yerushalmy from Israel. This presentation dealt with generating visual examples as a tool for learning about student knowledge throughout feedback from automated formative assessment systems. The two presentations generated a rich discussion about the foundations and the innovations in the field.

The second meeting ran in two parallel meetings and included six presentations. In Session A, Amy Lin from Canada suggested us to look at spatial reasoning of young students in her presentation: *Go figure: can actions promote visual and spatial reasoning?* Lin concentrated on gestural interfaces such as touchscreens that provide a more hands-on experience for the student as a potential support for cognitive processes and mathematical thinking. The research questions addressed in this study concern with the types of gestures (iconic, deictic, metaphoric, rhythmic) emerging when children are solving spatial reasoning problems and the study follow differences between non-spontaneously produced actions through gestural interfaces and learning supported with spontaneous actions. Ulrike Dreher from Germany looks at the multiple representations aspects supported by technology in her presentation titled *Factors that influence representational choice: students' mathematical abilities, self-efficacy and preference* Dreher (with Leuders & Holzäpfel) enquire is there a relationship between students' mathematical skills (translating/working with representations), their self-efficacy beliefs and their preferences for individual representations. The intention was to identify and analyze the relationships between various factors, specifically, the preferences of learners for different representations; the factors of their mathematical and representational self-efficacy as well as their meta-representational competence. The artefact that Natthapoj Vincent Trakulphadetkrai from UK is studying are picture books. In his presentation *Enhancing children's visualisation of multiplication through their self-generated mathematics picture books* he described the extent to which having primary school students create their own mathematics picture books enhance their ability to visualize multiplicative word problems and number sentences. The finding shows that children in the intervention class significantly outperformed their peers in the comparison class in both accuracy and ability to visualize word problems and number sentences.

In session B three presenters made their communications. The first communication *Seeing: an intuitive and creative way to solve a problem* was presented by Teresa Pimentel (with Isabel Vale and Ana Barbosa) from Portugal. This presentation discussed the potentialities of visual solutions and their connections with creativity. The second one, *Visual Patterns: a creative path to generalization,* was

presented by Ana Barbosa and Isabel Vale. The presenters reported a study involving 80 pre-service teachers in Portugal solving pattern generalizing tasks. The presentation discussed the teachers' strategies and difficulties, and highlighted the power of visualization to reach a generalization. The last one, *The role of visualization in the mathematical working space of teachers; differentiation of reasoning*, was presented by Carolina Henriquez Rivas from Chile. Using a theoretical framework known as Mathematical Working Space that characterized the epistemological and cognitive aspects activated by teachers when they are engaged in reasoning, the presentation discussed the Chilean mathematics teachers' reasoning when solving a geometrical task which favors a particular way of visualization. The results suggested deepening the theoretical aspect by proposing different categories of tasks and identifying the types of visualization involved.

The first communication of the third meeting was: *Analyzing students' visual thinking in solving selected concepts of mathematical analysis involving the concept of infinity*, was presented by Jonatan Muzangwa (with Ugorjio Ogbonnaya & David Mogari) from South Africa. This presentation analyzed the visual thinking of undergraduates majoring in Mathematics when they solved problems involving the concept of infinity. The presenters claimed that visual thinking was not helpful to the undergraduates in the topic of mathematical analysis. The second one, *Reflex: an educational representation of complex functions*, was presented by Mikael Mayer (with Lucas Willems) from France. In this presentation, the presenters showed the importance of using the REFLEX technology when learning complex functions. With the REFLEX tools students were able to visualize functions through color graphic representations, and connect its graphic representations with mathematical expression. The last one, *The relationship between teacher lens and teacher noticing of students' strategies in figural patterns,* was presented by Rabih El Mouhayar from Lebanon. This presentation discussed how teachers analyze students' strategies when are solving figural growing patterns, focusing on the lens that the teachers used and what they notice in their students' productions. This session saw a lively discussion among the three presenters and all the participants.

The last meeting session had two live presentations and one video presentation from Turkey, Nazan Sezen Yuksel, who was unable to attend ICME due to unforeseen political circumstances in her country. The first presentation, *Eliciting visualisation with techno-modelling tasks*, was presented by Johanna Kotze (with Gerrie Jacobs and Erica Spangenberg) from South Africa. Their study examined the influence of techno-modelling tasks on the visualization of 80 engineering students. The presenter reported positive findings such as the abilities of the students to reverse known processes and to make inferences based on their visual intuition and reasoning. The second presentation, *Onto-semiotic analysis of visualization and diagrammatic reasoning tasks*, was presented by Belen Giacomone (with Goldino, Wilhelmi, Blaanco and Contreras) from Spain. In this presentation, a training framework designed to help mathematics teachers gain competence in epistemic and cognitive analysis of mathematics instruction processes was discussed. The role of visual and analytical languages in establishing mathematical objects was highlighted. In the final presentation via a pre-recorded video, Nazan Sezen Yuksel

(with Bulbul) from Turkey presented their paper, *Investigation of development on mental cutting ability by latent growth*. The presenter reported on their study involving over 70 students that examined their mental cutting ability (a component of spatial ability) in three different mathematical activities. Their data analysis showed an increment in the level of students' mental cutting ability.

The group aim was to offer an integrative view on research and practice in the field of visualization, and indeed, participants offered a wide range of approaches and proposals for further development of the field in terms of mathematical foundations, research methods and technology.