A Direct Proof of the Strong Hanani–Tutte Theorem on the Projective Plane

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Abstract. We reprove the strong Hanani–Tutte theorem on the projective plane. In contrast to the previous proof by Pelsmajer, Schaefer and Stasi, our method is constructive and does not rely on the characterization of forbidden minors, which gives hope to extend it to other surfaces. Moreover, our approach can be used to provide an efficient algorithm turning a Hanani–Tutte drawing on the projective plane into an embedding.

Keywords: Graph drawing · Graph embedding · Hanani–Tutte theorem · Projective plane · Topological graph theory

1 Introduction

A drawing of a graph on a surface is a *Hanani–Tutte drawing* (shortly an *HT-drawing*) if no two vertex-disjoint edges cross an odd number of times. We call vertex-disjoint edges *independent*.

Pelsmajer, Schaefer and Stasi [14] proved the following theorem via consideration of the forbidden minors for the projective plane.

Theorem 1 (Strong Hanani–Tutte for the projective plane, [14]). *A graph G can be embedded into the projective plane if and only if it admits an HT-drawing on the projective plane.*[1]

The project was partially supported by the Czech-French collaboration project EMBEDS (CZ: 7AMB15FR003, FR: 33936TF). É.C.V. was partially supported by the French ANR Blanc project ANR- 12-BS02-005 (RDAM). V.K. was partially supported by the project GAUK 926416. V.K. and M.T. were partially supported by the project GAČR 16-01602Y. P.P. was supported by the ERC Advanced grant no. 320924. Z.P. was partially supported by Israel Science Foundation grant ISF-768/12.

[1] Of course, the "only if" part is trivial.

Y. Hu and M. Nöllenburg (Eds.): GD 2016, LNCS 9801, pp. 454–467, 2016.
DOI: 10.1007/978-3-319-50106-2_35

Our main result is a constructive proof of Theorem 1. The need for a constructive proof is motivated by the strong Hanani–Tutte conjecture, which states that an analogous result is valid on an arbitrary (closed) surface. This conjecture is known to be valid only on the sphere (plane) and on the projective plane. The approach via forbidden minors is relatively simple on the projective plane; however, this approach does not seem applicable to other surfaces, because there is no reasonable characterization of forbidden minors for them. (Already for the torus or the Klein bottle, the exact list is not known.)

On the other hand, our approach reveals a number of difficulties that have to be overcome in order to obtain a constructive proof. If the conjecture is true, our approach may serve as a basis for its proof on a general surface. If the conjecture is not true, then our approach may perhaps help to reveal appropriate structure needed for a construction of a counterexample.

The Hanani–Tutte Theorem on the Plane and Related Results. Let us now briefly describe the history of the problem; for complete history and relevant results we refer to a nice survey by Schaefer [17]. Following the work of Hanani [2], Tutte [19] made a remarkable observation now known as the (strong) Hanani–Tutte theorem: a graph is planar if and only if it admits an HT-drawing in the plane. The theorem has also a parallel history in algebraic topology, where it follows from the ideas of van Kampen, Flores, Shapiro and Wu [11,20,21].

It is a natural question whether the strong Hanani–Tutte theorem can be extended to graphs on other surfaces; as we already said before, it has been confirmed only for the projective plane [14] so far. On general surfaces, only the weak version [1,16] of the theorem is known to be true: if a graph is drawn on a surface so that every pair of edges crosses an even number of times[2], then the graph can be embedded into the surface while preserving the cyclic order of the edges at all vertices. Note that in the strong version we require that only independent edges cross even number of times, while in the weak version this condition has to hold for all pairs of edges.

We remark that other variants of the Hanani–Tutte theorem generalizing the notion of embedding in the plane have also been considered. For instance, the strong Hanani–Tutte theorem was proved for partially embedded graphs [18] and both weak and strong Hanani–Tutte theorem were proved also for 2-clustered graphs [6].

The strong Hanani–Tutte theorem is important from the algorithmic point of view, since it implies the Trémaux crossing theorem, which is used to prove de Fraysseix-Rosenstiehl's planarity criterion [4]. This criterion has been used to justify the linear time planarity algorithms including the Hopcroft-Tarjan [8] and the Left-Right [3] algorithms. For more details we again refer to [17].

One of the reasons why the strong Hanani–Tutte theorem is so important is that it turns planarity question into a system of linear equations. For general surfaces, the question whether there exists a Hanani–Tutte drawing of G leads to a system of quadratic equations [11] over $\mathbb{Z}_2$. If the strong Hanani–Tutte theorem

[2] Including 0 times.

is true for the surface, any solution to the system then serves as a certificate that G is embeddable. Moreover, if the proof of the Hanani–Tutte theorem is constructive, it gives a recipe how to turn the solution into an actual embedding. Unfortunately, solving systems of quadratic equations is NP-complete.

For completeness we mention that for each surface there exists a polynomial time algorithm that decides whether a graph can be embedded into that surface [9,12]; however, the hidden constant depends exponentially on the genus.

The original proofs of the strong Hanani–Tutte theorem in the plane used Kuratowski's theorem [10], and therefore are non-constructive. In 2007, Pelsmajer, Schaefer and Štefankovič [15] published a constructive proof. They showed a sequence of moves that change an HT-drawing into an embedding.

A key step in their proof is their Theorem 2.1. We say that an edge is *even* if it crosses every other edge an even number of times (including the adjacent edges).

Theorem 2 (Theorem 2.1 of [15]). *If D is a drawing of a graph G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

Unfortunately, an analogous result is simply not true on other surfaces, as is shown in [16]. In particular, this is an obstacle for a constructive proof of Theorem 1. The key step of our approach is to provide a suitable replacement of Theorem 2 on the projective plane. This is provided by Theorem 7 in Sect. 3.

The version of the paper identical to the present one can be found on arXiv: http://arxiv.org/abs/1608.07855v1. We refer to the full version of this paper in the subsequent submission on arXiv, which contains many details missing in this extended abstract.

2 Hanani–Tutte Drawings

In this section, we consider Hanani–Tutte drawings of graphs on the sphere and on the projective plane. We use the standard notation from graph theory. Namely, if G is a graph, then $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. Given a vertex v or an edge e, by $G - v$ or $G - e$ we denote the graph obtained from G by removing v or e, respectively.

Regarding drawings of graphs, first, let us recall a few standard definitions considered on an arbitrary surface. We put the standard general position assumptions on the drawings. That is, we consider only drawings of graphs on a surface such that no edge contains a vertex in its interior and every pair of edges meets only in a finite number of points, where they *cross* transversally. However, we allow three or more edges meeting in a single point.[3]

Let D be a drawing of a graph G on a surface S. Given two distinct edges e and f of G by $\mathrm{cr}(e,f) = \mathrm{cr}_D(e,f)$ we denote the number of crossings between e

[3] We do not mind them because we study pairwise interactions of edges only.

and f in D modulo 2. We say that an edge e of G is *even* if $\mathrm{cr}(e, f) = 0$ for any $f \in E(G)$ distinct from e. We emphasize that we consider the crossing number as an element of $\mathbb{Z}_2$ and all computations throughout the paper involving it are done in $\mathbb{Z}_2$.

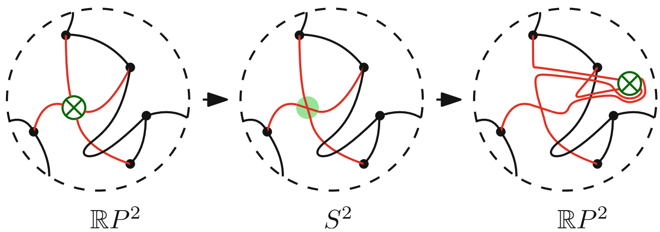
HT-Drawings on $\mathbb{R}P^2$. It is convenient for us to set up some conventions for working with the HT-drawings on the (real) projective plane, $\mathbb{R}P^2$. There are various ways to represent $\mathbb{R}P^2$. Our convention will be the following: we consider the sphere S^2 and a disk (2-ball) B in it. We remove the interior of B and identify the opposite points on the boundary ∂B. This way, we obtain a representation of $\mathbb{R}P^2$. Let γ be the curve coming from ∂B after the identification. We call this curve a *crosscap*. It is a homologically (homotopically) non-trivial simple cycle (loop) in $\mathbb{R}P^2$, and conversely, any homologically (homotopically) nontrivial simple cycle (loop) may serve as a crosscap up to a self-homeomorphism of $\mathbb{R}P^2$. In drawings, we use the symbol $\otimes$ for the crosscap coming from the removal of the disk 'inside' this symbol.

Given an HT-drawing of a graph on $\mathbb{R}P^2$, it can be slightly shifted so that it meets the crosscap in a finite number of points and only transversally, still keeping the property that we have an HT-drawing. Therefore, we may add to our conventions that this is the case for our HT-drawings on $\mathbb{R}P^2$.

Now, we consider a map $\lambda\colon E(G) \to \mathbb{Z}_2$. For an edge e, we let $\lambda(e)$ be the number of crossings of e and the crosscap γ modulo 2. We emphasize that λ depends on the choice of the crosscap.

Given a (graph-theoretic) cycle Z in G, we can distinguish whether Z is drawn as a homologically nontrivial cycle by checking the value $\lambda(Z) := \sum \lambda(e) \in \mathbb{Z}_2$ where the sum is over all edges of Z. The cycle Z is homologically nontrivial if and only if $\lambda(Z) = 1$. In particular, it follows that $\lambda(Z)$ does not depend on the choice of the crosscap.

Projective HT-Drawings on S^2. Let D be an HT-drawing of a graph G on $\mathbb{R}P^2$. It is not hard to deduce a drawing D' of the same graph on S^2 such that every pair (e, f) of *independent* edges satisfies $\mathrm{cr}(e, f) = \lambda(e)\lambda(f)$. Indeed, it is sufficient to 'undo' the crosscap, glue back the disk B and then let the edges intersect on B. See the two leftmost pictures below.



This motivates the following definition.

Definition 3. *Let D be a drawing of a graph G on S^2 and $\lambda\colon E(G) \to \mathbb{Z}_2$ be a function. Then the pair (D, λ) is a* projective HT-drawing of G *on* S^2 *if* $\mathrm{cr}(e, f) = \lambda(e)\lambda(f)$ *for any pair of independent edges e and f of G.*

It turns out that a projective HT-drawing on S^2 can also be transformed to an HT-drawing on $\mathbb{R}P^2$.

Proposition 4. *A graph G admits a projective HT-drawing on S^2 (with respect to some function $\lambda\colon E(G) \to \mathbb{Z}_2$) if and only if it admits an HT-drawing on $\mathbb{R}P^2$.*

The full proof of the missing implication is not too difficult and it is given in the full version of the paper (see Corollary 5). The core of the proof can be deduced from the two rightmost pictures above.

The main strength of Proposition 4 relies in the fact that in projective HT-drawings on S^2 we can ignore the actual geometric position of the crosscap and work in S^2 instead, which is simpler. This is especially helpful when we need to merge two drawings.

In order to distinguish the usual HT-drawings on S^2 from the projective HT-drawings, we will sometimes refer to the former as to the *ordinary* HT-drawings on S^2.

Nontrivial Walks. Let (D, λ) be a projective HT-drawing of a graph G and ω be a walk in G. We define $\lambda(\omega) := \sum_{e \in E(\omega)} \lambda(e)$ where $E(\omega)$ is the multiset of edges appearing in ω. Equivalently, it is sufficient to consider only the edges appearing an odd number of times in ω, because $2\lambda(e) = 0$ for any edge e. We say that ω is *trivial* if $\lambda(\omega) = 0$ and *nontrivial* otherwise. We often use this terminology in special cases when ω is an edge, a path, or a cycle.

Now let us consider a subgraph P of G such that every cycle in P is trivial. Then P essentially behaves as a planar subgraph of G, which we make more precise by the following lemma. For its proof, see Lemma 8 in the full version of the paper.

Lemma 5. *Let (D, λ) be a projective HT-drawing of G on S^2 and let P be a subgraph of G such that every cycle in P is trivial. Then there is a projective HT-drawing (D', λ') of G on S^2 such that $\lambda'(e) = 0$ for any edge e of $E(P)$.*

3 Separation Theorem

In this section, we state the replacement of Theorem 2 announced in the introduction. First we introduce some terminology; as we see from the definition below, a simple cycle Z such that every edge of Z is even splits G into two parts. This fact is analogous to the crucial step in the proof of Theorem 2.

Definition 6. *Let G be a graph and D be a drawing of G on S^2. Let us assume that Z is a cycle of G such that every edge of Z is even and it is drawn as a simple cycle in D. Let S^+ and S^- be the two components of $S^2 \setminus D(Z)$. We call a vertex $v \in V(G) \setminus V(Z)$ an* inside *vertex if it belongs to S^+ and an* outside *vertex otherwise. Given an edge* $e = uv \in E(G) \setminus E(Z)$*, we say that* e *is an* inside *edge if either* u *is an inside vertex or if* $u \in V(Z)$ *and* $D(e)$ *points locally to* S^+ *next to* $D(u)$*. Analogously we define an* outside *edge.*[4] *We let* V^+ *and* E^+ *be the sets of the inside vertices and the inside edges, respectively. Analogously, we define* V^- *and* E^-*. We also define the graphs* $G^{+0} := (V^+ \cup V(Z), E^+ \cup E(Z))$ *and* $G^{-0} := (V^- \cup V(Z), E^- \cup E(Z))$*.*

Now, we may formulate our main technical tool—the separation theorem for projective HT-drawings.

Theorem 7. *Let* (D, λ) *be a projective HT-drawing of a* 2*-connected graph* G *on* S^2 *and* Z *a cycle of* G *that is simple in* D *and such that every edge of* Z *is even. Moreover, we assume that every edge* e *of* Z *is trivial, that is,* $\lambda(e) = 0$*. Then there is a projective HT-drawing* (D', λ') *of* G *on* S^2 *satisfying the following properties.*

- *The drawings* D *and* D' *coincide on* Z*;*
- *the cycle* Z *is free of crossings and all of its edges are trivial in* D'*;*
- $D'(G^{+0})$ *is contained in* $S^+ \cup D'(Z)$*;*
- $D'(G^{-0})$ *is contained in* $S^- \cup D'(Z)$*; and*
- *either all edges of* G^{+0} *or all edges of* G^{-0} *are trivial (according to* λ'*); that is, at least one of the drawings* $D'(G^{+0})$ *or* $D'(G^{-0})$ *is an ordinary HT-drawing on* S^2*.*

In the remainder of this section, we describe the main ingredients of the proof of Theorem 7 and we also derive this theorem from the ingredients. We will often encounter the setting when G, (D, λ) and Z satisfy the assumptions of Theorem 7. Therefore, we say that G, (D, λ) and Z satisfy the *separation assumptions* if (1) G is a 2-connected graph; (2) (D, λ) is a projective HT-drawing of G; (3) Z is a cycle in G drawn as a simple cycle in D; (4) every edge of Z is even in D and trivial.

Arrow Graph. From now on, let us fix G, (D, λ) and Z satisfying the separation assumptions. This also fixes the distinction between the outside and the inside.

Definition 8. *A* bridge B *of* G *(with respect to* Z*) is a subgraph of* G *that is either an edge not in* Z *but with both endpoints in* Z *(and its endpoints also belong to* B*), or a connected component of* $G - V(Z)$ *together with all edges (and their endpoints in* Z*) with one endpoint in that component and the other endpoint in* Z*.*[5]

We say that B *is an* inside bridge *if it is a subgraph of* G^{+0}*, and an* outside bridge *if it is a subgraph of* G^{-0} *(every bridge is thus either an inside bridge or an outside bridge).*

A walk ω *in* G *is a* proper walk *if no vertex in* ω *belongs to* $V(Z)$*, except possibly its endpoints, and no edge of* ω *belongs to* $E(Z)$*. In particular, each proper walk belongs to a single bridge.*

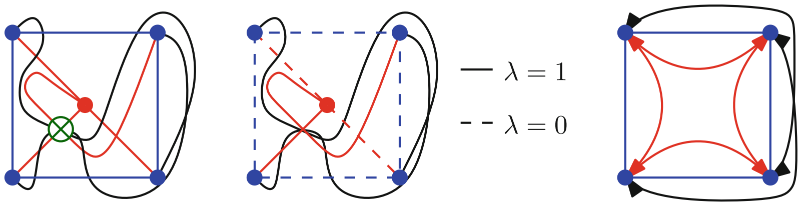
[4] It turns out that every edge $e \in E(G) \setminus E(Z)$ is either an outside edge or an inside edge, because every edge of Z is even.

[5] This is a standard definition; see, e.g., Mohar and Thomassen [13, p. 7].

Since we assume that G is 2-connected, every inside bridge contains at least two vertices of Z. The bridges induce partitions of $E(G) \setminus E(Z)$ and of $V(G) \setminus V(Z)$.

We want to record which pairs of vertices on $V(Z)$ are connected with a nontrivial and proper walk inside or outside. For this purpose, we create two new graphs A^+ and A^-, possibly with loops but without multiple edges. In order to distinguish these graphs from G, we draw their edges with double arrows and we call these graphs an *inside arrow graph* and an *outside arrow graph*, respectively. The edges of these graphs are called the *inside/outside arrows*. We set $V(A^+) = V(A^-) = V(Z)$.

Now we describe the *arrows*, that is, $E(A^+)$ and $E(A^-)$. Let u and v be two vertices of $V(Z)$, not necessarily distinct. By W^+_{uv} we denote the set of all proper nontrivial walks in G^{+0} with endpoints u and v. We have an *inside arrow* connecting u and v in $E(A^+)$ if and only if W^+_{uv} is nonempty. In order to distinguish the edges of G from the arrows, we denote an arrow by $\overline{uv} = \overline{vu}$. An arrow which is a loop at a vertex v is denoted by $\overline{vv}$. (This convention will allow us to work with arrows $\overline{uv}$ without a distinction whether $u = v$ or $u \neq v$.) Analogously, we define the set W^-_{uv} and the *outside arrows*. Below, we provide an example of an unusual HT-drawing of K_5 on $\mathbb{R}P^2$, the corresponding projective HT-drawing on S^2 and the corresponding arrow graphs.



Given an inside arrow $\overline{uv}$ and an inside bridge B, we say that B *induces* $\overline{uv}$ if there is a walk in B which belongs to W^+_{uv}. An inside bridge B is *nontrivial* if it induces at least one arrow. Given two inside arrows $\overline{uv}$ and $\overline{xy}$, we say that $\overline{uv}$ and $\overline{xy}$ *are induced by different bridges* if there are two different inside bridges B and B' such that B induces $\overline{uv}$ and B' induces $\overline{xy}$. As usual, we define analogous notions for the outside as well.

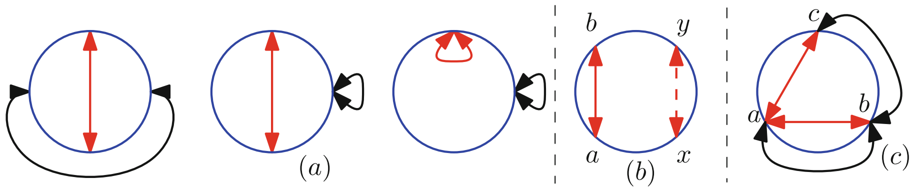
Possible Configurations of Arrows. Now, we utilize the arrow graph to show that certain configurations of arrows are not possible.

Lemma 9.

(a) Every inside arrow shares a vertex with every outside arrow.
(b) Let $\overline{ab}$ and $\overline{xy}$ be two arrows induced by different inside bridges of G^{+0}. If the two arrows do not share an endpoint, their endpoints have to interleave along Z.
(c) There are no three vertices a, b, c on Z, an inside bridge B^+, and an outside bridge B^- such that B^+ induces the arrows $\overline{ab}$ and $\overline{ac}$ (and no other arrows) and B^- induces the arrows $\overline{ab}$ and $\overline{bc}$ (and no other arrows).

For proof, see Lemmas 12, 13 and 14 in the full version of the paper.

Lemma 9 is, of course, also valid if we swap the inside and the outside. Schematically, the forbidden configurations from Lemma 9 are drawn in the picture below. The cyclic order in (a) may be arbitrary whereas it is important in (b) that the arrows there do not interleave. Different dashing of lines in (b) correspond to arrows induced by different inside bridges. The arrows of the same colour in (c) are induced by the same bridge.



Now we describe important configurations that may occur.

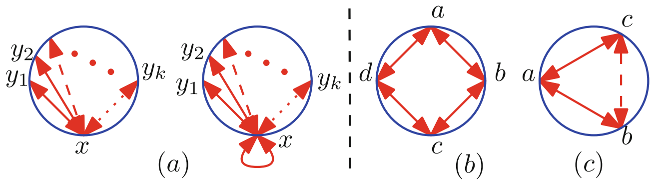
Definition 10. *We say that G forms*

(a) *an* inside fan *if there is a vertex common to all inside arrows. (The arrows may come from various inside bridges.)*
(b) *an* inside square *if it contains four vertices a, b, c and d ordered in this cyclic order along Z and the inside arrows are precisely $\overline{ab}$, $\overline{bc}$, $\overline{cd}$ and $\overline{ad}$. In addition, we require that the inside graph G^{+0} has only one nontrivial inside bridge.*
(c) *an* inside split triangle *if there exist three vertices a, b and c such that the arrows of G are $\overline{ab}$, $\overline{ac}$ and $\overline{bc}$. In addition, we require that every nontrivial inside bridge induces either the two arrows $\overline{ab}$ and $\overline{ac}$, or just a single arrow.*

We have analogous definitions for an outside fan*,* outside square *and* outside split triangle*.*

More precisely the notions in Definition 10 depend on G, (D, λ) and Z satisfying the separation assumptions.

The picture below shows schematic drawings of the configurations of arrows from Definition 10. Different dashing of lines correspond to different inside bridges. The loop in the right drawing in (a) is an inside loop (drawn outside due to lack of space). The drawing in (c) is only one instance of an inside split triangle.



A relatively direct case analysis, using Lemma 9, reveals the following fact.

Proposition 11. *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G satisfying the separation assumptions. Then G forms an (inside or outside) fan, square, or split triangle.*

For proof, see Proposition 16 in the full version of the paper. On the other hand, any configuration from Definition 10 can be redrawn without using the crosscap:

Proposition 12. *Let (D, λ) be a projective HT-drawing of G^{+0} on S^2 and Z be a cycle satisfying the separation assumptions. Moreover, let us assume that $D(G^{+0}) \cap S^- = \emptyset$ (that is, G^{+0} is fully drawn on $S^+ \cup D(Z)$). Let us also assume that G^{+0} forms an inside fan, an inside square or an inside split triangle. Then there is an ordinary HT-drawing D' of G^{+0} on S^2 such that D coincides with D' on Z and $D'(G^{+0}) \cap S^- = \emptyset$.*

For proof, see Proposition 17 in the full version of the paper.

Now we are missing only one tool to finish the proof of Theorem 7. This tool is the "redrawing procedure" of Pelsmajer, Schaefer and Štefankovič [15]. More concretely, we need the following variant of Theorem 2. (Note that the theorem below is not in the setting of projective HT-drawings. However, the notions used in the statement are still well defined according to Definition 6.)

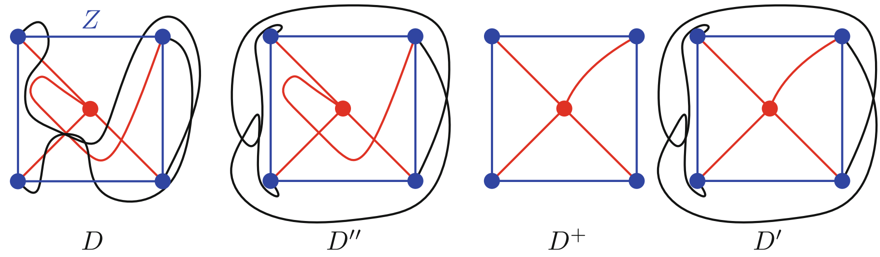
Theorem 13. *Let D be a drawing of a graph G on the sphere S^2. Let Z be a cycle in G such that every edge of Z is even and Z is drawn as a simple cycle. Then there is a drawing D'' of G such that*

- *D'' coincides with D on Z;*
- *$D''(G^{+0})$ belongs to $S^+ \cup D(Z)$ and $D''(G^{-0})$ belongs to $S^- \cup D(Z)$;*
- *whenever (e, f) is a pair of edges such that both e and f are inside edges or both e and f are outside edges, then $\mathrm{cr}_{D''}(e, f) = \mathrm{cr}_D(e, f)$.*

It is easy to check that the proof of Theorem 2 in [15] proves Theorem 13 as well. Additionally, we note that an alternative proof of Theorem 2 in [7, Lemma 3] can also be extended to yield Theorem 13. For completeness, we provide its proof in Sect. 8 of the full version of the paper.

Finally, we prove Theorem 7, assuming the validity of the aforementioned auxiliary results.

Proof sketch (of Theorem 7). First, we use Theorem 13 to G and D to obtain a drawing D'' keeping in mind that all edges of Z are even. By Proposition 11, G forms one of the redrawable configurations from Definition 10 on one of the sides. Without loss of generality, it appears inside. It means that D'' restricted to G^{+0} satisfies the assumptions of Proposition 12. Therefore, there is an ordinary HT-drawing D^+ of G^{+0} satisfying the conclusions of Proposition 12. Finally, we let D' be the drawing of G on S^2 which coincides with D^+ on G^{+0} and with D'' on G^{-0}. Both D'' and D^+ coincide with D on Z; therefore, D' is well defined. We set λ' so that $\lambda'(e) := \lambda(e)$ for an edge $e \in E^-$ and $\lambda'(e) := 0$ for any other edge. Now, it is easy to verify that (D', λ') is the required projective HT-drawing. The picture below provides an example of the drawings in the proof. □



4 Proof of the Strong Hanani–Tutte Theorem on $\mathbb{R}\boldsymbol{P}^2$

In this section we sketch a proof of Theorem 1 from Theorem 7 and the auxiliary results from the previous section.

Given a graph G that admits an HT-drawing on the projective plane, we need to show that G is actually projective-planar. By Proposition 4, we may assume that G admits a projective HT-drawing (D, λ) on S^2. We head for using Theorem 7. For this, we need that G is 2-connected and contains a suitable trivial cycle Z that may be redrawn so that it satisfies the assumptions of Theorem 7. Therefore, we start with auxiliary claims that will bring us to this setting. Many of them are similar to auxiliary steps in [15] (sometimes they are almost identical, adapted to a new setting). The proofs are at the beginning of Sect. 4 of the full version of the paper.

Before we state the next lemma, we recall the well known fact that any graph admits a (unique) decomposition into *blocks of 2-connectivity* [5, Chap. 3]. Here, we also allow the case that G is disconnected.

Lemma 14. *If G admits a projective HT-drawing on S^2, then at most one block of 2-connectivity in G is non-planar. Moreover, if all blocks are planar, G is planar as well.*

Observation 15. *Let (D, λ) be a drawing of a 2-connected graph. If D does not contain any trivial cycle, then G is planar.*

Lemma 16. *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G. Then G can be redrawn only by local changes next to the vertices of Z to a projective HT-drawing D' on S^2 so that λ remains unchanged and $\mathrm{cr}_{D'}(e, f) = \lambda(e)\lambda(f)$, for any pair $(e, f) \in E(Z) \times E(G)$ of distinct (not necessarily independent) edges. In particular, if $\lambda(e) = 0$ for every edge e of Z, then every edge of Z becomes even in D'.*

Once we know that the edges of a cycle can be made even we also need to know that such a cycle can be made simple.

Lemma 17. *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G such that each of its edges is even. Then G can be redrawn so that Z becomes a simple cycle, its edges remain even and the resulting drawing is still a projective HT-drawing (with λ unchanged).*

Proposition 18 below is our main tool for deriving Theorem 1 from Theorem 7. It is set up in such a way that it can be inductively proved from Theorem 7. Then it implies Theorem 1, using the auxiliary lemmas from the beginning of this section, relatively easily.

Proposition 18. *Let* (D, λ) *be a projective HT-drawing of a* 2*-connected graph* G *on* S^2 *and* Z *a cycle in* G *that is completely free of crossings in* D *and such that each of its edges is trivial in* D*. Assume that* (V^+, E^+) *or* (V^-, E^-) *is empty. Then* G *can be embedded into* $\mathbb{R}P^2$ *so that* Z *bounds a disk face of the resulting embedding. If, in addition,* D *is an ordinary HT-drawing on* S^2*, then* G *can be embedded into* S^2 *so that* Z *bounds a face of the resulting embedding.*[6]

First we prove Theorem 1 assuming the validity of Proposition 18. Then, we sketch a proof of the proposition. See Proposition 24 in the full version of the paper for the complete proof.

Proof (of Theorem 1). We prove the result by induction on the number of vertices of G. We can trivially assume that G has at least three vertices.

If G has at least two blocks of 2-connectivity, G can be written as $G_1 \cup G_2$, where $G_1 \cap G_2$ is a minimal cut of G, and therefore, has at most one vertex. By Lemma 14, we may assume that G_1 is planar and G_2 non-planar. By induction, there exists an embedding D_2 of G_2 into $\mathbb{R}P^2$. So G_1 is planar, G_2 is embeddable in $\mathbb{R}P^2$, and $G_1 \cap G_2$ has at most one vertex. From these two embeddings, we easily derive an embedding of $G = G_1 \cup G_2$ in $\mathbb{R}P^2$.

We are left with the case when G is 2-connected. By Observation 15, we may assume that there is at least one trivial cycle Z in (D, λ). We can also make each of its edges trivial by Lemma 5 and even by Lemma 16. In addition, we make Z simple using Lemma 17. Hence G, Z and the current projective HT-drawing satisfy the separation assumptions.

Then we use Z to redraw G as follows. At first, we apply Theorem 7 to get a projective HT-drawing (D', λ') that separates G^{+0} and G^{-0}. We define $D^+ := D'(G^{+0})$ and $D^- := D'(G^{-0})$—without loss of generality, D^- is an ordinary HT-drawing on S^2, while D^+ is a projective HT-drawing on S^2.

Next, we apply Proposition 18 above to D^+ and D^- separately. Thus, we get embeddings of G^{+0} and G^{-0}—one of them in S^2, the other one in $\mathbb{R}P^2$. In addition, Z bounds a face in both of them; hence, we can easily glue them to get an embedding of the whole graph G into $\mathbb{R}P^2$. □

Proof sketch (of Proposition 18). The proof proceeds by induction on the number of edges of G. The base case is when G is a cycle.

Without loss of generality, we assume that (V^-, E^-) is empty. That is, $G = G^{+0}$. If (V^+, E^+) is also empty, G consists only of Z and such a graph can easily be embedded into the plane or projective plane as required. Therefore, we assume that (V^+, E^+) is nonempty.

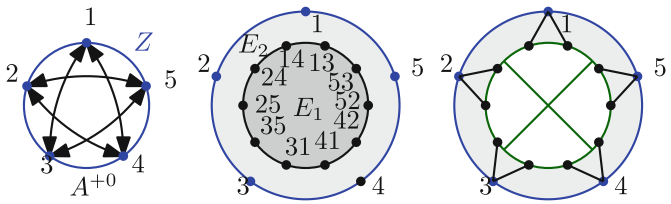
[6] We need to consider the case of ordinary HT-drawings in this proposition for a well working induction.

We find a path γ in $(V(G^{+0}), E(G^{+0}) \setminus E(Z))$ connecting two points x and y lying on Z. We may choose x, y so that $x \neq y$ since G is 2-connected.

Case 1: There exists a trivial γ. We provide only a very brief sketch for this case. First, we achieve, without redrawing Z, that γ is drawn as a simple path and every edge of γ is even and trivial. This can be done by steps similar to those in the proof of Theorem 1. Since γ is inside Z, now it splits the interior of Z into two disks. Once we carefully identify the two arcs of Z determined by the endpoints of γ, we get a cycle (in a different graph) separating the two disks. This way, we achieve essentially the same situation as in the proof of Theorem 1 and we can resolve it using Theorem 7.

Case 2: All choices of γ *are nontrivial.* Now, we need to resolve the case when all possible choices of γ are nontrivial. Let A^{+0} be the graph obtained from the inside arrow graph A^+ by adding the edges of Z (in particular, Z is a subgraph of A^{+0}). We aim to show that A^{+0} admits an embedding in $\mathbb{R}P^2$ such that Z bounds a disk face. As soon as we show this, we aim to replace the embedding of each arrow of A^{+0} by an embedding of the inside bridges inducing this arrow (if there are more such bridges, we embed them in parallel). The key fact that makes it possible is that each inside bridge meets Z in exactly two points and induces a single arrow and no loop. (Here, we leave this fact without a proof.) We also need to check that each of the bridges, together with Z, admits an embedding. This follows from Proposition 12 for inside fans and from Case 1 of this proof.

It remains to sketch why A^{+0} admits the required embedding. We know that any two disjoint arrows interleave using Lemma 9(b). Let us consider two concentric closed disks E_1 and E_2 such that E_1 belongs to the interior of E_2. Let us draw Z to the boundary of E_2. Let a be the number of arrows of A^+ and let us consider $2a$ points on the boundary of E_1 forming the vertices of a regular $2a$-gon. These points will be marked by ordered pairs xy where $\overline{xy}$ is an inside arrow. We mark the points so that the cyclic order of the points respects the cyclic order on Z in the first coordinate (the pairs with the same first coordinate are consecutive). However, for a fixed x, the pairs $xy_1, \ldots, xy_k$ corresponding to all arrows emanating from x are ordered in the reverted order when compared with the order of $y_1, \ldots, y_k$ on Z.



It is not hard to check that the points marked xy and yx are precisely the opposite points. Now, we get the required drawing in the following way. For any arrow $\overline{xy}$ we connect x with the point marked xy and y with yx. We can do all the connections simultaneously for all arrows without introducing any crossing since we have respected the cyclic order in the first coordinate. We remove the interior of E_1 and identify the opposite points on its boundary. This way we introduce a crosscap. Finally, we glue another disk along its boundary to Z and we get the required drawing on $\mathbb{R}P^2$. □

Acknowledgment. We would like to thank Alfredo Hubard for fruitful discussions and valuable comments.

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