On the Density of Non-simple 3-Planar Graphs

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Abstract. A *k-planar graph* is a graph that can be drawn in the plane such that every edge is crossed at most k times. For $k \leq 4$, Pach and Tóth [20] proved a bound of $(k+3)(n-2)$ on the total number of edges of a k-planar graph, which is tight for $k = 1, 2$. For $k = 3$, the bound of $6n-12$ has been improved to $\frac{11}{2}n - 11$ in [19] and has been shown to be optimal up to an additive constant for simple graphs. In this paper, we prove that the bound of $\frac{11}{2}n-11$ edges also holds for non-simple 3-planar graphs that admit drawings in which non-homotopic parallel edges and self-loops are allowed. Based on this result, a characterization of *optimal 3-planar graphs* (that is, 3-planar graphs with n vertices and exactly $\frac{11}{2}n - 11$ edges) might be possible, as to the best of our knowledge the densest known simple 3-planar is not known to be optimal.

1 Introduction

Planar graphs play an important role in graph drawing and visualization, as the avoidance of crossings and occlusions is central objective in almost all applications [10,18]. The theory of planar graphs [15] could be very nicely applied and used for developing great layout algorithms [13,22,23] based on the planarity concepts. Unfortunately, real-world graphs are usually not planar despite of their sparsity. With this background, an initiative has formed in recent years to develop a suitable theory for *nearly planar graphs*, that is, graphs with various restrictions on their crossings, such as limitations on the number of crossings per edge (e.g., k-planar graphs [21]), avoidance of local crossing configurations (e.g., quasi planar graphs [2], fan-crossing free graphs [9], fan-planar graphs [17]) or restrictions on the crossing angles (e.g., RAC graphs [11], LAC graphs [12]). For precise definitions, we refer to the literature mentioned above.

The most prominent is clearly the concept of k-planar graphs, namely graphs that allow drawings in the plane such that each edge is crossed at most k times by other edges. The simplest case $k = 1$, i.e., 1-planar graphs [21], has been subject of intensive research in the past and it is quite well understood, see e.g. [4,6–8,14,20]. For $k \geq 2$, the picture is much less clear. Only few papers on special cases appeared, see e.g., [3,16].

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Pach and Tóth's paper [20] stands out and contributed a lot to the understanding of nearly planar graphs. The paper considers the number of edges in simple k-planar graphs for general k. Note the well-known bound of $3n-6$ edges for planar graphs deducible from Euler's formula. For small $k = 1, 2, 3$ and 4, bounds of $4n - 8$, $5n - 10$, $6n - 12$ and $7n - 14$ respectively, are proven which are tight for $k = 1$ and $k = 2$. This sequence seems to suggest a bound of $O(kn)$ for general k, but Pach and Tóth also gave an upper bound of $4.1208\sqrt{k}n$. Unfortunately, this bound is still quite large even for medium k (for $k = 9$, it gives $12.36n$). Meanwhile for $k = 3$ and $k = 4$, the bounds above have been improved to $5.5n-11$ and $6n-12$ in [19] and [1], respectively. In this paper, we prove that the bound on the number of edges for $k = 3$ also holds for non-simple 3-planar graphs that do not contain homotopic parallel edges and homotopic self-loops. Our extension required substantially different approaches and relies more on geometric techniques than the more combinatorial ones given in [19] and [1]. We believe that it might also be central for the characterization of *optimal* 3-planar graphs (that is, 3-planar graphs with n vertices and exactly $\frac{11}{2}n - 11$ edges), since the densest known simple 3-planar graph has only $\frac{11n}{2} - 15$ edges and does not reach the known bound.

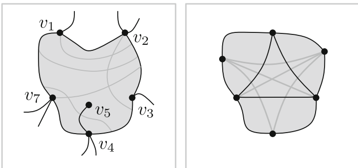
The remaining of this paper is structured as follows: Some definitions and preliminaries are given in Sect. 2. In Sects. 3 and 4, we give significant insights in structural properties of 3-planar graphs in order to prove that 3-planar graphs on n vertices cannot have more than $\frac{11}{2}n-11$ edges. We conclude in Sect. 5 with open problems.

2 Preliminaries

A *drawing* of a graph G is a representation of G in the plane, where the vertices of G are represented by distinct points and its edges by Jordan curves joining the corresponding pairs of points, so that: (i) no edge passes through a vertex different from its endpoints, (ii) no edge crosses itself and (iii) no two edges meet tangentially. In the case where G has multi-edges, we will further assume that both the bounded and the unbounded closed regions defined by any pair of self-loops or parallel edges of G contain at least one vertex of G in their interior. Hence, the drawing of G has no *homotopic* edges. In the following when referring to 3-planar graphs we will mean that non-homotopic edges are allowed in the corresponding drawings. We call such graphs *non-simple*.

Following standard naming conventions, we refer to a 3-planar graph with n vertices and maximum possible number of edges as *optimal* 3*-planar*. Let H be an optimal 3-planar graph on n vertices together with a corresponding 3-planar drawing $\Gamma(H)$. Let also H_p be a subgraph of H with the largest number of edges, such that in the drawing of H_p (that is inherited from $\Gamma(H)$) no two edges cross each other. We call H_p a *maximal planar substructure* of H. Among all possible optimal 3-planar graphs on n vertices, let $G = (V, E)$ be the one with the following two properties: (a) its maximal planar substructure, say $G_p = (V, E_p)$, has maximum number of edges among all possible planar substructures of all optimal 3-planar graphs, (b) the number of crossings in the drawing of G is minimized over all optimal 3-planar graphs subject to (a). We refer to G as *crossing-minimal optimal* 3*-planar graph*.

With slight abuse of notation, let $G - G_p$ be obtained from G by removing only the edges of G_p and let e be an edge of $G - G_p$. Since G_p is maximal, edge e must cross at least one edge of G_p. We refer to the part of e between an endpoint of e and the nearest crossing with an edge of G_p as *stick*. The parts of e between two consecutive crossings with G_p are called *middle parts*. Clearly, e consists of exactly 2 sticks and 0, 1, or 2 middle parts. A stick of e lies completely in a face of G_p and crosses at most two other edges of $G - G_p$ and an edge of this particular face. A stick of e is called *short*, if there is a walk along the face boundary from the endpoint of the stick to the nearest crossing point with G_p, which contains only one other vertex of the face boundary. Otherwise, the stick of e is called *long*; see Fig. 1a. A middle part of e also lies in a face of G_p. We say that e *passes through* a face of G_p, if there exists a middle part of e that completely lies in the interior of this particular face. We refer to a middle part of an edge that crosses consecutive edges of a face of G_p as *short middle part*. Otherwise, we call it *far middle part*.



(a) **(b)**

Fig. 1. (a) Illustration of a non-simple face $\{v_1, v_2, \ldots, v_7\}$; v_6 is identified with v_4. The sticks from v_1 and v_2 are short, while the one from v_7 is long. All other edge segments are middle-parts. (b) The case, where two triangles of type $(3, 0, 0)$ are associated to the same triangle.

Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$ be a face of G_p with $s \geq 3$. The order of the vertices (and subsequently the order of the edges) of $\mathcal{F}_s$ is determined by a walk around the boundary of $\mathcal{F}_s$ in clockwise direction. Since $\mathcal{F}_s$ is not necessarily simple, a vertex (or an edge, respectively) may appear more than once in this order; see Fig. 1a. We say that $\mathcal{F}_s$ is of type $(\tau_1, \tau_2, \ldots, \tau_s)$ if for each $i = 1, 2, \ldots, s$ vertex v_i is incident to τ_i sticks of $\mathcal{F}_s$ that lie between (v_{i-1}, v_i) and (v_i, v_{i+1})[1].

Lemma 1 (Pach and Tóth [20]). *A triangular face of G_p contains at most* 3 *sticks.*

Proof. Consider a triangular face $\mathcal{T}$ of G_p of type (τ_1, τ_2, τ_3). Clearly, $\tau_1, \tau_2, \tau_3 \leq 3$, as otherwise an edge of G_p has more than three crossings. Since a stick of $\mathcal{T}$ cannot cross more than two other sticks of $\mathcal{T}$, it follows that $\tau_1 + \tau_2 + \tau_3 \leq 3$. □

[1] In the remainder of the paper, all indices are subject to $(mod\ s) + 1$.

3 The Density of Non-simple 3-Planar Graphs

Let $G = (V, E)$ be a crossing-minimal optimal 3-planar graph with n vertices drawn in the plane. Let also $G_p = (V, E_p)$ be the maximal planar substructure of G. In this section, we will prove that G cannot have more than $\frac{11n}{2} - 11$ edges, assuming that G_p is fully triangulated, i.e., $|E_p| = 3n - 6$. This assumption will be proved in Sect. 4. Next, we prove that the number of triangular faces of G_p with exactly 3 sticks cannot be larger than those with at most 2 sticks.

Lemma 2. *We can uniquely associate each triangular face of G_p with 3 sticks to a neighboring triangular face of G_p with at most 2 sticks.*

Proof. Let $\mathcal{T} = \{v_1, v_2, v_3\}$ be a triangular face of G_p. By Lemma 1, we have to consider three types for $\mathcal{T}$: $(3, 0, 0)$, $(2, 1, 0)$ and $(1, 1, 1)$.

- $\mathcal{T}$ *is of type* $(3, 0, 0)$: Since v_1 is incident to 3 sticks of $\mathcal{T}$, edge (v_2, v_3) is crossed three times. Let $\mathcal{T}'$ be the triangular face of G_p neighboring $\mathcal{T}$ along (v_2, v_3). We have to consider two cases: (a) one of the sticks of $\mathcal{T}$ ends at a corner of $\mathcal{T}'$, and (b) none of the sticks of $\mathcal{T}$ ends at a corner of $\mathcal{T}'$. In Case (a), the two remaining sticks of $\mathcal{T}$ might use the same or different sides of $\mathcal{T}'$ to exit it. In both subcases, it is not difficult to see that $\mathcal{T}'$ can have at most two sticks. In Case (b), we again have to consider two subcases, depending on whether all sticks of $\mathcal{T}$ use the same side of $\mathcal{T}'$ to pass through it or two different ones. In the former case, it is not difficult to see that $\mathcal{T}'$ cannot have any stick, while in the later $\mathcal{T}'$ can have at most one stick. In all aforementioned cases, we associate $\mathcal{T}$ with $\mathcal{T}'$.
- $\mathcal{T}$ *is of type* $(2, 1, 0)$: Since v_2 is incident to one stick of $\mathcal{T}$, edge (v_1, v_3) is crossed at least once. We associate $\mathcal{T}$ with the triangular face $\mathcal{T}'$ of G_p neighboring $\mathcal{T}$ along (v_1, v_3). Since the stick of $\mathcal{T}$ that is incident to v_2 has three crossings in $\mathcal{T}$, $\mathcal{T}'$ has no sticks emanating from v_1 or v_3. In particular, $\mathcal{T}'$ can have at most one additional stick emanating from its third vertex.
- $\mathcal{T}$ *is of type* $(1, 1, 1)$: This actually cannot occur. Indeed, if $\mathcal{T}$ is of type $(1, 1, 1)$, then all sticks of $\mathcal{T}$ have already three crossings each. Hence, the three triangular faces adjacent to $\mathcal{T}$ define a 6-gon in G_p, which contains only six interior edges. So, we can easily remove them and replace them with 8 interior edges (see, e.g., Fig. 1b), contradicting thus the optimality of G.

Note that our analysis also holds for non-simple triangular faces. We now show that the assignment is unique. This holds for triangular faces of type $(2, 1, 0)$, since a triangular face that is associated with one of type $(2, 1, 0)$ cannot contain two sides each with two crossings, which implies that it cannot be associated with another triangular face with three sticks. This leaves only the case that two $(3, 0, 0)$ triangles are associated with the same triangle $\mathcal{T}'$ (see, e.g., the triangle with the gray-colored edges in Fig. 1b). In this case, there exists another triangular face (bottommost in Fig. 1b), which has exactly two sticks because of 3-planarity. In addition, this face cannot be associated with some other triangular face. Hence, one of the two type-$(3, 0, 0)$ triangular faces associated with $\mathcal{T}'$ can be assigned to this triangular face instead resolving the conflict. □

We are now ready to prove the main theorem of this section.

Theorem 1. *A 3-planar graph of n vertices has at most $\frac{11}{2}n - 11$ edges, which is a tight bound.*

Proof. Let t_i be the number of triangular faces of G_p with exactly i sticks, $0 \leq i \leq 3$. The argument starts by counting the number of triangular faces of G_p with exactly 3 sticks. From Lemma 2, we conclude that the number t_3 of triangular faces of G_p with exactly 3 sticks is at most as large as the number of triangular faces of G_p with 0, 1 or 2 sticks. Hence $t_3 \leq t_0 + t_1 + t_2$. We conclude that $t_3 \leq t_p/2$, where t_p denotes the number of triangular faces in G_p, since $t_0 + t_1 + t_2 + t_3 = t_p$. Note that by Euler's formula $t_p = 2n - 4$. Hence, $t_3 \leq n - 2$. Thus, we have: $|E| - |E_p| = (t_1 + 2t_2 + 3t_3)/2 = (t_1 + t_2 + t_3) + (t_3 - t_1)/2 = (t_p - t_0) + (t_3 - t_1)/2 \leq t_p + t_3/2 \leq 5t_p/4$. So, the total number of edges of G is at most: $|E| \leq |E_p| + 5t_p/4 \leq 3n - 6 + 5(2n - 4)/4 = 11n/2 - 11$. In [5] we prove that our bound is tight by a construction similar to the one of Pach et al. [19]. □

4 The Density of the Planar Substructure

Let $G = (V, E)$ be a crossing-minimal optimal 3-planar graph with n vertices drawn in the plane. Let also $G_p = (V, E_p)$ be the maximal planar substructure of G. In this section, we will prove that G_p is fully triangulated, i.e., $|E_p| = 3n - 6$ (see Theorem 2). To do so, we will explore several structural properties of G_p (see Lemmas 3–13), assuming that G_p has at least one non-triangular face, say $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$ with $s \geq 4$. In the first observations, we do not require that G_p is connected. This is proved in Lemma 6. Recall that in general $\mathcal{F}_s$ is not necessarily simple, which means that a vertex may appear more than once along $\mathcal{F}_s$. Our goal is to contradict either the *optimality* of G (that is, the fact that G contains the maximum number of edges among all 3-planar graphs with n vertices) or the *maximality* of G_p (that is, the fact that G_p has the maximum number of edges among all planar substructures of all optimal 3-planar graphs with n vertices) or the *crossing minimality* of G (that is, the fact that G has the minimum number of crossings subject to the size of the planar substructure).

Lemma 3. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, each stick of $\mathcal{F}_s$ is crossed at least once within $\mathcal{F}_s$.*

Proof (Sketch). Assume to the contrary that there exists a stick of $\mathcal{F}_s$ that is not crossed within $\mathcal{F}_s$. W.l.o.g. let (v_1, v_1') be the edge containing this stick and assume that (v_1, v_1') emanates from vertex v_1 and leads to vertex v_1' by crossing the edge (v_i, v_{i+1}) of $\mathcal{F}_s$. We initially prove that $i + 1 = s$. Next, we show that there exist two edges e_1 and e_2 which cross (v_i, v_{i+1}) and are not sticks emanating from v_1. The desired contradiction follows from the observation that we can remove edges e_1, e_2 and (v_1, v_1') from G and replace them with the chord (v_1, v_{s-1}) and two additional edges that are both sticks either at v_1 or at v_s. In this way, a new graph is obtained, whose maximal planar substructure has more edges than G_p, which contradicts the maximality of G_p. The detailed proof is given in [5]. □

Lemma 4. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, each middle part of $\mathcal{F}_s$ is short, i.e., it crosses consecutive edges of $\mathcal{F}_s$.*

Proof. (Sketch). For a proof by contradiction, assume that (u, u') is an edge that defines a middle part of $\mathcal{F}_s$ which crosses two non-consecutive edges of $\mathcal{F}_s$, say w.l.o.g. (v_1, v_2) and (v_i, v_{i+1}), where $i \neq 2$ and $i + 1 \neq s$. We distinguish two main cases. Either (u, u') is not involved in crossings in the interior of $\mathcal{F}_s$ or (u, u') is crossed by an edge, say e, within $\mathcal{F}_s$. In both cases, it is possible to lead to a contradiction to the maximality of G_p; refer to [5] for more details. □

Lemma 5. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, each stick of $\mathcal{F}_s$ is short.*

Proof. Assume for a contradiction that there exists a far stick. Let w.l.o.g. (v_1, v_1') be the edge containing this stick and assume that (v_1, v_1') emanates from vertex v_1 and leads to vertex v_1' by crossing the edge (v_i, v_{i+1}) of $\mathcal{F}_s$, where $i \neq 2$ and $i + 1 \neq s$. If we can replace (v_1, v_1') either with chord (v_1, v_i) or with chord (v_1, v_{i+1}), then the maximal planar substructure of the derived graph would have more edges than G_p; contradicting the maximality of G_p. Thus, there exist two edges, say e_1 and e_2, that cross (v_i, v_{i+1}) to the left and to the right of (v_1, v_1'), respectively; see Fig. 2a. By Lemma 3, edge (v_1, v_1') is crossed by at least one other edge, say e, inside $\mathcal{F}_s$. Note that by 3-planarity edge (v_1, v_1') might also be crossed by a second edge, say e', inside $\mathcal{F}_s$. Suppose first, that (v_1, v_1') has a single crossing inside $\mathcal{F}_s$. To cope with this case, we propose two alternatives: (a) replace e_1 with chord (v_1, v_{i+1}) and make vertex v_{i+1} an endpoint of e, or (b) replace e_2 with chord (v_1, v_i) and make vertex v_i an endpoint of both e; see Figs. 2b and c, respectively. Since e and (v_i, v_{i+1}) are not homotopic, it follows that at least one of the two alternatives can be applied, contradicting the maximality of G_p.

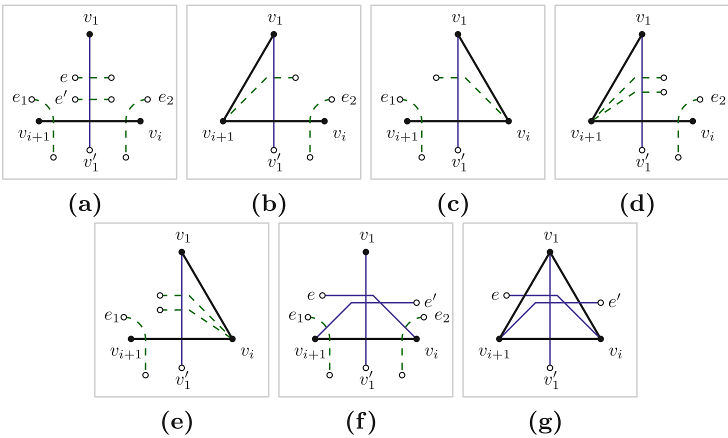


Fig. 2. Different configurations used in the proof of Lemma 5.

Consider now the case where (v_1, v'_1) has two crossings inside $\mathcal{F}_s$, with edges e and e'. Similarly to the previous case, we propose two alternatives: (a) replace e_1 with chord (v_1, v_{i+1}) and make vertex v_{i+1} an endpoint of both e and e', or (b) replace e_2 with chord (v_1, v_i) and make vertex v_i an endpoint of both e and e'; see Figs. 2d and e, respectively. Note that in both alternatives the maximal planar substructure of the derived graph has more edges than G_p, contradicting the maximality of G_p. Since e and e' are not homotopic, it follows that one of the two alternatives is always applicable, as long as, e and e' are not simultaneously sticks from v_i and v_{i+1}, respectively; see Fig. 2f. In this scenario, both alternatives would lead to a situation, where (v_i, v_{i+1}) has two homotopic copies. To cope with this case, we observe that e, e' and (v_1, v'_1) are three mutually crossing edges inside $\mathcal{F}_s$. We proceed by removing from G edges e_1 and e_2, which we replace by (v_1, v_i) and (v_1, v_{i+1}); see Fig. 2g. In the derived graph the maximal planar substructure contains more edges than G_p (in particular, edges (v_1, v_i) and (v_1, v_{i+1})), contradicting its maximality. □

Lemma 6. *The planar substructure G_p of a crossing-minimal optimal 3-planar graph G is connected.*

Proof. Assume to the contrary that the maximum planar substructure G_p of G is not connected and let G'_p be a connected component of G_p. Since G is connected, there is an edge of $G - G_p$ that bridges G'_p with $G_p - G'_p$. By definition, this edge is either a stick or a passing through edge for the common face of G'_p and $G - G'_p$. In both cases, it has to be short (by Lemmas 4 and 5); a contradiction. □

In the next two lemmas, we consider the case where a non-triangular face $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ of G_p has no sticks. Let $br(\mathcal{F}_s)$ and $\overline{br}(\mathcal{F}_s)$ be the set of bridges and non-bridges of $\mathcal{F}_s$, respectively (in Fig. 1a, edge (v_4, v_5) is a bridge). In the absence of sticks, a passing through edge of $\mathcal{F}_s$ *originates* from one of its end-vertices, crosses an edge of $\overline{br}(\mathcal{F}_s)$ to *enter* $\mathcal{F}_s$, passes through $\mathcal{F}_s$ (possibly by defining two middle parts, if it crosses an edge of $br(\mathcal{F}_s)$), crosses another edge of $\overline{br}(\mathcal{F}_s)$ to *exit* $\mathcal{F}_s$ and *terminates* to its other end-vertex. We *associate* the edge of $\overline{br}(\mathcal{F}_s)$ that is used by the passing through edge to enter (exit) $\mathcal{F}_s$ with the origin (terminal) of this passing through edge. Let $\overline{s_b}$ and s_b be the number of edges in $\overline{br}(\mathcal{F}_s)$ and $br(\mathcal{F}_s)$, respectively. Let also $\widehat{s_b}$ be the number of edges of $\overline{br}(\mathcal{F}_s)$ that are crossed by no passing through edge of $\mathcal{F}_s$. Clearly, $\widehat{s_b} \leq \overline{s_b}$ and $s = \overline{s_b} + 2s_b$.

Lemma 7. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p that has no sticks. Then, the number $\widehat{s_b}$ of non-bridges of $\mathcal{F}_s$ that are crossed by no passing through edge of $\mathcal{F}_s$ is strictly less than half the number $\overline{s_b}$ of of non-bridges of $\mathcal{F}_s$, that is, $\widehat{s_b} < \frac{\overline{s_b}}{2}$.*

Proof. For a proof by contradiction assume that $\widehat{s_b} \geq \frac{\overline{s_b}}{2}$. Since at most $\frac{\overline{s_b}}{2}$ edges of $\mathcal{F}_s$ can be crossed (each of which at most three times) and each passing through edge of $\mathcal{F}_s$ crosses two edges of $\overline{br}(\mathcal{F}_s)$, it follows that $|pt(\mathcal{F}_s)| \leq \lfloor \frac{3\overline{s_b}}{4} \rfloor$, where $pt(\mathcal{F}_s)$ denotes the set of passing through edges of $\mathcal{F}_s$. To obtain a contradiction, we remove from G all edges that pass through $\mathcal{F}_s$ and we introduce $2s-6$ edges $\{(v_1, v_i) : 2 < i < s\} \cup \{(v_i, v_i + 2) : 2 \leq i \leq s-2\}$ that lie completely in the interior of $\mathcal{F}_s$. This simple operation will lead to a larger graph (and therefore to a contradiction to the optimality of G) or to a graph of the same size but with larger planar substructure (and therefore to a contradiction to the maximality of G_p) as long as $s > 4$. For $s = 4$, we need a different argument. By Lemma 4, we may assume that all three passing through edges of $\mathcal{F}_s$ cross two consecutive edges of $\mathcal{F}_s$, say w.l.o.g. (v_1, v_2) and (v_2, v_3). This implies that chord (v_1, v_3) can be safely added to G; a contradiction to the optimality of G. □

Lemma 8. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, $\mathcal{F}_s$ has at least one stick.*

Proof (Sketch). For a proof by contradiction, assume that $\mathcal{F}_s$ has no sticks. By Lemma 7, it follows that there exist at least two incident edges of $\overline{br}(\mathcal{F}_s)$ that are crossed by passing through edges of $\mathcal{F}_s$, say w.l.o.g. (v_s, v_1) and (v_1, v_2). Note that these two edges are not bridges of $\mathcal{F}_s$. If $s + \widehat{s_b} + 2s_b \geq 6$, then as in the proof of Lemma 7, it is possible to construct a graph that is larger than G or of equal size as G but with larger planar substructure. The same holds when $s + \widehat{s_b} + 2s_b = 5$ (that is, $s = 5$ and $\widehat{s_b} = s_b = 0$ or $s = 4$, $\widehat{s_b} = 1$ and $s_b = 0$). Both cases, contradict either the optimality of G or the maximality of G_p. The case where $s + \widehat{s_b} + 2s_b = 4$ is slightly more involved; refer to [5]. □

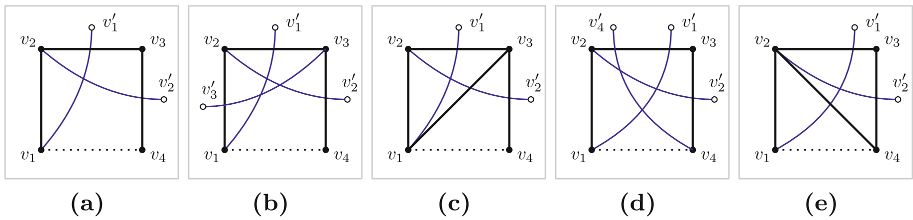


Fig. 3. Different configurations used in Lemma 9.

By Lemma 5, all sticks of $\mathcal{F}_s$ are short. A stick (v_i, v_i') of $\mathcal{F}_s$ is called *right*, if it crosses edge (v_{i+1}, v_{i+2}) of $\mathcal{F}_s$. Otherwise, stick (v_i, v_i') is called *left*. Two sticks are called *opposite*, if one is left and the other one is right.

Lemma 9. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, $\mathcal{F}_s$ has not three mutually crossing sticks.*

Proof. Suppose to the contrary that there exist three mutually crossing sticks of $\mathcal{F}_s$ and let e_i, for $i = 1, 2, 3$ be the edges containing these sticks. W.l.o.g. we assume that at least two of them are right sticks, say e_1 and e_2. Let $e_1 = (v_1, v_1')$. Then, $e_2 = (v_2, v_2')$; see Fig. 3a. Since e_1, e_2 and e_3 mutually cross, e_3 can only contain a left stick. By Lemma 5 its endpoint on $\mathcal{F}_s$ is v_3 or v_4. The first case is illustrated in Fig. 3b. Observe that (v_1, v_2) of $\mathcal{F}_s$ is only crossed by e_3. Indeed, if there was another edge crossing (v_1, v_2), then it would also cross e_1 or e_2, both of which have three crossings. Hence, e_3 can be replaced with (v_1, v_3); see Fig. 3c. The maximal planar substructure of the derived graph would have more edges than G_p, contradicting the maximality of G_p. The case where v_4 is the endpoint of e_3 on $\mathcal{F}_s$ is illustrated in Fig. 3e. Suppose that there exists an edge crossing (v_2, v_3) of $\mathcal{F}_s$ to the left of e_3. This edge should also cross e_2 or e_3, which is not possible since both edges have three crossings. So, we can replace e_3 with chord (v_2, v_4) as in Fig. 3e, contradicting the maximality of G_p. □

Lemma 10. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, each stick of $\mathcal{F}_s$ is crossed exactly once within $\mathcal{F}_s$.*

Proof (Sketch). The detailed proof is given in [5]. By Lemma 3, each stick of $\mathcal{F}_s$ is crossed at least once within $\mathcal{F}_s$. So, the proof is given by contradiction either to the optimality of G or to the maximality of G_p, assuming the existence of a stick of $\mathcal{F}_s$ that is crossed twice within $\mathcal{F}_s$, say by edges e_1 and e_2. Note that by 3-planarity a stick of $\mathcal{F}_s$ cannot be further crossed within $\mathcal{F}_s$. First, we prove that e_1 and e_2 do not cross each other. Then, we show that e_1 and e_2 cannot be simultaneously passing through $\mathcal{F}_s$. The desired contradiction is obtained by considering two main cases: Either e_1 passes through $\mathcal{F}_s$ (and therefore, e_2 is a stick of $\mathcal{F}_s$) or both e_1 and e_2 are sticks of $\mathcal{F}_s$. □

Lemma 11. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, there are no crossings between sticks and middle parts of $\mathcal{F}_s$.*

Proof. Assume to the contrary that there exists a stick, say of edge (v_1, v_1') that emanates from vertex v_1 of $\mathcal{F}_s$ (towards v_1'), which is crossed by a middle part of (u, u') of $\mathcal{F}_s$. By Lemma 10, this stick cannot have another crossing within $\mathcal{F}_s$. By Lemma 5, we can assume w.l.o.g. that (v_1, v_1') is a right stick, i.e., (v_1, v_1') crosses (v_2, v_3). By Lemma 4, edge (u, u') crosses two consecutive edges of $\mathcal{F}_s$. We distinguish two cases based on whether (v_1, v_1') crosses (v_s, v_1) and (v_1, v_2) of $\mathcal{F}_s$ or (v_1, v_1') crosses (v_1, v_2) and (v_2, v_3) of $\mathcal{F}_s$; see Figs. 4a and c respectively.

In the first case, we can assume w.l.o.g. that u is the vertex associated with (v_1, v_2), while u' is the one associated with (v_s, v_1). Hence, there exists an edge, say f_1, that crosses (v_1, v_2) to the right of (u, u'), as otherwise we could replace (u, u') with stick (v_2, u') and reduce the total number of crossings by one, contradicting the crossing minimality of G. Edge f_1 passes through $\mathcal{F}_s$ and also crosses

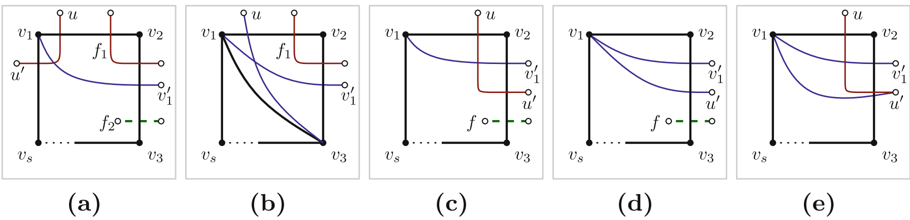


Fig. 4. Different configurations used in Lemma 11.

edge (v_2, v_3) above (v_1, v_1'). Similarly, there exists an edge f_2 that crosses (v_2, v_3) below (v_1, v_1'), as otherwise replacing (v_1, v_1') with chord (v_1, v_3) would contradict the maximality of G_p. We proceed by removing edges (u, u') and f_2 from G and by replacing them with (v_3, u) and chord (v_1, v_3); see Fig. 4b. The maximal planar substructure of the derived graph is larger than G_p; a contradiction.

In the second case, we assume that u is associated with (v_1, v_2) and u' with (v_2, v_3); see Fig. 4c. In this scenario, there exists an edge, say f, that crosses (v_2, v_3) below (v_1, v_1'), as otherwise we could replace (v_1, v_1') with chord (v_1, v_3), contradicting the maximality of G_p. If (v_1, u') does not belong to G, then we remove (u, u') from G and replace it with stick (v_1, u'); see Fig. 4d. In this way, the derived graph has fewer crossings than G; a contradiction. Note that (v_1, v_1') and (v_1, u') cannot be homotopic (if $v_1' = u'$), as otherwise edge (v_1, v_1') and (u, u') would not cross in the initial configuration. Hence, edge (v_1, u') already exists in G. In this case, f is identified with (v_1, u'); see Fig. 4e. But, in this case f is an uncrossed stick of $\mathcal{F}_s$, contradicting Lemma 3. □

Lemma 12. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, any stick of $\mathcal{F}_s$ is only crossed by some opposite stick of $\mathcal{F}_s$.*

Proof. By Lemma 5, each stick of $\mathcal{F}_s$ is short. By Lemma 10, each stick of $\mathcal{F}_s$ is crossed exactly once within $\mathcal{F}_s$ and this crossing is not with a middle part due to Lemma 11. For a proof by contradiction, consider two crossing sticks that are not opposite and assume w.l.o.g. that the first stick emanates from vertex v_1 (towards vertex v_1') and crosses edge (v_2, v_3), while the second stick emanates from vertex v_2 (towards vertex v_2') and crosses edge (v_3, v_4); see Fig. 5a.

If we can replace (v_1, v_1') with the chord (v_1, v_3), then the maximal planar substructure of the derived graph would have more edges than G_p; contradicting the maximality of G_p. Thus, there exists an edge, say e, that crosses (v_2, v_3) below (v_1, v_1'). By Lemma 11, edge e is passing through $\mathcal{F}_s$. Symmetrically, we can prove that there exists an edge, say e', which crosses (v_3, v_4) right next to v_4, that is, e' defines the closest crossing point to v_4 along (v_3, v_4). Note that e' can be either a passing through edge or a stick of $\mathcal{F}_s$. We proceed by removing from G edges e' and (v_1, v_1') and by replacing them by the chord (v_2, v_4) and edge (v_4, v_1'); see Fig. 5b. The maximal planar substructure of the derived graph has more edges than G_p (in the presence of edge (v_2, v_4)), a contradiction. □

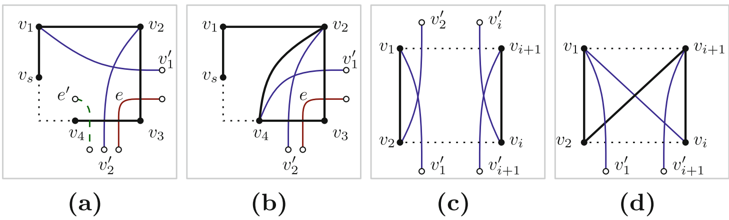


Fig. 5. Different configurations used in (a)–(b) Lemma 12 and (c)–(d) Lemma 13.

Lemma 13. *Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$ be a non-triangular face of G_p. Then, $\mathcal{F}_s$ has exactly two sticks.*

Proof. By Lemmas 8 and 12 there exists at least one pair of opposite crossing sticks. To prove the uniqueness, assume that $\mathcal{F}_s$ has two pairs of crossing opposite sticks, say (v_1, v_1'), (v_2, v_2') and (v_i, v_i'), (v_{i+1}, v_{i+1}'), $2 < i < s$; see Fig. 5c. We remove edges (v_2, v_2') and (v_i, v_i') and replace them by (v_1, v_i) and (v_2, v_{i+1}); see Fig. 5d. By Lemmas 4 and 5, the newly introduced edges cannot be involved in crossings. The maximal planar substructure of the derived graph has more edges than G_p (in the presence of (v_1, v_i) or (v_2, v_{i+1})); a contradiction. □

We are ready to state the main theorem of this section.

Theorem 2. *The planar substructure G_p of a crossing-minimal optimal 3-planar graph G is fully triangulated.*

Proof. For a proof by contradiction, assume that G_p has a non-triangular face $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$, $s \geq 4$. By Lemmas 10, 12 and 13, face $\mathcal{F}_s$ has exactly two opposite sticks, that cross each other. Assume w.l.o.g. that these two sticks emanate from v_1 and v_2 (towards v_1' and v_2') and exit $\mathcal{F}_s$ by crossing (v_2, v_3) and (v_1, v_s), respectively; recall that by Lemma 5 all sticks are short; see Fig. 6a.

If we can replace (v_1, v_1') with the chord (v_1, v_3), then the maximal planar substructure of the derived graph would have more edges than G_p; contradicting the maximality of G_p. Thus, there exists an edge, say e, that crosses (v_2, v_3) below (v_1, v_1'). By Lemma 13, edge e is passing through $\mathcal{F}_s$. We consider two cases: (a) edge (v_2, v_3) is only crossed by e and (v_1, v_1'), (b) there is a third edge, say e', that crosses (v_2, v_3) (which by Lemma 13 is also passing through $\mathcal{F}_s$).

In Case (a), we can remove from G edges e and (v_1, v_1'), and replace them by (v_1, v_3) and the edge from v_2 to the endpoint of e that is below (v_3, v_4); see Fig. 6b. In Case (b), there has to be a (passing through) edge, say e'', surrounding v_4 (see Fig. 6c), as otherwise we could replace e' with a stick emanating from v_4 towards the endpoint of e' that is to the right of (v_2, v_3), which contradicts Lemma 13. We proceed by removing from G edges e'' and (v_1, v_1') and by replacing them by (v_2, v_4) and the edge from v_2 to the endpoint of e'' that is associated with (v_3, v_4); see Fig. 6d. The maximal planar substructure of the derived graph has more edges than G_p (in the presence of (v_1, v_2) in Case (a) and (v_2, v_4) in

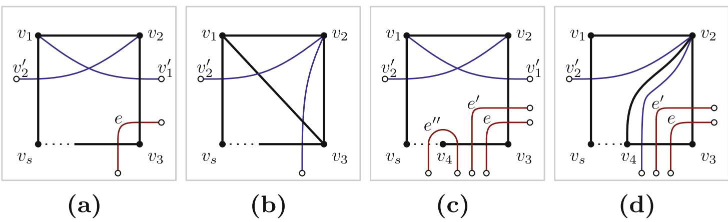


Fig. 6. Different configurations used in Theorem 2.

Case (b)), which contradicts the maximality of G_p. Since G_p is connected, there cannot exist a face consisting of only two vertices. □

5 Discussion and Conclusion

This paper establishes a tight upper bound on the number of edges of non-simple 3-planar graphs containing no homotopic parallel edges or self-loops. Our work is towards a complete characterization of all optimal such graphs. In addition, we believe that our technique can be used to achieve better bounds for larger values of k. We demonstrate it for the case where $k = 4$, where the known bound for simple graphs is due to Ackerman [1].

If we could prove that a crossing-minimal optimal 4-planar graph $G = (V, E)$ has always a fully triangulated planar substructure $G_p = (V, E_p)$ (as we proved in Theorem 2 for the corresponding 3-planar ones), then it is not difficult to prove a tight bound on the number of edges for 4-planar graphs. Similar to Lemma 1, we can argue that no triangle of G_p has more than 4 sticks. Then, we associate each triangle of G_p with 4 sticks to a neighboring triangle with at most 2 sticks. This would imply $t_4 \leq t_1 + t_2$, where t_i denotes the number of triangles of G_p with exactly i sticks. So, we would have $|E| - |E_p| = (4t_4 + 3t_3 + 2t_2 + t_1)/2 \leq 3(t_4 + t_3 + t_2 + t_1)/2 = 3(2n - 4)/2 = 3n - 6$. Hence, the number of edges of a 4-planar graph G is at most $6n - 12$. We conclude with some open questions.

- A nice consequence of our work would be the complete characterization of optimal 3-planar graphs, as exactly those graphs that admit drawings where the set of crossing-free edges form hexagonal faces which contain 8 additional edges each
- We also believe that for simple 3-planar graphs (i.e., where even non-homotopic parallel edges are not allowed) the corresponding bound is $5.5n - 15$.
- We conjecture that the maximum number of edges of 5- and 6-planar graphs are $\frac{19}{3}n - O(1)$ and $7n - 14$, respectively.
- More generally, is there a closed function on k which describes the maximum number of edges of a k-planar graph for $k > 3$? Recall the general upper bound of $4.1208\sqrt{k}n$ by Pach and Tóth [20].

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References

1. Ackerman, E.: On topological graphs with at most four crossings per edge. CoRR abs/1509.01932 (2015)
2. Agarwal, P.K., Aronov, B., Pach, J., Pollack, R., Sharir, M.: Quasi-planar graphs have a linear number of edges. Combinatorica **17**(1), 1–9 (1997)
3. Auer, C., Brandenburg, F.J., Gleißner, A., Hanauer, K.: On sparse maximal 2-planar graphs. In: Didimo, W., Patrignani, M. (eds.) GD 2012. LNCS, vol. 7704, pp. 555–556. Springer, Heidelberg (2013). doi:10.1007/978-3-642-36763-2_50

4. Bekos, M.A., Bruckdorfer, T., Kaufmann, M., Raftopoulou, C.: 1-planar graphs have constant book thickness. In: Bansal, N., Finocchi, I. (eds.) ESA 2015. LNCS, vol. 9294, pp. 130–141. Springer, Heidelberg (2015). doi:10.1007/978-3-662-48350-3_12
5. Bekos, M.A., Kaufmann, M., Raftopoulou, C.N.: On the density of 3-planar graphs. CoRR abs/1602.04995v3 (2016)
6. Borodin, O.V.: A new proof of the 6 color theorem. J. Graph Theory **19**(4), 507–521 (1995)
7. Brandenburg, F.J.: 1-visibility representations of 1-planar graphs. J. Graph Algorithms Appl. **18**(3), 421–438 (2014)
8. Brandenburg, F.J., Eppstein, D., Gleißner, A., Goodrich, M.T., Hanauer, K., Reislhuber, J.: On the density of maximal 1-planar graphs. In: Didimo, W., Patrignani, M. (eds.) GD 2012. LNCS, vol. 7704, pp. 327–338. Springer, Heidelberg (2013). doi:10.1007/978-3-642-36763-2_29
9. Cheong, O., Har-Peled, S., Kim, H., Kim, H.-S.: On the number of edges of fan-crossing free graphs. In: Cai, L., Cheng, S.-W., Lam, T.-W. (eds.) ISAAC 2013. LNCS, vol. 8283, pp. 163–173. Springer, Heidelberg (2013). doi:10.1007/978-3-642-45030-3_16
10. Di Battista, G., Eades, P., Tamassia, R., Tollis, I.G.: Graph Drawing: Algorithms for the Visualization of Graphs. Prentice-Hall, Upper Saddle River (1999)
11. Didimo, W., Eades, P., Liotta, G.: Drawing graphs with right angle crossings. Theoret. Comput. Sci. **412**(39), 5156–5166 (2011)
12. Dujmovic, V., Gudmundsson, J., Morin, P., Wolle, T.: Notes on large angle crossing graphs. Chicago J. Theor. Comput. Sci. **4**, 1–14 (2011)
13. de Fraysseix, H., Pach, J., Pollack, R.: How to draw a planar graph on a grid. Combinatorica **10**(1), 41–51 (1990)
14. Grigoriev, A., Bodlaender, H.L.: Algorithms for graphs embeddable with few crossings per edge. Algorithmica **49**(1), 1–11 (2007)
15. Harary, F.: Graph Theory. Addison-Wesley, Boston (1991)
16. Hong, S.-H., Nagamochi, H.: Testing full outer-2-planarity in linear time. In: Mayr, E.W. (ed.) WG 2015. LNCS, vol. 9224, pp. 406–421. Springer, Heidelberg (2016). doi:10.1007/978-3-662-53174-7_29
17. Kaufmann, M., Ueckerdt, T.: The density of fan-planar graphs. CoRR abs/1403.6184 (2014)
18. Kaufmann, M., Wagner, D. (eds.): Drawing Graphs, Methods and Models. LNCS, vol. 2025. Springer, Heidelberg (2001)
19. Pach, J., Radoicic, R., Tardos, G., Tóth, G.: Improving the crossing lemma by finding more crossings in sparse graphs. Discrete Comput. Geom. **36**(4), 527–552 (2006)
20. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. Combinatorica **17**(3), 427–439 (1997)
21. Ringel, G.: Ein sechsfarbenproblem auf der kugel. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (in German) **29**, 107–117 (1965)
22. Tamassia, R.: On embedding a graph in the grid with the minimum number of bends. SIAM J. Comput. **16**(3), 421–444 (1987)
23. Tutte, W.T.: How to draw a graph. Proc. London Math. Soc. **3**(13), 743–767 (1963)