Chapter 27
And More Recently?

Summarizing what we have seen so far, the *genus* is a measure of the complexity of abstract objects: algebraic curves (defined over an arbitrary field, etc.) and topological surfaces (orientable, etc.) This is in contrast with the *degree*, which partially measures the embedding into an ambient space.

To conclude the part of this story which deals with algebraic curves and topological surfaces, I would like to state two more recent very deep theorems concerning the genus, seen from both viewpoints. The first one is the affirmative answer given by Faltings [75] to a conjecture of Mordell, and the second one is the affirmative answer given by Kronheimer and Mrowka [124] to a conjecture of Thom.

Theorem 27.1 *Any irreducible curve of genus $p > 1$ which is defined by equations with rational coefficients admits a* finite *number of points with rational coordinates.*

This is to be contrasted with the curves of genera 0 or 1, which may have an infinite number of points with rational coordinates (see Chap. 25).

Theorem 27.2 *If a smooth, compact, connected and oriented surface without boundary is embedded in the complex projective plane and if it is homologous there to a smooth complex algebraic curve of degree n, then its genus is at least equal to that of the curve, that is, to* $\frac{(n-1)(n-2)}{2}$.

Recall that we stated the previous formula for the genus of a smooth algebraic curve of degree n in Theorem 20.1. In what concerns the notion of *homology*, it is treated briefly in Chaps. 38 and 39. Note that all the smooth algebraic curves of the same degree are not only homologous, but also continuously deformable into one another inside the complex projective plane, that is, they are *isotopic*.

The previous theorem is perhaps evidence of a certain "*principle of economy of algebraic geometry*", still rather mysterious, which states that in order to construct some kinds of objects, one cannot proceed in a simpler way (here, with a smaller genus) than using algebraic geometry.

P. Popescu-Pampu, *What is the Genus?*, Lecture Notes in Mathematics 2162,
DOI 10.1007/978-3-319-42312-8_27