Derivative Pricing for a Multi-curve Extension of the Gaussian, Exponentially Quadratic Short Rate Model

Zorana Grbac, Laura Meneghello and Wolfgang J. Runggaldier

Abstract The recent financial crisis has led to so-called multi-curve models for the term structure. Here we study a multi-curve extension of short rate models where, in addition to the short rate itself, we introduce short rate spreads. In particular, we consider a Gaussian factor model where the short rate and the spreads are second order polynomials of Gaussian factor processes. This leads to an exponentially quadratic model class that is less well known than the exponentially affine class. In the latter class the factors enter linearly and for positivity one considers square root factor processes. While the square root factors in the affine class have more involved distributions, in the quadratic class the factors remain Gaussian and this leads to various advantages, in particular for derivative pricing. After some preliminaries on martingale modeling in the multi-curve setup, we concentrate on pricing of linear and optional derivatives. For linear derivatives, we exhibit an adjustment factor that allows one to pass from pre-crisis single curve values to the corresponding post-crisis multi-curve values.

Keywords Multi-curve models · Short rate models · Short rate spreads · Gaussian exponentially quadratic models · Pricing of linear and optional interest rate derivatives · Riccati equations · Adjustment factors

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1   Introduction

The recent financial crisis has heavily impacted the financial market and the fixed income markets in particular. Key features put forward by the crisis are counterparty and liquidity/funding risk. In interest rate derivatives the underlying rates are typically Libor/Euribor. These are determined by a panel of banks and thus reflect various risks in the interbank market, in particular counterparty and liquidity risk. The standard no-arbitrage relations between Libor rates of different maturities have broken down and significant spreads have been observed between Libor rates of different tenors, as well as between Libor and OIS swap rates, where OIS stands for Overnight Indexed Swap. For more details on this issue see Eqs. (5)–(7) and the paragraph following them, as well as the paper by Bormetti et al. [1] and a corresponding version in this volume. This has led practitioners and academics alike to construct multi-curve models where future cash flows are generated through curves associated to the underlying rates (typically the Libor, one for each tenor structure), but are discounted by another curve.

For the pre-crisis single-curve setup various interest rate models have been proposed. Some of the standard model classes are: the short rate models; the instantaneous forward rate models in an Heath–Jarrow–Morton (HJM) setup; the market forward rate models (Libor market models). In this paper we consider a possible multi-curve extension of the short rate model class that, with respect to the other model classes, has in particular the advantage of leading more easily to a Markovian structure. Other multi-curve extensions of short rate models have appeared in the literature such as Kijima et al. [22], Kenyon [20], Filipović and Trolle [14], Morino and Runggaldier [27]. The present paper considers an exponentially quadratic model, whereas the models in the mentioned papers concern mainly the exponentially affine framework, except for [22] in which the exponentially quadratic models are mentioned. More details on the difference between the exponentially affine and exponentially quadratic short rate models will be provided below.

Inspired by a credit risk analogy, but also by a common practice of deriving multi-curve quantities by adding a spread over the corresponding single-curve risk-free quantities, we shall consider, next to the short rate itself, a short rate spread to be added to the short rate, one for each possible tenor structure. Notice that these spreads are added from the outset.

To discuss the basic ideas in an as simple as possible way, we consider just a two-curve model, namely with one curve for discounting and one for generating future cash flows; in other words, we shall consider a single tenor structure. We shall thus concentrate on the short rate r_t and a single short rate spread s_t and, for their dynamics, introduce a factor model. In the pre-crisis single-curve setting there are two basic factor model classes for the short rate: the exponentially affine and the exponentially quadratic model classes. Here we shall concentrate on the less common quadratic class with Gaussian factors. In the exponentially affine class where, to guarantee positivity of rates and spreads, one considers generally square root models for the

factors, the distribution of the factors is χ^2. In the exponentially quadratic class the factors have a more convenient Gaussian distribution.

The paper is structured as follows. In the preliminary Sect. 2 we mainly discuss issues related to martingale modeling. In Sect. 3 we introduce the multi-curve Gaussian, exponentially quadratic model class. In Sect. 4 we deal with pricing of linear interest rate derivatives and, finally, in Sect. 5 with nonlinear/optional interest rate derivatives.

2 Preliminaries

2.1 Discount Curve and Collateralization

In the presence of multiple curves, the choice of the curve for discounting the future cash flows, and a related choice of the numeraire for the standard martingale measure used for pricing, in other words, the question of absence of arbitrage, becomes nontrivial (see e.g. the discussion in Kijima and Muromachi [21]). To avoid issues of arbitrage, one should possibly have a common discount curve to be applied to all future cash flows independently of the tenor. A choice, which has been widely accepted and became practically standard, is given by the OIS-curve $T \mapsto p(t, T) = p^{OIS}(t, T)$ that can be stripped from OIS rates, namely the fair rates in an OIS. The arguments justifying this choice and which are typically evoked in practice, are the fact that the majority of the traded interest rate derivatives are nowadays being collateralized and the rate used for remuneration of the collateral is exactly the overnight rate, which is the rate the OIS are based on. Moreover, the overnight rate bears very little risk due to its short maturity and therefore can be considered relatively risk-free. In this context we also point out that prices, corresponding to fully collateralized transactions, are considered as clean prices (this terminology was first introduced by Crépey [6] and Crépey et al. [9]). Since collateralization is by now applied in the majority of cases, one may thus ignore counterparty and liquidity risk between individual parties when pricing interest rate derivatives, but cannot ignore the counterparty and liquidity risk in the interbank market as a whole. These risks are often jointly referred to as interbank risk and they are main drivers of the multi-curve phenomenon, as documented in the literature (see e.g. Crépey and Douady [7], Filipović and Trolle [14], and Gallitschke et al. [15]). We shall thus consider only *clean valuation* formulas, which take into account the multi-curve issue. Possible ways to account for counterparty risk and funding issues between individual counterparties in a contract are, among others, to follow a global valuation approach that leads to nonlinear derivative valuation (see Brigo et al. [3, 4] and other references therein, and in particular Pallavicini and Brigo [28] for a global valuation approach applied specifically to interest rate modeling), or to consider various valuation adjustments that are generally computed on top of the clean prices (see Crépey [6]). A fully nonlinear valuation is preferable, but is more difficult to achieve. On the other hand,

valuation adjustments are more consolidated and also used in practice and this gives a further justification to still look for clean prices. Concerning the explicit role of collateral in the pricing of interest rate derivatives, we refer to the above-mentioned paper by Pallavicini and Brigo [28].

2.2 Martingale Measures

The fundamental theorem of asset pricing links the economic principle of absence of arbitrage with the notion of a martingale measure. As it is well known, this is a measure, under which the traded asset prices, expressed in units of a same numeraire, are local martingales. Models for interest rate markets are typically incomplete so that absence of arbitrage admits many martingale measures. A common approach in interest rate modeling is to perform martingale modeling, namely to model the quantities of interest directly under a generic martingale measure; one has then to perform a calibration in order to single out the specific martingale measure of interest. The modeling under a martingale measure now imposes some conditions on the model and, in interest rate theory, a typical such condition is the Heath–Jarrow–Morton (HJM) drift condition.

Starting from the OIS bonds, we shall first derive a suitable numeraire and then consider as martingale measure a measure Q under which not only the OIS bonds, but also the FRA contracts seen as basic quantities in the bond market, are local martingales when expressed in units of the given numeraire. To this basic market one can then add various derivatives imposing that their prices, expressed in units of the numeraire, are local martingales under Q.

Having made the choice of the OIS curve $T \mapsto p(t, T)$ as the discount curve, consider the instantaneous forward rates $f(t, T) := -\frac{\partial}{\partial T} \log p(t, T)$ and let $r_t = f(t, t)$ be the corresponding short rate at the generic time t. Define the OIS bank account as

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{1}$$

and, as usual, the standard martingale measure Q as the measure, equivalent to the physical measure P, that is associated to the bank account B_t as numeraire. Hence the arbitrage-free prices of all assets, discounted by B_t, have to be local martingales with respect to Q. For derivative pricing, among them also FRA pricing, it is often more convenient to use, equivalently, the forward measure Q^T associated to the OIS bond $p(t, T)$ as numeraire. The two measures Q and Q^T are related by their Radon–Nikodym density process

$$\left.\frac{d Q^T}{d Q}\right|_{\mathscr{F}_t} = \frac{p(t, T)}{B_t p(0, T)} \qquad 0 \leq t \leq T. \tag{2}$$

As already mentioned, we shall follow the traditional *martingale modeling*, whereby the model dynamics are assigned under the martingale measure Q. This leads to defining the OIS bond prices according to

$$p(t, T) = E^Q \left\{ \exp\left[-\int_t^T r_u du \right] \mid \mathscr{F}_t \right\} \qquad (3)$$

after having specified the $Q-$dynamics of r.

Coming now to the FRA contracts, recall that they concern a forward rate agreement, established at a time t for a future interval $[T, T + \Delta]$, where at time $T + \Delta$ the interest corresponding to a floating rate is received in exchange for the interest corresponding to a fixed rate R. There exist various possible conventions concerning the timing of the payments. Here we choose payment in arrears, which in this case means at time $T + \Delta$. Typically, the floating rate is given by the Libor rate and, having assumed payments in arrears, we also assume that the rate is fixed at the beginning of the interval of interest, here at T. Recall that for expository simplicity we had reduced ourselves to a two-curve setup involving just a single Libor for a given tenor Δ. The floating rate received at $T + \Delta$ is therefore the rate $L(T; T, T + \Delta)$, fixed at the inception time T. For a unitary notional, and using the $(T + \Delta)$-forward measure $Q^{T+\Delta}$ as the pricing measure, the arbitrage-free price at $t \leq T$ of the FRA contract is then

$$P^{FRA}(t; T, T + \Delta, R) = \Delta p(t, T + \Delta) E^{T+\Delta} \{L(T; T, T + \Delta) - R \mid \mathscr{F}_t\}, \qquad (4)$$

where $E^{T+\Delta}$ denotes the expectation with respect to the measure $Q^{T+\Delta}$. From this expression it follows that the value of the fixed rate R that makes the contract fair at time t is given by

$$R_t = E^{T+\Delta} \{L(T; T, T + \Delta) \mid \mathscr{F}_t\} := L(t; T, T + \Delta) \qquad (5)$$

and we shall call $L(t; T, T + \Delta)$ the *forward Libor rate*. Note that $L(\cdot; T, T + \Delta)$ is a $Q^{T+\Delta}-$martingale by construction.

In view of developing a model for $L(T; T, T + \Delta)$, recall that, by absence of arbitrage arguments, the classical discrete compounding forward rate at time t for the future time interval $[T, T + \Delta]$ is given by

$$F(t; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} - 1 \right),$$

where $p(t, T)$ represents here the price of a risk-free zero coupon bond. This expression can be justified also by the fact that it represents the fair fixed rate in a forward rate agreement, where the floating rate received at $T + \Delta$ is

$$F(T; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{1}{p(T, T + \Delta)} - 1 \right) \qquad (6)$$

and we have

$$F(t; T, T + \Delta) = E^{T+\Delta} \{F(T; T, T + \Delta) \mid \mathscr{F}_t\}. \tag{7}$$

This makes the forward rate coherent with the risk-free bond prices, where the latter represent the expectation of the market concerning the future value of money.

Before the financial crisis, $L(T; T, T + \Delta)$ was assumed to be equal to $F(T; T, T + \Delta)$, an assumption that allowed for various simplifications in the determination of derivative prices. After the crisis $L(T; T, T + \Delta)$ is no longer equal to $F(T; T, T + \Delta)$ and what one considers for $F(T; T, T + \Delta)$ is in fact the *OIS discretely compounded rate*, which is based on the OIS bonds, even though the OIS bonds are not necessarily equal to the risk-free bonds (see Sects. 1.3.1 and 1.3.2 of Grbac and Runggaldier [18] for more details on this issue). In particular, the Libor rate $L(T; T, T + \Delta)$ cannot be expressed by the right-hand side of (6). The fact that $L(T; T, T + \Delta) \neq F(T; T, T + \Delta)$ implies by (5) and (7) that also $L(t; T, T + \Delta) \neq F(t; T, T + \Delta)$ for all $t \leq T$ and this leads to a *Libor-OIS spread* $L(t; T, T + \Delta) - F(t; T, T + \Delta)$.

Following some of the recent literature (see e.g. Kijima et al. [22], Crépey et al. [8], Filipović and Trolle [14]), one possibility is now to keep the classical relationship (6) also for $L(T; T, T + \Delta)$ thereby replacing however the bonds $p(t, T)$ by fictitious risky ones $\bar{p}(t, T)$ that are assumed to be affected by the same factors as the Libor rates. Such a bond can be seen as an average bond issued by a representative bank from the Libor group and it is therefore sometimes referred to in the literature as a *Libor bond.* This leads to

$$L(T; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{1}{\bar{p}(T, T + \Delta)} - 1 \right). \tag{8}$$

Recall that, for simplicity of exposition, we consider a single Libor for a single tenor Δ and so also a single fictitious bond. In general, one has one Libor and one fictitious bond for each tenor, i.e. $L^{\Delta}(T; T, T + \Delta)$ and $\bar{p}^{\Delta}(T, T + \Delta)$. Note that we shall model the bond prices $\bar{p}(t, T)$, for all t and T with $t \leq T$, even though only the prices $\bar{p}(T, T + \Delta)$, for all T, are needed in relation (8). Moreover, keeping in mind that the bonds $\bar{p}(t, T)$ are fictitious, they do not have to satisfy the boundary condition $\bar{p}(T, T) = 1$, but we still assume this condition in order to simplify the modeling.

To derive a dynamic model for $L(t; T, T + \Delta)$, we may now derive a dynamic model for $\bar{p}(t, T + \Delta)$, where we have to keep in mind that the latter is not a traded quantity. Inspired by a credit-risk analogy, but also by a common practice of deriving multi-curve quantities by adding a spread over the corresponding single-curve (risk-free) quantities, which in this case is the short rate r_t, let us define then the Libor (risky) bond prices as

$$\bar{p}(t, T) = E^Q \left\{ \exp \left[- \int_t^T (r_u + s_u) du \right] \mid \mathscr{F}_t \right\}, \tag{9}$$

with s_t representing the short rate spread. In case of default risk alone, s_t corresponds to the hazard rate/default intensity, but here it corresponds more generally to all the factors affecting the Libor rate, namely besides credit risk, also liquidity risk, etc. Notice also that the spread is introduced here from the outset. Having for simplicity considered a single tenor Δ and thus a single $\bar{p}(t, T)$, we shall also consider only a single spread s_t. In general, however, one has a spread s_t^Δ for each tenor Δ.

We need now a dynamical model for both r_t and s_t and we shall define this model directly under the martingale measure Q (*martingale modeling*).

3 Short Rate Model

3.1 The Model

As mentioned, we shall consider a dynamical model for r_t and the single spread s_t under the martingale measure Q that, in practice, has to be calibrated to the market. For this purpose we shall consider a factor model with several factors driving r_t and s_t.

The two basic factor model classes for the short rate in the pre-crisis single-curve setup, namely the exponentially affine and the exponentially quadratic model classes, both allow for flexibility and analytical tractability and this in turn allows for closed or semi-closed formulas for linear and optional interest rate derivatives. The former class is usually better known than the latter, but the latter has its own advantages. In fact, for the exponentially affine class one would consider r_t and s_t as given by a linear combination of the factors and so, in order to obtain positivity, one has to consider a square root model for the factors. On the other hand, in the Gaussian exponentially quadratic class, one considers mean reverting Gaussian factor models, but at least some of the factors in the linear combination for r_t and s_t appear as a square. In this way the distribution of the factors remains always Gaussian; in a square-root model it is a non-central χ^2-distribution. Notice also that the exponentially quadratic models can be seen as dual to the square root exponentially affine models.

In the pre-crisis single-curve setting, the exponentially quadratic models have been considered, e.g. in El Karoui et al. [12], Pelsser [29], Gombani and Runggaldier [17], Leippold and Wu [24], Chen et al. [5], and Gaspar [16]. However, since the pre-crisis exponentially affine models are more common, there have also been more attempts to extend them to a post-crisis multi-curve setting (for an overview and details see e.g. Grbac and Runggaldier [18]). A first extension of exponentially quadratic models to a multi-curve setting can be found in Kijima et al. [22] and the present paper is devoted to a possibly full extension.

Let us now present the model for r_t and s_t, where we consider not only the short rate r_t itself, but also its spread s_t to be given by a linear combination of the factors, where at least some of the factors appear as a square. To keep the presentation simple, we shall consider a small number of factors and, in order to model also a possible

correlation between r_t and s_t, the minimal number of factors is three. It also follows from some of the econometric literature that a small number of factors may suffice to adequately model most situations (see also Duffee [10] and Duffie and Gârleanu [11]).

Given three independent affine factor processes Ψ_t^i, $i = 1, 2, 3$, having under Q the Gaussian dynamics

$$d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i dw_t^i, \quad i = 1, 2, 3, \tag{10}$$

with b_i, $\sigma_i > 0$ and w_t^i, $i = 1, 2, 3$, independent $Q-$Wiener processes, we let

$$\begin{cases} r_t = \Psi_t^1 + (\Psi_t^2)^2 \\ s_t = \kappa \Psi_t^1 + (\Psi_t^3)^2 \end{cases}, \tag{11}$$

where Ψ_t^1 is the common systematic factor allowing for instantaneous correlation between r_t and s_t with correlation intensity κ and Ψ_t^2 and Ψ_t^3 are the idiosyncratic factors. Other factors may be added to drive s_t, but the minimal model containing common and idiosyncratic components requires three factors, as explained above. The common factor is particularly important because we want to take into account the realistic feature of non-zero correlation between r_t and s_t in the model.

Remark 3.1 The zero mean-reversion level is here considered only for convenience of simpler formulas, but can be easily taken to be positive, so that short rates and spreads can become negative only with small probability (see Kijima and Muromachi [21] for an alternative representation of the spreads in terms of Gaussian factors that guarantee the spreads to remain nonnegative and still allows for correlation between r_t and s_t). Note, however, that given the current market situation where the observed interest rates are very close to zero and sometimes also negative, even models with negative mean-reversion level have been considered, as well as models allowing for regime-switching in the mean reversion parameter.

Remark 3.2 For the short rate itself one could also consider the model $r_t = \phi_t + \Psi_t^1 + (\Psi_t^2)^2$ where ϕ_t is a *deterministic shift extension* (see Brigo and Mercurio [2]) that allows for a good fit to the initial term structure in short rate models even with constant model parameters.

In the model (11) we have included a linear term Ψ_t^1 which may lead to negative values of rates and spreads, although only with small probability in the case of models of the type (10) with a positive mean reversion level. The advantage of including this linear term is more generality and flexibility in the model. Moreover, it allows to express $\bar{p}(t, T)$ in terms of $p(t, T)$ multiplied by a factor. This property will lead to an *adjustment factor* by which one can express post-crisis quantities in terms of corresponding pre-crisis quantities, see Morino and Runggaldier [27] in which this idea has been first proposed in the context of exponentially affine short rate models for multiple curves.

3.2 Bond Prices (OIS and Libor Bonds)

In this subsection we derive explicit pricing formulas for the OIS bonds $p(t, T)$ as defined in (3) and the fictitious Libor bonds $\bar{p}(t, T)$ as defined in (9). Thereby, r_t and s_t are supposed to be given by (11) with the factor processes Ψ_t^i evolving under the standard martingale measure Q according to (10). Defining the matrices

$$F = \begin{bmatrix} -b^1 & 0 & 0 \\ 0 & -b^2 & 0 \\ 0 & 0 & -b^3 \end{bmatrix} , \ D = \begin{bmatrix} \sigma^1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^3 \end{bmatrix} \tag{12}$$

and considering the vector factor process $\Psi_t := [\Psi_t^1, \Psi_t^2, \Psi_t^3]'$ as well as the multivariate Wiener process $W_t := [w_t^1, w_t^2, w_t^3]'$, where $'$ denotes transposition, the dynamics (10) can be rewritten in synthetic form as

$$d\Psi_t = F\Psi_t dt + D dW_t. \tag{13}$$

Using results on exponential quadratic term structures (see Gombani and Runggaldier [17], Filipović [13]), we have

$$p(t, T) = E^Q \left\{ e^{-\int_t^T r_u du} \bigg| \mathscr{F}_t \right\} = E^Q \left\{ e^{-\int_t^T (\Psi_u^1 + (\Psi_u^2)^2) du} \bigg| \mathscr{F}_t \right\}$$
$$= \exp\left[-A(t, T) - B'(t, T)\Psi_t - \Psi_t' C(t, T)\Psi_t \right] \tag{14}$$

and, setting $R_t := r_t + s_t$,

$$\bar{p}(t, T) = E^Q \left\{ e^{-\int_t^T R_u du} \bigg| \mathscr{F}_t \right\} = E^Q \left\{ e^{-\int_t^T ((1+\kappa)\Psi_u^1 + (\Psi_u^2)^2 + (\Psi_u^3)^2) du} \bigg| \mathscr{F}_t \right\}$$
$$= \exp\left[-\bar{A}(t, T) - \bar{B}'(t, T)\Psi_t - \Psi_t' \bar{C}(t, T)\Psi_t \right], \tag{15}$$

where $A(t, T)$, $\bar{A}(t, T)$, $B(t, T)$, $\bar{B}(t, T)$, $C(t, T)$ and $\bar{C}(t, T)$ are scalar, vector, and matrix-valued deterministic functions to be determined.

For this purpose we recall the Heath–Jarrow–Morton (HJM) approach for the case when $p(t, T)$ in (14) represents the price of a risk-free zero coupon bond. The HJM approach leads to the so-called HJM drift conditions that impose conditions on the coefficients in (14) so that the resulting prices $p(t, T)$ do not imply arbitrage possibilities. Since the risk-free bonds are traded, the no-arbitrage condition is expressed by requiring $\frac{p(t,T)}{B_t}$ to be a $Q-$martingale for B_t defined as in (1) and it is exactly this martingality property to yield the drift condition. In our case, $p(t, T)$ is the price of an OIS bond that is not necessarily traded and in general does not coincide with the price of a risk-free bond. However, whether the OIS bond is traded or not, $\frac{p(t,T)}{B_t}$ is a $Q-$martingale by the very definition of $p(t, T)$ in (14) (see the first equality in (14)) and so we can follow the same HJM approach to obtain conditions on the coefficients in (14) also in our case.

For what concerns, on the other hand, the coefficients in (15), recall that $\bar{p}(t, T)$ is a fictitious asset that is not traded and thus is not subject to any no-arbitrage condition. Notice, however, that by analogy to $p(t, T)$ in (14), by its very definition given in the first equality in (15), $\frac{\bar{p}(t,T)}{\bar{B}_t}$ is a Q−martingale for $\bar{B}_t$ given by $\bar{B}_t := \exp \int_0^t R_u du$. The two cases $p(t, T)$ and $\bar{p}(t, T)$ can thus be treated in complete analogy provided that we use for $\bar{p}(t, T)$ the numeraire $\bar{B}_t$.

We shall next derive from the Q−martingality of $\frac{p(t,T)}{B_t}$ and $\frac{\bar{p}(t,T)}{B_t}$ conditions on the coefficients in (14) and (15) that correspond to the classical HJM drift condition and lead thus to ODEs for these coefficients. For this purpose we shall proceed by analogy to Sect. 2 in [17], in particular to the proof of Proposition 2.1 therein, to which we also refer for more detail.

Introducing the "instantaneous forward rates" $f(t, T) := -\frac{\partial}{\partial T} \log p(t, T)$ and $\bar{f}(t, T) := -\frac{\partial}{\partial T} \log \bar{p}(t, T)$, and setting

$$a(t, T) := \frac{\partial}{\partial T} A(t, T), \quad b(t, T) := \frac{\partial}{\partial T} B(t, T), \quad c(t, T) := \frac{\partial}{\partial T} C(t, T) \quad (16)$$

and analogously for $\bar{a}(t, T), \bar{b}(t, T), \bar{c}(t, T)$, from (14) and (15) we obtain

$$f(t, T) = a(t, T) + b'(t, T)\Psi_t + \Psi_t' c(t, T)\Psi_t, \tag{17}$$

$$\bar{f}(t, T) = \bar{a}(t, T) + \bar{b}'(t, T)\Psi_t + \Psi_t' \bar{c}(t, T)\Psi_t. \tag{18}$$

Recalling that $r_t = f(t, t)$ and $R_t = \bar{f}(t, t)$, this implies, with $a(t) := a(t, t)$, $b(t) := b(t, t), c(t) := c(t, t)$ and analogously for the corresponding quantities with a bar, that

$$r_t = a(t) + b'(t)\Psi_t + \Psi_t' c(t)\Psi_t \tag{19}$$

and

$$R_t = r_t + s_t = \bar{a}(t) + \bar{b}'(t)\Psi_t + \Psi_t' \bar{c}(t)\Psi_t. \tag{20}$$

Comparing (19) and (20) with (11), we obtain the following conditions where $i, j = 1, 2, 3$, namely

$$\begin{cases} a(t) = 0 \\ b^i(t) = \mathbf{1}_{\{i=1\}} \\ c^{ij}(t) = \mathbf{1}_{\{i=j=2\}} \end{cases} \quad \begin{cases} \bar{a}(t) = 0 \\ \bar{b}^i(t) = (1 + \kappa)\mathbf{1}_{\{i=1\}} \\ \bar{c}^{ij}(t) = \mathbf{1}_{\{i=j=2\}\cup\{i=j=3\}}. \end{cases}$$

Using next the fact that

$$p(t, T) = \exp\left[-\int_t^T f(t, s)ds\right], \quad \bar{p}(t, T) = \exp\left[-\int_t^T \bar{f}(t, s)ds\right],$$

and imposing $\frac{p(t,T)}{B_t}$ and $\frac{\bar{p}(t,T)}{\bar{B}_t}$ to be $Q-$martingales, one obtains ordinary differential equations to be satisfied by $c(t, T)$, $b(t, T)$, $a(t, T)$ and analogously for the quantities with a bar. Integrating these ODEs with respect to the second variable and recalling (16) one obtains (for the details see the proof of Proposition 2.1 in [17])

$$\begin{cases} C_t(t, T) + 2FC(t, T) - 2C(t, T)DDC(t, T) + c(t) = 0, & C(T, T) = 0 \\ \bar{C}_t(t, T) + 2F\bar{C}(t, T) - 2\bar{C}(t, T)DD\bar{C}(t, T) + \bar{c}(t) = 0, & \bar{C}(T, T) = 0 \end{cases} \quad (21)$$

with

$$c(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{c}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (22)$$

The special forms of $F, D, c(t)$ and $\bar{c}(t)$ together with boundary conditions $C(T, T) = 0$ and $\bar{C}(T, T) = 0$ imply that only $C^{22}, \bar{C}^{22}, \bar{C}^{33}$ are non-zero and satisfy

$$\begin{cases} C_t^{22}(t, T) - 2b^2 C^{22}(t, T) - 2(\sigma^2)^2 (C^{22}(t, T))^2 + 1 = 0, & C^{22}(T, T) = 0 \\ \bar{C}_t^{22}(t, T) - 2b^2 \bar{C}^{22}(t, T) - 2(\sigma^2)^2 (\bar{C}^{22}(t, T))^2 + 1 = 0, & \bar{C}^{22}(T, T) = 0 \\ \bar{C}_t^{33}(t, T) - 2b^3 \bar{C}^{33}(t, T) - 2(\sigma^3)^2 (\bar{C}^{33}(t, T))^2 + 1 = 0, & \bar{C}^{33}(T, T) = 0 \end{cases} \quad (23)$$

that can be shown to have as solution

$$\begin{cases} C^{22}(t, T) = \bar{C}^{22}(t, T) = \frac{2(e^{(T-t)h^2} - 1)}{2h^2 + (2b^2 + h^2)(e^{(T-t)h^2} - 1)} \\ \bar{C}^{33}(t, T) = \frac{2(e^{(T-t)h^3} - 1)}{2h^3 + (2b^3 + h^3)(e^{(T-t)h^3} - 1)} \end{cases} \quad (24)$$

with $h^i = \sqrt{4(b^i)^2 + 8(\sigma^i)^2} > 0$, $i = 2, 3$.

Next, always by analogy to the proof of Proposition 2.1 in [17], the vectors of coefficients $B(t, T)$ and $\bar{B}(t, T)$ of the first order terms can be seen to satisfy the following system

$$\begin{cases} B_t(t, T) + B(t, T)F - 2B(t, T)DDC(t, T) + b(t) = 0, & B(T, T) = 0 \\ \bar{B}_t(t, T) + \bar{B}(t, T)F - 2\bar{B}(t, T)DD\bar{C}(t, T) + \bar{b}(t) = 0, & \bar{B}(T, T) = 0 \end{cases} \quad (25)$$

with

$$b(t) = [1, 0, 0] \quad \bar{b}(t) = [(1 + \kappa), 0, 0].$$

Noticing similarly as above that only $B^1(t, T), \bar{B}^1(t, T)$ are non-zero, system (25) becomes

$$\begin{cases} B_t^1(t, T) - b^1 B^1(t, T) + 1 = 0 & B^1(T, T) = 0 \\ \bar{B}_t^1(t, T) - b^1 \bar{B}^1(t, T) + (1 + \kappa) = 0 & \bar{B}^1(T, T) = 0 \end{cases} \quad (26)$$

leading to the explicit solution

$$\begin{cases} B^1(t, T) = \frac{1}{b^1}\left(1 - e^{-b^1(T-t)}\right) \\ \bar{B}^1(t, T) = \frac{1+\kappa}{b^1}\left(1 - e^{-b^1(T-t)}\right) = (1+\kappa)B^1(t, T). \end{cases} \tag{27}$$

Finally, $A(t, T)$ and $\bar{A}(t, T)$ have to satisfy

$$\begin{cases} A_t(t, T) + (\sigma^2)^2 C^{22}(t, T) - \frac{1}{2}(\sigma^1)^2(B^1(t, T))^2 = 0, \\ \bar{A}_t(t, T) + (\sigma^2)^2 \bar{C}^{22}(t, T) + (\sigma^3)^2 \bar{C}^{33}(t, T) - \frac{1}{2}(\sigma^1)^2(\bar{B}^1(t, T))^2 = 0 \end{cases} \tag{28}$$

with boundary conditions $A(T, T) = 0, \bar{A}(T, T) = 0$. The explicit expressions can be obtained simply by integrating the above equations.

Summarizing, we have proved the following:

Proposition 3.1 *Assume that the OIS short rate r and the spread s are given by* (*11*) *with the factor processes* Ψ_t^i, $i = 1, 2, 3$, *evolving according to* (*10*) *under the standard martingale measure Q. The time-t price of the OIS bond p(t, T), as defined in* (*3*), *is given by*

$$p(t, T) = \exp[-A(t, T) - B^1(t, T)\Psi_t^1 - C^{22}(t, T)(\Psi_t^2)^2], \tag{29}$$

and the time-t price of the fictitious Libor bond $\bar{p}(t, T)$, *as defined in* (*9*), *by*

$$\begin{aligned} \bar{p}(t, T) &= \exp[-\bar{A}(t, T) - (\kappa + 1)B^1(t, T)\Psi_t^1 - C^{22}(t, T)(\Psi_t^2)^2 - \bar{C}^{33}(t, T)(\Psi_t^3)^2] \\ &= p(t, T)\exp[-\tilde{A}(t, T) - \kappa B^1(t, T)\Psi_t^1 - \bar{C}^{33}(t, T)(\Psi_t^3)^2], \end{aligned} \tag{30}$$

where $\tilde{A}(t, T) := \bar{A}(t, T) - A(t, T)$ *with* $A(t, T)$ *and* $\bar{A}(t, T)$ *given by* (*28*), $B^1(t, T)$ *given by* (*27*) *and* $C^{22}(t, T)$ *and* $C^{33}(t, T)$ *given by* (*24*).

In particular, expression (*30*) gives $\bar{p}(t, T)$ in terms of $p(t, T)$. Based on this we shall derive in the following section the announced *adjustment factor* allowing to pass from pre-crisis quantities to the corresponding post-crisis quantities.

3.3 Forward Measure

The underlying factor model was defined in (*10*) under the standard martingale measure Q. For derivative prices, which we shall determine in the following two sections, it will be convenient to work under forward measures, for which, given the single tenor Δ, we shall consider a generic $(T + \Delta)$-forward measure. The density process to change the measure from Q to $Q^{T+\Delta}$ is

$$\mathcal{L}_t := \frac{d\,Q^{T+\Delta}}{d\,Q}\bigg|_{\mathscr{F}_t} = \frac{p(t, T+\Delta)}{p(0, T+\Delta)} \frac{1}{B_t} \tag{31}$$

from which it follows by (29) and the martingale property of $\left(\frac{p(t,T+\Delta)}{B_t}\right)_{t \le T+\Delta}$ that

$$d\mathcal{L}_t = \mathcal{L}_t \left(-B^1(t, T+\Delta)\sigma^1 dw_t^1 - 2C^{22}(t, T+\Delta)\Psi_t^2 \sigma^2 dw_t^2\right).$$

This implies by Girsanov's theorem that

$$\begin{cases} dw_t^{1,T+\Delta} = dw_t^1 + \sigma^1 B^1(t, T+\Delta)dt \\ dw_t^{2,T+\Delta} = dw_t^2 + 2C^{22}(t, T+\Delta)\Psi_t^2 \sigma^2 dt \\ dw_t^{3,T+\Delta} = dw_t^3 \end{cases} \tag{32}$$

are $Q^{T+\Delta}$−Wiener processes. From the Q−dynamics (10) we then obtain the following $Q^{T+\Delta}$−dynamics for the factors

$$\begin{aligned} d\Psi_t^1 &= -\left[b^1\Psi_t^1 + (\sigma^1)^2 B^1(t, T+\Delta)\right]dt + \sigma^1 dw_t^{1,T+\Delta} \\ d\Psi_t^2 &= -\left[b^2\Psi_t^2 + 2(\sigma^2)^2 C^{22}(t, T+\Delta)\Psi_t^2\right]dt + \sigma^2 dw_t^{2,T+\Delta} \\ d\Psi_t^3 &= -b^3\Psi_t^3 dt + \sigma^3 dw_t^{3,T+\Delta}. \end{aligned} \tag{33}$$

Remark 3.3 While in the dynamics (10) for Ψ_t^i, $(i = 1, 2, 3)$ under Q we had for simplicity assumed a zero mean-reversion level, under the $(T + \Delta)$-forward measure the mean-reversion level is for Ψ_t^1 now different from zero due to the measure transformation.

Lemma 3.1 *Analogously to the case when $p(t, T)$ represents the price of a risk-free zero coupon bond, also for $p(t, T)$ viewed as OIS bond we have that $\frac{p(t,T)}{p(t,T+\Delta)}$ is a $Q^{T+\Delta}$−martingale.*

Proof We have seen that also for OIS bonds as defined in (3) we have that, with B_t as in (1), the ratio $\frac{p(t,T)}{B_t}$ is a Q−martingale. From Bayes' formula we then have

$$E^{T+\Delta}\left\{\frac{p(T,T)}{p(T,T+\Delta)}\mid \mathscr{F}_t\right\} = \frac{E^Q\left\{\frac{1}{p(0,T+\Delta)}\frac{1}{B_{T+\Delta}}\frac{p(T,T)}{p(T,T+\Delta)}\mid \mathscr{F}_t\right\}}{E^Q\left\{\frac{1}{p(0,T+\Delta)}\frac{1}{B_{T+\Delta}}\mid \mathscr{F}_t\right\}}$$

$$= \frac{E^Q\left\{\frac{p(T,T)}{p(T,T+\Delta)}E^Q\left\{\frac{1}{B_{T+\Delta}}\mid \mathscr{F}_T\right\}\mid \mathscr{F}_t\right\}}{\frac{p(t,T+\Delta)}{B_t}} = \frac{B_t E^Q\left\{\frac{p(T,T)}{p(T,T+\Delta)}\frac{p(T,T+\Delta)}{B_T}\mid \mathscr{F}_t\right\}}{p(t,T+\Delta)}$$

$$= \frac{B_t E^Q\left\{\frac{p(T,T)}{B_T}\mid \mathscr{F}_t\right\}}{p(t,T+\Delta)} = \frac{p(t,T)}{p(t,T+\Delta)},$$

thus proving the statement of the lemma. $\qquad\square$

We recall that we denote the expectation with respect to the measure $Q^{T+\Delta}$ by $E^{T+\Delta}\{\cdot\}$. The dynamics in (33) lead to Gaussian distributions for Ψ_t^i, $i = 1, 2, 3$ that, given $B^1(\cdot)$ and $C^{22}(\cdot)$, have mean and variance

$$E^{T+\Delta}\{\Psi_t^i\} = \bar{\alpha}_t^i = \bar{\alpha}_t^i(b^i, \sigma^i) \quad, \quad Var^{T+\Delta}\{\Psi_t^i\} = \bar{\beta}_t^i = \bar{\beta}_t^i(b^i, \sigma^i),$$

which can be explicitly computed. More precisely, we have

$$
\begin{cases}
\bar{\alpha}_t^1 &= e^{-b^1 t}\left[\Psi_0^1 - \frac{(\sigma^1)^2}{2(b^1)^2}e^{-b^1(T+\Delta)}(1 - e^{2b^1 t}) - \frac{(\sigma^1)^2}{(b^1)^2}(1 - e^{b^1 t})\right] \\
\bar{\beta}_t^1 &= e^{-2b^1 t}(e^{2b^1 t} - 1)\frac{(\sigma^1)^2}{2(b^1)} \\
\bar{\alpha}_t^2 &= e^{-(b^2 t + 2(\sigma^2)^2 \tilde{C}^{22}(t, T+\Delta))}\Psi_0^2 \\
\bar{\beta}_t^2 &= e^{-(2b^2 t + 4(\sigma^2)^2 \tilde{C}^{22}(t, T+\Delta))}\int_0^t e^{2b^2 s + 4(\sigma^2)^2 \tilde{C}^{22}(s, T+\Delta)}(\sigma^2)^2 ds \\
\bar{\alpha}_t^3 &= e^{-b^3 t}\Psi_0^3 \\
\bar{\beta}_t^3 &= e^{-2b^3 t}\frac{(\sigma^3)^2}{2b^3}(e^{2b^3 t} - 1),
\end{cases}
\tag{34}
$$

with

$$
\begin{aligned}
\tilde{C}^{22}(t, T+\Delta) &= \frac{2(2\log(2b^2(e^{(T+\Delta-t)h^2} - 1) + h^2(e^{(T+\Delta-t)h^2} + 1)) + t(2b^2 + h^2))}{(2b^2 + h^2)(2b^2 - h^2)} \\
&\quad - \frac{2(2\log(2b^2(e^{(T+\Delta)h^2} - 1) + h^2(e^{(T+\Delta)h^2} + 1)))}{(2b^2 + h^2)(2b^2 - h^2)}
\end{aligned}
\tag{35}
$$

and $h^2 = \sqrt{(2b^2)^2 + 8(\sigma^2)^2}$, and where we have assumed deterministic initial values Ψ_0^1, Ψ_0^2 and Ψ_0^3. For details of the above computation see the proof of Corollary 4.1.3. in Meneghello [25].

4 Pricing of Linear Interest Rate Derivatives

We have discussed in Sect. 3.2 the pricing of OIS and Libor bonds in the Gaussian, exponentially quadratic short rate model introduced in Sect. 3.1. In the remaining part of the paper we shall be concerned with the pricing of interest rate derivatives, namely with derivatives having the Libor rate as underlying rate. In the present section we shall deal with the basic linear derivatives, namely FRAs and interest rate swaps, while nonlinear derivatives will then be dealt with in the following Sect. 5. For the FRA rates discussed in the next Sect. 4.1 we shall in Sect. 4.1.1 exhibit an *adjustment factor* allowing to pass from the single-curve FRA rate to the multi-curve FRA rate.

4.1 FRAs

We start by recalling the definition of a standard forward rate agreement. We emphasize that we use a text-book definition which differs slightly from a market definition, see Mercurio [26].

Definition 4.1 Given the time points $0 \leq t \leq T < T + \Delta$, a *forward rate agreement* (FRA) is an OTC derivative that allows the holder to lock in at the generic date $t \leq T$ the interest rate between the inception date T and the maturity $T + \Delta$ at a fixed value R. At maturity $T + \Delta$ a payment based on the interest rate R, applied to a notional amount of N, is made and the one based on the relevant floating rate *(generally the spot Libor rate $L(T; T, T + \Delta)$)* is received.

Recalling that for the Libor rate we had postulated the relation (8) to hold at the inception time T, namely

$$L(T; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{1}{\bar{p}(T, T + \Delta)} - 1 \right),$$

the price, at $t \leq T$, of the FRA with fixed rate R and notional N can be computed under the $(T + \Delta)$-forward measure as

$$
\begin{aligned}
P^{FRA}&(t; T, T + \Delta, R, N) \\
&= N \Delta p(t, T + \Delta) E^{T+\Delta} \{ L(T; T, T + \Delta) - R \mid \mathscr{F}_t \} \\
&= N p(t, T + \Delta) E^{T+\Delta} \left\{ \frac{1}{\bar{p}(T, T + \Delta)} - (1 + \Delta R) \mid \mathscr{F}_t \right\},
\end{aligned}
\tag{36}
$$

Defining

$$\bar{\nu}_{t,T} := E^{T+\Delta} \left\{ \frac{1}{\bar{p}(T, T + \Delta)} \mid \mathscr{F}_t \right\}, \tag{37}$$

it is easily seen from (36) that the *fair rate of the FRA, namely the FRA rate*, is given by

$$\bar{R}_t = \frac{1}{\Delta} \left(\bar{\nu}_{t,T} - 1 \right). \tag{38}$$

In the *single-curve case* we have instead

$$R_t = \frac{1}{\Delta} \left(\nu_{t,T} - 1 \right), \tag{39}$$

where, given that $\frac{p(\cdot, T)}{p(\cdot, T + \Delta)}$ is a $Q^{T+\Delta}$-martingale (see Lemma 3.1),

$$\nu_{t,T} := E^{T+\Delta} \left\{ \frac{1}{p(T, T + \Delta)} \mid \mathscr{F}_t \right\} = \frac{p(t, T)}{p(t, T + \Delta)}, \tag{40}$$

which is the classical expression for the FRA rate in the single-curve case. Notice that, contrary to (37), the expression in (40) can be explicitly computed on the basis of bond price data without requiring an interest rate model.

4.1.1 Adjustment Factor

We shall show here the following:

Proposition 4.1 *We have the relationship*

$$\bar{\nu}_{t,T} = \nu_{t,T} \cdot Ad_t^{T,\Delta} \cdot Res_t^{T,\Delta} \tag{41}$$

with

$$
Ad_t^{T,\Delta} := E^Q \left\{ \frac{p(T, T + \Delta)}{\bar{p}(T, T + \Delta)} \mid \mathscr{F}_t \right\} = E^Q \left\{ \exp \Big[\tilde{A}(T, T + \Delta) \right.
$$
$$
\left. + \kappa B^1(T, T + \Delta)\Psi_T^1 + \bar{C}^{33}(T, T + \Delta)(\Psi_T^3)^2 \Big] \mid \mathscr{F}_t \right\} \tag{42}
$$

and

$$
Res_t^{T,\Delta} = \exp \left[-\kappa \frac{(\sigma^1)^2}{2(b^1)^3} \left(1 - e^{-b^1 \Delta} \right) \left(1 - e^{-b^1(T-t)} \right)^2 \right], \tag{43}
$$

where $\tilde{A}(t, T)$ is defined after (30), $B^1(t, T)$ in (27) and $\bar{C}^{33}(t, T)$ in (24).

Proof Firstly, from (30) we obtain

$$
\frac{p(T, T + \Delta)}{\bar{p}(T, T + \Delta)} = e^{\tilde{A}(T,T+\Delta)+\kappa B^1(T,T+\Delta)\Psi_T^1+\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2}. \tag{44}
$$

In (37) we now change back from the $(T + \Delta)$-forward measure to the standard martingale measure using the density process $\mathscr{L}_t$ given in (31). Using furthermore the above expression for the ratio of the OIS and the Libor bond prices and taking into account the definition of the short rate r_t in terms of the factor processes, we obtain

$$
\bar{\nu}_{t,T} = E^{T+\Delta} \left\{ \frac{1}{\bar{p}(T, T + \Delta)} \Big| \mathscr{F}_t \right\} = \mathscr{L}_t^{-1} E^Q \left\{ \frac{\mathscr{L}_T}{\bar{p}(T, T + \Delta)} \Big| \mathscr{F}_t \right\}
$$
$$
= \frac{1}{p(t, T + \Delta)} E^Q \left\{ \exp\left(-\int_t^T r_u du \right) \frac{p(T, T + \Delta)}{\bar{p}(T, T + \Delta)} \Big| \mathscr{F}_t \right\}
$$
$$
= \frac{1}{p(t, T + \Delta)} \exp[\tilde{A}(T, T + \Delta)] E^Q \left\{ e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2} \Big| \mathscr{F}_t \right\}
$$

$$\cdot E^Q \left\{ e^{- \int_t^T (\Psi_u^1 + (\Psi_u^2)^2) du} e^{\kappa B^1(T, T+\Delta) \Psi_T^1} \Big| \mathscr{F}_t \right\}$$

$$= \frac{1}{p(t, T+\Delta)} \exp[\tilde{A}(T, T+\Delta)] E^Q \left\{ e^{\bar{C}^{33}(T, T+\Delta)(\Psi_T^3)^2} \Big| \mathscr{F}_t \right\}$$

$$\cdot E^Q \left\{ e^{- \int_t^T \Psi_u^1 du} e^{\kappa B^1(T, T+\Delta) \Psi_T^1} \Big| \mathscr{F}_t \right\} E^Q \left\{ e^{- \int_t^T (\Psi_u^2)^2 du} \Big| \mathscr{F}_t \right\}, \qquad (45)$$

where we have used the independence of the factors Ψ^i, $i = 1, 2, 3$ under Q.

Recall now from the theory of affine processes (see e.g. Lemma 2.1 in Grbac and Runggaldier [18]) that, for a process Ψ_t^1 satisfying (10), we have for all $\delta, K \in \mathbb{R}$

$$E^Q \left\{ \exp\left[- \int_t^T \delta \Psi_u^1 du - K \Psi_T^1 \right] \mid \mathscr{F}_t \right\} = \exp[\alpha^1(t, T) - \beta^1(t, T) \Psi_t^1], \qquad (46)$$

where

$$\begin{cases} \beta^1(t, T) = K e^{-b^1(T-t)} - \frac{\delta}{b^1} \left(e^{-b^1(T-t)} - 1 \right) \\ \alpha^1(t, T) = \frac{(\sigma^1)^2}{2} \int_t^T (\beta^1(u, T))^2 du. \end{cases}$$

Setting $K = -\kappa B^1(T, T+\Delta)$ and $\delta = 1$, and recalling from (27) that $B^1(t, T) = \frac{1}{b^1} \left(1 - e^{-b^1(T-t)} \right)$, this leads to

$$E^Q \left\{ e^{- \int_t^T \Psi_u^1 du} e^{\kappa B^1(T, T+\Delta) \Psi_T^1} \Big| \mathscr{F}_t \right\}$$

$$= \exp\left[\frac{(\sigma^1)^2}{2} (\kappa B^1(T, T+\Delta))^2 \int_t^T e^{-2b^1(T-u)} du \right.$$

$$- \kappa B^1(T, T+\Delta)(\sigma^1)^2 \int_t^T B^1(u, T) e^{-b^1(T-u)} du + \frac{(\sigma^1)^2}{2} \int_t^T (B^1(u, T))^2 du$$

$$+ \left(\kappa B^1(T, T+\Delta) e^{-b^1(T-t)} - B^1(t, T) \right) \Psi_t^1 \Bigg]. \qquad (47)$$

On the other hand, from the results of Sect. 3.2 we also have that, for a process Ψ_t^2 satisfying (10),

$$E^Q \left\{ \exp\left[- \int_t^T (\Psi_u^2)^2 du \right] \mid \mathscr{F}_t \right\} = \exp\left[-\alpha^2(t, T) - C^{22}(t, T)(\Psi_t^2)^2 \right],$$

where $C^{22}(t, T)$ corresponds to (24) and (see (28))

$$\alpha^2(t, T) = (\sigma^2)^2 \int_t^T C^{22}(u, T) du.$$

This implies that

$$E^Q \left\{ \exp\left[-\int_t^T (\Psi_u^2)^2 du \right] \mid \mathscr{F}_t \right\}$$

$$= \exp\left[-(\sigma^2)^2 \int_t^T C^{22}(u,T) du - C^{22}(t,T) \left(\Psi_t^2 \right)^2 \right]. \qquad (48)$$

Replacing (47) and (48) into (45), and recalling the expression for $p(t,T)$ in (29) with $A(\cdot)$, $B^1(\cdot)$, $C^{22}(\cdot)$ according to (28), (27) and (24) respectively, we obtain

$$\bar{\nu}_{t,T} = \frac{p(t,T)}{p(t,T+\Delta)} e^{\tilde{A}(T,T+\Delta)} E^Q \left[e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2} \mid \mathscr{F}_t \right]$$

$$\cdot \exp\left[\frac{(\sigma^1)^2}{2} (\kappa B^1(T,T+\Delta))^2 \int_t^T e^{-2b^1(T-u)} du + \kappa B^1(T,T+\Delta) e^{-b^1(T-t)} \Psi_t^1 \right]$$

$$\cdot \exp\left[-\kappa B^1(T,T+\Delta)(\sigma^1)^2 \int_t^T B^1(u,T) e^{-b^1(T-u)} du \right]. \qquad (49)$$

We recall the expression (44) for $\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)}$ and the fact that, according to (46), we have

$$E^Q \left\{ e^{\kappa B^1(T,T+\Delta)\Psi_T^1} \mid \mathscr{F}_t \right\}$$

$$= \exp\left[\frac{(\sigma^1)^2}{2} (\kappa B^1(T,T+\Delta))^2 \int_t^T e^{-2b^1(T-u)} du + \kappa B^1(T,T+\Delta) e^{-b^1(T-t)} \Psi_t^1 \right].$$

Inserting these expressions into (49) we obtain the result, namely

$$\bar{\nu}_{t,T} = \frac{p(t,T)}{p(t,T+\Delta)} E^Q \left\{ \frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \mid \mathscr{F}_t \right\}$$

$$\cdot \exp\left[-\kappa B^1(T,T+\Delta)(\sigma^1)^2 \int_t^T B^1(u,T) e^{-b^1(T-u)} du \right]$$

$$= \frac{p(t,T)}{p(t,T+\Delta)} E^Q \left\{ \frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \mid \mathscr{F}_t \right\}$$

$$\cdot \exp\left[-\frac{\kappa}{b^1} (e^{-b^1\Delta} - 1)(\sigma^1)^2 \left(\frac{1}{2(b^1)^2} (1 - e^{-2b^1(T-t)}) - \frac{1}{(b^1)^2} (1 - e^{-b^1(T-t)}) \right) \right], \qquad (50)$$

where we have also used the fact that

$$\int_t^T B^1(u,T) e^{-b^1(T-u)} du = \int_t^T \frac{1}{b^1} \left(1 - e^{-b^1(T-u)} \right) e^{-b^1(T-u)} du$$

$$= -\frac{1}{2(b^1)^2} \left(1 - e^{-2b^1(T-t)} \right) + \frac{1}{(b^1)^2} \left(1 - e^{-b^1(T-t)} \right).$$

$$\square$$

Remark 4.1 The adjustment factor $Ad_t^{T,\Delta}$ allows for some intuitive interpretations. Here we mention only the easiest one for the case when $\kappa = 0$ (independence of r_t and s_t). In this case we have $r_t + s_t > r_t$ implying that $\bar{p}(T, T + \Delta) < p(T, T + \Delta)$ so that $Ad_t^{T,\Delta} \geq 1$. Furthermore, always for $\kappa = 0$, the residual factor has value $Res_t^{T,\Delta} = 1$. All this in turn implies $\bar{\nu}_{t,T} \geq \nu_{t,T}$ and with it $\bar{R}_t \geq R_t$, which is what one would expect to be the case.

Remark 4.2 (*Calibration to the initial term structure*). The parameters in the model (10) for the factors Ψ_t^i and thus also in the model (11) for the short rate r_t and the spread s_t are the coefficients b^i and σ^i for $i = 1, 2, 3$. From (14) notice that, for $i = 1, 2$, these coefficients enter the expressions for the OIS bond prices $p(t, T)$ that can be assumed to be observable since they can be bootstrapped from the market quotes for the OIS swap rates. We may thus assume that these coefficients, i.e. b^i and σ^i for $i = 1, 2$, can be calibrated as in the pre-crisis single-curve short rate models. It remains to calibrate b^3, σ^3 and, possibly the correlation coefficient κ. Via (15) they affect the prices of the fictitious Libor bonds $\bar{p}(t, T)$ that are, however, not observable. One may observe though the FRA rates R_t and $\bar{R}_t$ and thus also $\nu_{t,T}$, as well as $\bar{\nu}_{t,T}$. Via (41) this would then allow one to calibrate also the remaining parameters. This task would turn out to be even simpler if one would have access to the value of κ by other means.

We emphasize that in order to ensure a good fit to the initial bond term structure, a deterministic shift extension of the model or time-dependent coefficients b^i could be considered. We recall also that we have assumed the mean-reversion level equal to zero for simplicity; in practice it would be one more coefficient to be calibrated for each factor Ψ_t^i.

4.2 Interest Rate Swaps

We first recall the notion of a *(payer) interest rate swap*. Given a collection of dates $0 \leq T_0 < T_1 < \cdots < T_n$ with $\gamma \equiv \gamma_k := T_k - T_{k-1}$ $(k = 1, \cdots, n)$, as well as a notional amount N, a payer swap is a financial contract, where a stream of interest payments on the notional N is made at a fixed rate R in exchange for receiving an analogous stream corresponding to the Libor rate. Among the various possible conventions concerning the fixing for the Libor and the payment dates, we choose here the one where, for each interval $[T_{k-1}, T_k]$, the Libor rates are fixed in advance and the payments are made in arrears. The swap is thus initiated at T_0 and the first payment is made at T_1. A *receiver swap* is completely symmetric with the interest at the fixed rate being received; here we concentrate on payer swaps.

The arbitrage-free price of the swap, evaluated at $t \leq T_0$, is given by the following expression where, analogously to $E^{T+\Delta}\{\cdot\}$, we denote by $E^{T_k}\{\cdot\}$ the expectation with respect to the forward measure Q^{T_k} $(k = 1, \cdots, n)$

$$P^{Sw}(t; T_0, T_n, R) = \gamma \sum_{k=1}^{n} p(t, T_k) E^{T_k} \{L(T_{k-1}; T_{k-1}, T_k) - R|\mathscr{F}_t\}$$

$$= \gamma \sum_{k=1}^{n} p(t, T_k) (L(t; T_{k-1}, T_k) - R). \tag{51}$$

For easier notation we have assumed the notional to be 1, i.e. $N = 1$.

We shall next obtain an explicit expression for $P^{Sw}(t; T_0, T_n, R)$ starting from the first equality in (51). To this effect, recalling from (24) that $C^{22}(t, T) = \bar{C}^{22}(t, T)$, introduce again some shorthand notation, namely

$$A_k := \bar{A}(T_{k-1}, T_k), B_k^1 := B^1(T_{k-1}, T_k),$$
$$C_k^{22} := C^{22}(T_{k-1}, T_k) = \bar{C}^{22}(T_{k-1}, T_k), \bar{C}_k^{33} := \bar{C}^{33}(T_{k-1}, T_k). \tag{52}$$

The crucial quantity to be computed in (51) is the following one

$$E^{T_k}\{\gamma L(T_{k-1}; T_{k-1}, T_k)|\mathscr{F}_t\} = E^{T_k}\left\{\frac{1}{\bar{p}(T_{k-1}, T_k)}|\mathscr{F}_t\right\} - 1$$
$$= e^{A_k} E^{T_k}\{\exp((\kappa + 1)B_k^1 \Psi_{T_{k-1}}^1 + C_k^{22}(\Psi_{T_{k-1}}^2)^2 + \bar{C}_k^{33}(\Psi_{T_{k-1}}^3)^2)|\mathscr{F}_t\} - 1, \tag{53}$$

where we have used the first relation on the right in (30). The expectations in (53) have to be computed under the measures Q^{T_k}, under which, by analogy to (33), the factors have the dynamics

$$\begin{aligned} d\Psi_t^1 &= -\left[b^1\Psi_t^1 + (\sigma^1)^2 B^1(t, T_k)\right] dt + \sigma^1 dw_t^{1,k} \\ d\Psi_t^2 &= -\left[b^2\Psi_t^2 + 2(\sigma^2)^2 C^{22}(t, T_k)\Psi_t^2\right] dt + \sigma^2 dw_t^{2,k} \\ d\Psi_t^3 &= -b^3\Psi_t^3 dt + \sigma^3 dw_t^{3,k}. \end{aligned} \tag{54}$$

where $w^{i,k}$, $i = 1, 2, 3$, are independent Wiener processes with respect to Q^{T_k}. A straightforward generalization of (46) to the case where the factor process Ψ_t^1 satisfies the following affine Hull–White model

$$d\Psi_t^1 = (a^1(t) - b^1\Psi_t^1)dt + \sigma^1 dw_t$$

can be obtained as follows

$$E^Q\left\{\exp\left[-\int_t^T \delta\Psi_u^1 du - K\Psi_T^1\right] | \mathscr{F}_t\right\} = \exp[\alpha^1(t, T) - \beta^1(t, T)\Psi_t^1], \tag{55}$$

with

$$\begin{cases} \beta^1(t, T) = Ke^{-b^1(T-t)} - \frac{\delta}{b^1}\left(e^{-b^1(T-t)} - 1\right) \\ \alpha^1(t, T) = \frac{(\sigma^1)^2}{2} \int_t^T (\beta^1(u, T))^2 du - \int_t^T a^1(u)\beta^1(u, T)du. \end{cases} \tag{56}$$

We apply this result to our situation where under Q^{T_k} the process Ψ_t^1 satisfies the first SDE in (54) and thus corresponds to the above dynamics with $a^1(t) = -(\sigma^1)^2 B^1(t, T_k)$. Furthermore, setting $K = -(\kappa + 1) B_k^1$ and $\delta = 0$, we obtain for the first expectation in the second line of (53)

$$E^{T_k}\{\exp((\kappa + 1)B_k^1\Psi_{T_{k-1}}^1 | \mathscr{F}_t\} = \exp[\Gamma^1(t, T_k) - \rho^1(t, T_k)\,\Psi_t^1], \qquad (57)$$

with

$$\begin{cases} \rho^1(t, T_k) = -(\kappa + 1)B_k^1 e^{-b^1(T_k - t)} \\ \Gamma^1(t, T_k) = \frac{(\sigma^1)^2}{2} \int_t^{T_k} \left(\rho^1(u, T_k)\right)^2 du + (\sigma^1)^2 \int_t^{T_k} B^1(u, T_k)\rho^1(u, T_k)du. \end{cases} \qquad (58)$$

For the remaining two expectations in the second line of (53) we shall use the following:

Lemma 4.1 *Let a generic process Ψ_t satisfy the dynamics*

$$d\Psi_t = b(t)\Psi_t dt + \sigma\,dw_t \qquad (59)$$

with w_t a Wiener process. Then, for all $C \in \mathbb{R}$ such that $E^Q\left\{\exp\left[C\,(\Psi_T)^2\right]\right\} < \infty$, we have

$$E^Q\left\{\exp\left[C\,(\Psi_T)^2\right] | \mathscr{F}_t\right\} = \exp\left[\Gamma(t, T) - \rho(t, T)\,(\Psi_t)^2\right] \qquad (60)$$

with $\rho(t, T)$ and $\Gamma(t, T)$ satisfying

$$\begin{cases} \rho_t(t, T) + 2b(t)\rho(t, T) - 2(\sigma)^2\,(\rho(t, T))^2 = 0; \quad \rho(T, T) = -C \\ \Gamma_t(t, T) = (\sigma)^2\rho(t, T). \end{cases} \qquad (61)$$

Proof An application of Itô's formula yields that the nonnegative process $\Phi_t := (\Psi_t)^2$ satisfies the following SDE

$$d\Phi_t = \left((\sigma)^2 + 2b(t)\,\Phi_t\right)dt + 2\sigma\,\sqrt{\Phi_t}\,dw_t.$$

We recall that a process Φ_t given in general form by

$$d\Phi_t = (a + \lambda(t)\Phi_t)dt + \eta\,\sqrt{\Phi_t}\,dw_t,$$

with $a, \eta > 0$ and $\lambda(t)$ a deterministic function, is a CIR process. Thus, $(\Psi_t)^2$ is equivalent in distribution to a CIR process with coefficients given by

$$\lambda(t) = 2b(t)\quad,\quad \eta = 2\sigma\quad,\quad a = (\sigma)^2.$$

From the theory of affine term structure models (see e.g. Lamberton and Lapeyre [23], or Lemma 2.2 in Grbac and Runggaldier [18]) it now follows that

$$E^Q\left\{\exp\left[C\,(\Psi_T)^2\right]\mid\mathscr{F}_t\right\} = E^Q\left\{\exp\left[C\,\Phi_T\right]\mid\mathscr{F}_t\right\} = \exp\left[\Gamma(t,T)-\rho(t,T)\,\Phi_t\right]$$
$$= \exp\left[\Gamma(t,T)-\rho(t,T)\,(\Psi_t)^2\right]$$

with $\rho(t,T)$ and $\Gamma(t,T)$ satisfying (61).

Corollary 4.1 *When $b(t)$ is constant with respect to time, i.e. $b(t)\equiv b$, so that also $\lambda(t)\equiv\lambda$, then the equations for $\rho(t,T)$ and $\Gamma(t,T)$ in (61) admit an explicit solution given by*

$$\begin{cases} \rho(t,T)=\dfrac{4bhe^{2b(T-t)}}{4(\sigma)^2he^{2b(T-t)}-1} & \text{with}\ \ h:=\dfrac{C}{4(\sigma)^2C+4b} \\[2mm] \Gamma(t,T)=-(\sigma)^2\displaystyle\int_t^T\rho(u,T)du. \end{cases} \tag{62}$$

Coming now to the second expectation in the second line of (53) and using the second equation in (54), we set

$$b(t):=-\left[b^2+2(\sigma^2)^2C^{22}(t,T_k)\right],\ \sigma:=\sigma^2,\ C=C_k^{22}$$

and apply Lemma 4.1, provided that the parameters b^2 and σ^2 of the process Ψ^2 are such that $C=C_k^{22}$ satisfies the assumption from the lemma. We thus obtain

$$E^{T_k}\{\exp(C_k^{22}(\Psi_{T_{k-1}}^2)^2)|\mathscr{F}_t\} = \exp[\Gamma^2(t,T_k)-\rho^2(t,T_k)(\Psi_t^2)^2], \tag{63}$$

with $\rho^2(t,T)$, $\Gamma^2(t,T)$ satisfying

$$\begin{cases} \rho_t^2(t,T)-2\left[b^2+2(\sigma^2)^2C^{22}(t,T_k)\right]\rho^2(t,T)-2(\sigma^2)^2(\rho^2(t,T))^2=0 \\[1mm] \rho^2(T_k,T_k)=-C_k^{22} \\[1mm] \Gamma^2(t,T)=-(\sigma^2)^2\displaystyle\int_t^T\rho^2(u,T)du. \end{cases} \tag{64}$$

Finally, for the third expectation in the second line of (53), we may take advantage of the fact that the dynamics of Ψ_t^3 do not change when passing from the measure Q to the forward measure Q^{T_k}. We can then apply Lemma 4.1, this time with (see the third equation in (54))

$$b(t):=-b^3,\ \sigma:=\sigma^3,\ C=\bar{C}_k^{33}$$

and ensuring that the parameters b^3 and σ^3 of the process Ψ^3 are such that $C=\bar{C}_k^{33}$ satisfies the assumption from the lemma. Since $b(t)$ is constant with respect to time, also Corollary 4.1 applies and we obtain

$$E^{T_k}\{\exp(\bar{C}_k^{33}(\Psi_{T_{k-1}}^3)^2)|\mathscr{F}_t\} = \exp[\Gamma^3(t,T_k)-\rho^3(t,T_k)(\Psi_t^3)^2],$$

where

$$\begin{cases} \rho^3(t, T_k) = \dfrac{-4b^3 h_k^3 e^{-2b^3(T_k-t)}}{4(\sigma^3)^2 h_k^3 e^{-2b^3(T_k-t)} - 1} \quad \text{with} \quad h_k^3 = \dfrac{\bar{C}_k^{33}}{4(\sigma^3)^2 \bar{C}_k^{33} - 4b^3} \\ \Gamma^3(t, T_k) = -(\sigma^3)^2 \displaystyle\int_t^{T_k} \rho^3(u, T_k)\,du. \end{cases} \tag{65}$$

With the use of the explicit expressions for the expectations in (53), and taking also into account the expression for $p(t, T)$ in (29), it follows immediately that the arbitrage-free swap price in (51) can be expressed according to the following

Proposition 4.2 *The price of a payer interest rate swap at $t \le T_0$ is given by*

$$\begin{aligned} P^{Sw}(t; T_0, T_n, R) &= \gamma \sum_{k=1}^n p(t, T_k) E^{T_k} \left\{ L(T_{k-1}; T_{k-1}, T_k) - R | \mathscr{F}_t \right\} \\ &= \sum_{k=1}^n p(t, T_k) \Big(D_{t,k} e^{-\rho^1(t,T_k)\Psi_t^1 - \rho^2(t,T_k)(\Psi_t^2)^2 - \rho^3(t,T_k)(\Psi_t^3)^2} - (R\gamma + 1) \Big) \\ &= \sum_{k=1}^n \Big(D_{t,k} e^{-A_{t,k}} e^{-\tilde{B}_{t,k}^1 \Psi_t^1 - \tilde{C}_{t,k}^{22}(\Psi_t^2)^2 - \tilde{C}_{t,k}^{33}(\Psi_t^3)^2} \\ &\qquad\quad - (R\gamma + 1) e^{-A_{t,k}} e^{-B_{t,k}^1 \Psi_t^1 - C_{t,k}^{22}(\Psi_t^2)^2} \Big), \end{aligned} \tag{66}$$

where

$$\begin{aligned} &A_{t,k} := A(t, T_k), \ B_{t,k}^1 := B^1(t, T_k), \ C_{t,k}^{22} := C^{22}(t, T_k) \\ &\tilde{B}_{t,k}^1 := B_{t,k}^1 + \rho^1(t, T_k), \ \tilde{C}_{t,k}^{22} := C_{t,k}^{22} + \rho^2(t, T_k), \ \tilde{C}_{t,k}^{33} := \rho^3(t, T_k) \\ &D_{t,k} := e^{A_k} \exp[\Gamma^1(t, T_k) + \Gamma^2(t, T_k) + \Gamma^3(t, T_k)], \end{aligned} \tag{67}$$

with $\rho^i(t, T_k)$, $\Gamma^i(t, T_k)$ $(i = 1, 2, 3)$ determined according to (58), (64), and (65) respectively and with A_k as in (52).

5 Nonlinear/optional Interest Rate Derivatives

In this section we consider the main nonlinear interest rate derivatives with the Libor rate as underlying. They are also called *optional derivatives* since they have the form of an option. In Sect. 5.1 we shall consider the case of caps and, symmetrically, that of floors. In the subsequent Sect. 5.2 we shall then concentrate on swaptions as options on a payer swap of the type discussed in Sect. 4.2.

5.1 Caps and Floors

Since floors can be treated in a completely symmetric way to the caps simply by interchanging the roles of the fixed rate and the Libor rate, we shall concentrate

here on caps. Furthermore, to keep the presentation simple, we consider here just a single caplet for the time interval $[T, T + \Delta]$ and for a fixed rate R (recall also that we consider just one tenor Δ). The payoff of the caplet at time $T + \Delta$ is thus $\Delta(L(T; T, T + \Delta) - R)^+$, assuming the notional $N = 1$, and its time-t price $P^{Cpl}(t; T + \Delta, R)$ is given by the following risk-neutral pricing formula under the forward measure $Q^{T+\Delta}$

$$P^{Cpl}(t; T + \Delta, R) = \Delta p(t, T + \Delta) E^{T+\Delta} \left\{ (L(T; T, T + \Delta) - R)^+ \mid \mathscr{F}_t \right\}.$$

In view of deriving pricing formulas, recall from Sect. 3.3 that, under the $(T + \Delta)-$ forward measure, at time T the factors Ψ_T^i have independent Gaussian distributions (see (34)) with mean and variance given, for $i = 1, 2, 3$, by

$$E^{T+\Delta}\{\Psi_t^i\} = \bar{\alpha}_t^i = \bar{\alpha}_t^i(b^i, \sigma^i), \qquad Var^{T+\Delta}\{\Psi_t^i\} = \bar{\beta}_t^i = \bar{\beta}_t^i(b^i, \sigma^i).$$

In the formulas below we shall consider the joint probability density function of $(\Psi_T^1, \Psi_T^2, \Psi_T^3)$ under the $T + \Delta$ forward measure, namely, using the independence of the processes Ψ_t^i, $(i = 1, 2, 3)$,

$$f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x_1, x_2, x_3) = \prod_{i=1}^{3} f_{\Psi_T^i}(x_i) = \prod_{i=1}^{3} \mathcal{N}(x_i, \bar{\alpha}_T^i, \bar{\beta}_T^i), \tag{68}$$

and use the shorthand notation $f_i(\cdot)$ for $f_{\Psi_T^i}(\cdot)$ in the sequel. We shall also write $\bar{A}, B^1, C^{22}, \bar{C}^{33}$ for the corresponding functions evaluated at $(T, T + \Delta)$ and given in (28), (27) and (24) respectively.

Setting $\tilde{R} := 1 + \Delta R$, and recalling the first equality in (30), the time-0 price of the caplet can be expressed as

$$\begin{aligned} P^{Cpl}(0; T + \Delta, R) &= \Delta p(0, T + \Delta) E^{T+\Delta} \left\{ (L(T; T, T + \Delta) - R)^+ \right\} \\ &= p(0, T + \Delta) E^{T+\Delta} \left\{ \left(\frac{1}{\bar{p}(T, T + \Delta)} - \tilde{R} \right)^+ \right\} \\ &= p(0, T + \Delta) E^{T+\Delta} \left\{ \left(e^{\bar{A}+(\kappa+1)B^1 \Psi_T^1 + C^{22}(\Psi_T^2)^2 + \bar{C}^{33}(\Psi_T^3)^2} - \tilde{R} \right)^+ \right\} \\ &= p(0, T + \Delta) \int_{\mathbb{R}^3} \left(e^{\bar{A}+(\kappa+1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R} \right)^+ \\ &\quad \cdot f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x, y, z) d(x, y, z). \end{aligned} \tag{69}$$

To proceed, we extend to the multi-curve context an idea suggested in Jamshidian [19] (where it is applied to the pricing of coupon bonds) by considering the function

$$g(x, y, z) := \exp[\bar{A} + (\kappa + 1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2]. \tag{70}$$

Noticing that $\bar{C}^{33}(T, T + \Delta) > 0$ (see (24) together with the fact that $h^3 > 0$ and $2b^3 + h^3 > 0$), for fixed x, y the function $g(x, y, z)$ can be seen to be continuous and increasing for $z \geq 0$ and decreasing for $z < 0$ with $\lim_{z \to \pm\infty} g(x, y, z) = +\infty$. It will now be convenient to introduce some objects according to the following:

Definition 5.1 Let a set $M \subset \mathbb{R}^2$ be given by

$$M := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq \tilde{R}\} \tag{71}$$

and let M^c be its complement. Furthermore, for $(x, y) \in M$ let

$$\bar{z}^1 = \bar{z}^1(x, y), \quad \bar{z}^2 = \bar{z}^2(x, y)$$

be the solutions of $g(x, y, z) = \tilde{R}$. They satisfy $\bar{z}^1 \leq 0 \leq \bar{z}^2$.

Notice that, for $z \leq \bar{z}^1 \leq 0$ and $z \geq \bar{z}^2 \geq 0$, we have $g(x, y, z) \geq g(x, y, \bar{z}^k) = \tilde{R}$, and for $z \in (\bar{z}^1, \bar{z}^2)$, we have $g(x, y, z) < \tilde{R}$. In M^c we have $g(x, y, z) \geq g(x, y, 0) > \tilde{R}$ and thus no solution of the equation $g(x, y, z) = \tilde{R}$.

In view of the main result of this subsection, given in Proposition 5.1 below, we prove as a preliminary the following:

Lemma 5.1 *Assuming that the (nonnegative) coefficients b^3, σ^3 in the dynamics (10) of the factor Ψ_t^3 satisfy the condition*

$$b^3 \geq \frac{\sigma^3}{\sqrt{2}}, \tag{72}$$

we have that $1 - 2\bar{\beta}_T^3 \bar{C}^{33} > 0$, where $\bar{C}^{33} = \bar{C}^{33}(T, T + \Delta)$ is given by (24) and where $\bar{\beta}_T^3 = \frac{(\sigma^3)^2}{2b^3}(1 - e^{-2b^3 T})$ according to (34).

Proof From the definitions of $\bar{\beta}_T^3$ and $\bar{C}^{33}$ we may write

$$1 - 2\bar{\beta}_T^3 \bar{C}^{33} = 1 - \left(1 - e^{-2b^3 T}\right) \frac{2\left(e^{\Delta h^3} - 1\right)}{2\frac{b^3 h^3}{(\sigma^3)^2} + \frac{b^3}{(\sigma^3)^2}(2b^3 + h^3)\left(e^{\Delta h^3} - 1\right)}. \tag{73}$$

Notice next that $b^3 > 0$ implies that $1 - e^{-2b^3 T} \in (0, 1)$ and that $\frac{b^3 h^3}{(\sigma^3)^2} \geq 0$. From (73) it then follows that a sufficient condition for $1 - 2\bar{\beta}_T^3 \bar{C}^{33} > 0$ to hold is that

$$2 \leq \frac{b^3}{(\sigma^3)^2}(2b^3 + h^3). \tag{74}$$

Given that, see definition after (24), $h^3 = 2\sqrt{(b^3)^2 + 2(\sigma^3)^2} \geq 2b^3$, the condition (74) is satisfied under our assumption (72). $\square$

Proposition 5.1 *Under assumption* (72) *we have that the time-0 price of the caplet for the time interval* $[T, T + \Delta]$ *and with fixed rate R is given by*

$$
\begin{aligned}
P^{Cpl}(0; T + \Delta, R) = p(0, T + \Delta)\Bigg[& \int_M e^{\bar{A}+(\kappa+1)B^1 x + C^{22}(y)^2} \\
& \cdot \Big[\gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33}) \left(\Phi(d^1(x, y)) + \Phi(-d^2(x, y)) \right) \\
& - e^{\bar{C}^{33}(\bar{z}^1(x,y))^2} \Phi(d^3(x, y)) + e^{\bar{C}^{33}(\bar{z}^2(x,y))^2} \Phi(-d^4(x, y)) \Big] \\
& \times f_1(x) f_2(y) dx dy + \gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33}) \int_{M^c} e^{\bar{A}+(\kappa+1)B^1 x + C^{22}(y)^2} \\
& \times f_1(x) f_2(y) dx dy - \tilde{R} Q^{T+\Delta} \left\{ (\Psi_T^1, \Psi_T^2) \in M^c \right\} \Bigg],
\end{aligned}
\tag{75}
$$

where $\Phi(\cdot)$ *is the cumulative standard Gaussian distribution function, M and* M^c *are as in Definition* 5.1,

$$
\begin{cases}
d^1(x, y) := \frac{\sqrt{1-2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^1(x,y) - (\bar{\alpha}_T^3 - \theta \bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}} \\
d^2(x, y) := \frac{\sqrt{1-2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^2(x,y) - (\bar{\alpha}_T^3 - \theta \bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}} \\
d^3(x, y) := \frac{\bar{z}^1(x,y) - \bar{\alpha}_T^3}{\sqrt{\bar{\beta}_T^3}} \\
d^4(x, y) := \frac{\bar{z}^2(x,y) - \bar{\alpha}_T^3}{\sqrt{\bar{\beta}_T^3}}
\end{cases}
\tag{76}
$$

with $\theta := \frac{\bar{\alpha}_T^3 \left(1 - 1/\sqrt{1-2\bar{\beta}_T^3 \bar{C}^{33}}\right)}{\bar{\beta}_T^3}$, *which by Lemma* 5.1 *is well defined under the given assumption* (72), *and with* $\gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33}) := \frac{e^{(\frac{1}{2}(\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)}}{\sqrt{1-2\bar{\beta}_T^3 \bar{C}^{33}}}$.

Remark 5.1 Notice that, once the set M and its complement M^c from Definition 5.1 are made explicit, the integrals, as well as the probability in (75), can be computed explicitly.

Proof On the basis of the sets M and M^c we can continue (69) as

$$
\begin{aligned}
P^{Cpl}(0; T + \Delta, R) = p(0, T + \Delta) & \int_{\mathbb{R}^3} \left(e^{\bar{A}+(\kappa+1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R} \right)^+ \\
& \cdot f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x, y, z) d(x, y, z) \\
= p(0, T + \Delta) & \int_{M \times \mathbb{R}} \left(e^{\bar{A}+(\kappa+1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R} \right)^+ \\
& \cdot f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x, y, z) d(x, y, z)
\end{aligned}
$$

$$+ p(0, T + \Delta) \int_{M^c \times \mathbb{R}} \left(e^{\bar{A} + (\kappa + 1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R}\right)^+$$

$$\cdot f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x, y, z) d(x, y, z)$$

$$=: P^1(0; T + \Delta) + P^2(0; T + \Delta). \tag{77}$$

We shall next compute separately the two terms in the last equality in (77) distinguishing between two cases according to whether $(x, y) \in M$ or $(x, y) \in M^c$.

Case (i): For $(x, y) \in M$ we have from Definition 5.1 that there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ so that for $z \in [\bar{z}^1, \bar{z}^2]$ we have $g(x, y, z) \leq g(x, y, \bar{z}^k) = \tilde{R}$. For $P^1(0; T + \Delta)$ we now obtain

$$P^1(0; T + \Delta) = p(0, T + \Delta)$$

$$\cdot \int_M e^{\bar{A} + (\kappa + 1)B^1 x + C^{22} y^2} \left(\int_{-\infty}^{\bar{z}^1(x,y)} (e^{\bar{C}^{33} z^2} - e^{\bar{C}^{33}(\bar{z}^1)^2}) f_3(z) dz \right.$$

$$\left. + \int_{\bar{z}^2(x,y)}^{+\infty} (e^{\bar{C}^{33} z^2} - e^{\bar{C}^{33}(\bar{z}^2)^2}) f_3(z) dz \right) f_2(y) f_1(x) dy dx. \tag{78}$$

Next, using the results of Sect. 3.3 concerning the Gaussian distribution $f_3(\cdot) = f_{\Psi_T^3}(\cdot)$, we obtain the calculations in (79) below, where, recalling Lemma 5.1, we make successively the following changes of variables: $\zeta := \sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} z$, $\theta := \frac{\bar{\alpha}_T^3 (1 - 1/\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}})}{\bar{\beta}_T^3}$, $s := \frac{\zeta - (\bar{\alpha}_T^3 - \theta \bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}}$ and where $d^i(x, y)$, $i = 1, \cdots, 4$ are as defined in (76)

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33} z^2} f_3(z) dz = \int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33} z^2} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(z - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} dz$$

$$= \int_{-\infty}^{\bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} z - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\frac{\bar{\alpha}_T^3 (\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} - 1)}{\bar{\beta}_T^3} z} dz$$

$$= \int_{-\infty}^{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\zeta - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\frac{\bar{\alpha}_T^3 (1 - 1/\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}})}{\bar{\beta}_T^3} \zeta} \frac{1}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} d\zeta$$

$$= \frac{1}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \int_{-\infty}^{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\zeta - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\theta \zeta} d\zeta$$

$$= \frac{e^{(\frac{1}{2}(\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)}}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \int_{-\infty}^{d^1(x,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds = \frac{e^{(\frac{1}{2}(\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)}}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \Phi(d^1(x, y)). \tag{79}$$

On the other hand, always using the results of Sect. 3.3 concerning the Gaussian distribution $f_3(\cdot) = f_{\Psi_T^3}(\cdot)$ and making this time the change of variables $\zeta := \frac{(z - \bar{\alpha}_T^3)}{\sqrt{\bar{\beta}_T^3}}$, we obtain

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33}(\bar{z}^1)^2} f_3(z)dz = e^{\bar{C}^{33}(\bar{z}^1)^2} \int_{-\infty}^{\bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(z - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} dz$$

$$= e^{\bar{C}^{33}(\bar{z}^1)^2} \int_{-\infty}^{d^3(x,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2} d\zeta = e^{\bar{C}^{33}(\bar{z}^1)^2} \Phi(d^3(x,y)). \tag{80}$$

Similarly, we have

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{\bar{C}^{33}z^2} f_3(z)dz = \frac{1}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} e^{(\frac{1}{2}(\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)} \Phi(-d^2(x,y))$$

$$\tag{81}$$

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{\bar{C}^{33}(\bar{z}^1)^2} f_3(z)dz = e^{\bar{C}^{33}(\bar{z}^2)^2} \Phi(-d^4(x,y)).$$

Case (ii): We come next to the case $(x, y) \in M^c$, for which $g(x, y, z) \geq g(x, y, 0) > \tilde{R}$. For $P^2(0; T + \Delta)$ we obtain

$$P^2(0; T + \Delta) = p(0, T + \Delta) \int_{M^c \times \mathbb{R}} \left(e^{\bar{A} + (\kappa+1)B^1 x + C^{22}y^2 + \bar{C}^{33}z^2} - \tilde{R} \right)$$

$$\cdot f_3(z) f_2(y) f_1(x)dzdydx$$

$$= p(0, T + \Delta) \left(e^{\bar{A}} \int_{M^c} e^{(\kappa+1)B^1 x + C^{22}y^2} f_1(x) f_2(y)dxdy \int_{\mathbb{R}} e^{\bar{C}^{33}z^2} f_3(z)dz \right.$$

$$\left. - \tilde{R} Q^{T+\Delta}[(\Psi_T^1, \Psi_T^2) \in M^c] \right)$$

$$= p(0, T + \Delta) \left(e^{\bar{A}} \left[\int_{M^c} e^{(\kappa+1)B^1 x + C^{22}y^2} f_1(x) f_2(y)dxdy \right] \frac{e^{(\frac{1}{2}(\theta^3)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta^3)}}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \right.$$

$$\left. - \tilde{R} Q^{T+\Delta}[(\Psi_T^1, \Psi_T^2) \in M^c] \right), \tag{82}$$

where we computed the integral over $\mathbb{R}$ analogously to (79).

Adding the two expressions derived for Cases (i) and (ii), we obtain the statement of the proposition. $\qquad\square$

5.2 Swaptions

We start by recalling some of the most relevant aspects of a (payer) swaption. Considering a swap (see Sect. 4.2) for a given collection of dates $0 \leq T_0 < T_1 < \cdots < T_n$,

a swaption is an option to enter the swap at a pre-specified initiation date $T \leq T_0$, which is thus also the maturity of the swaption and that, for simplicity of notation, we assume to coincide with T_0, i.e. $T = T_0$. The arbitrage-free swaption price at $t \leq T_0$ can be computed as

$$P^{Swn}(t; T_0, T_n, R) = p(t, T_0)E^{T_0}\left\{\left(P^{Sw}(T_0; T_n, R)\right)^+ | \mathscr{F}_t\right\}, \tag{83}$$

where we have used the shorthand notation $P^{Sw}(T_0; T_n, R) = P^{Sw}(T_0; T_0, T_n, R)$.

We first state the next Lemma, that follows immediately from the expression for $\rho^3(t, T_k)$ and the corresponding expression for h_k^3 in (65).

Lemma 5.2 *We have the equivalence*

$$\rho^3(t, T_k) > 0 \Leftrightarrow h_k^3 \in \left(0, \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}\right). \tag{84}$$

This lemma prompts us to split the swaption pricing problem into two cases:

$$
\begin{aligned}
\textbf{Case(1)}: \quad & h_k^3 < 0 \text{ or } h_k^3 > \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}} \\
\textbf{Case(2)}: \quad & 0 < h_k^3 < \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}.
\end{aligned}
\tag{85}
$$

Note from the definition of $\rho^3(t, T_k)$ that $h_k^3 \neq \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}$ and that $h_k^3 = 0$ would imply $\bar{C}_k^{33} = 0$ which corresponds to a trivial case in which the factor Ψ^3 is not present in the dynamics of the spread s, hence the inequalities in Case (1) and Case (2) above are indeed strict.

To proceed, we shall introduce some more notation. In particular, instead of only one function $g(x, y, z)$ as in (70), we shall consider also a function $h(x, y)$, more precisely, we shall define here the continuous functions

$$g(x, y, z) := \sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} e^{-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2 - \tilde{C}_{0,k}^{33} z^2} \tag{86}$$

$$h(x, y) := \sum_{k=1}^{n} (R\gamma + 1) e^{-A_{0,k}} e^{-B_{0,k}^1 x - C_{0,k}^{22} y^2}, \tag{87}$$

with the coefficients given by (67) for $t = T_0$. Note that by a slight abuse of notation we write $D_{0,k}$ for $D_{T_0,k}$ and similarly for other coefficients above, always meaning $t = T_0$ in (67). We distinguish the two cases specified in (85):

For *Case (1)* we have (see (67) and Lemma 5.2) that $\tilde{C}_{0,k}^{33} = \rho^3(T_0, T_k) < 0$ for all $k = 1, \cdots, n$, and so the function $g(x, y, z)$ in (86) is, for given (x, y), monotonically increasing for $z \geq 0$ and decreasing for $z < 0$ with

$$\lim_{z \to \pm\infty} g(x, y, z) = +\infty.$$

For *Case (2)* we have instead that $\tilde{C}_{0,k}^{33} = \rho^3(T_0, T_k) > 0$ for all $k = 1, \cdots, n$ and so the nonnegative function $g(x, y, z)$ in (86) is, for given (x, y), monotonically decreasing for $z \geq 0$ and increasing for $z < 0$ with

$$\lim_{z \to \pm\infty} g(x, y, z) = 0.$$

Analogously to Definition 5.1 we next introduce the following objects:

Definition 5.2 Let a set $\bar{M} \subset \mathbb{R}^2$ be given by

$$\bar{M} := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq h(x, y)\}. \tag{88}$$

Since $g(x, y, z)$ and $h(x, y)$ are continuous, $\bar{M}$ is closed, measurable and connected. Let $\bar{M}^c$ be its complement. Furthermore, we define two functions $\bar{z}^1(x, y)$ and $\bar{z}^2(x, y)$ distinguishing between the two Cases (1) and (2) as specified in (85).

Case (1) If $(x, y) \in \bar{M}$, we have $g(x, y, 0) \leq h(x, y)$ and so there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ for which, for $i = 1, 2$,

$$g(x, y, \bar{z}^i) = \sum_{k=1}^n D_{0,k} e^{-A_{0,k}} e^{-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2 - \tilde{C}_{0,k}^{33} (\bar{z}^i)^2}$$

$$= \sum_{k=1}^n (R\gamma + 1) e^{-A_{0,k}} e^{-B_{0,k}^1 x - C_{0,k}^{22} y^2} = h(x, y) \tag{89}$$

and, for $z \notin [\bar{z}^1, \bar{z}^2]$, one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$.
If $(x, y) \in \bar{M}^c$, we have $g(x, y, 0) > h(x, y)$ so that $g(x, y, z) \geq g(x, y, 0) > h(x, y)$ for all z and we have no points corresponding to $\bar{z}^1(x, y)$ and $\bar{z}^2(x, y)$ above.

Case (2) If $(x, y) \in \bar{M}$, we have, as for Case (1), $g(x, y, 0) \leq h(x, y)$ and so there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ for which, for $i = 1, 2$, (89) holds. However, this time it is for $z \in [\bar{z}^1, \bar{z}^2]$ that one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$.
If $(x, y) \in M^c$, then we are in the same situation as for Case (1).

Starting from (83) combined with (66) and taking into account the set $\bar{M}$ according to Definition 5.2, we can obtain the following expression for the swaption price at $t = 0$. As for the caps, here too we consider the joint Gaussian distribution $f_{(\psi_{T_0}^1, \psi_{T_0}^2, \psi_{T_0}^3)}(x, y, z)$ of the factors under the T_0-forward measure Q^{T_0} and we have

$$P^{Swn}(0; T_0, T_n, R) = p(0, T_0)E^{T_0}\left\{\left(P^{Sw}(T_0; T_n, R)\right)^+ | \mathscr{F}_0\right\}$$

$$= p(0, T_0)\int_{\mathbb{R}^3}\left[\sum_{k=1}^n D_{0,k}e^{-A_{0,k}}\exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22}y^2 - \tilde{C}_{0,k}^{33}z^2)\right.$$

$$\left. - \sum_{k=1}^n (R\gamma + 1)e^{-A_{0,k}}\exp(-B_{0,k}^1 x - C_{0,k}^{22}y^2)\right]^+ f_{(\Psi_{T_0}^1, \Psi_{T_0}^2, \Psi_{T_0}^3)}(x, y, z)dxdydz$$

$$= p(0, T_0)\int_{\bar{M}\times\mathbb{R}}\left[\sum_{k=1}^n D_{0,k}e^{-A_{0,k}}\exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22}y^2 - \tilde{C}_{0,k}^{33}z^2)\right.$$

$$\left. - \sum_{k=1}^n (R\gamma + 1)e^{-A_{0,k}}\exp(-B_{0,k}^1 x - C_{0,k}^{22}y^2)\right]^+ f_{(\Psi_{T_0}^1, \Psi_{T_0}^2, \Psi_{T_0}^3)}(x, y, z)dxdydz$$

$$+ p(0, T_0)\int_{\bar{M}^c\times\mathbb{R}}\left[\sum_{k=1}^n D_{0,k}e^{-A_{0,k}}\exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22}y^2 - \tilde{C}_{0,k}^{33}z^2)\right.$$

$$\left. - \sum_{k=1}^n (R\gamma + 1)e^{-A_{0,k}}\exp(-B_{0,k}^1 x - C_{0,k}^{22}y^2)\right]^+ f_{(\Psi_{T_0}^1, \Psi_{T_0}^2, \Psi_{T_0}^3)}(x, y, z)dxdydz$$

$$=: P^1(0; T_0, T_n, R) + P^2(0; T_0, T_n, R). \tag{90}$$

We can now state and prove the main result of this subsection consisting in a pricing formula for swaptions for the Gaussian exponentially quadratic model of this paper. We have

Proposition 5.2 *Assume that the parameters in the model are such that, if h_k^3 belongs to Case (1) in (85) and $h_k^3 > 0$, then $h_k^3 > \frac{1}{4(\sigma^3)^2 e^{-2b^3 T_k}}$. The arbitrage-free price at $t = 0$ of the swaption with payment dates $T_1 < \cdots < T_n$ such that $\gamma = \gamma_k := T_k - T_{k-1}$ $(k = 1, \cdots, n)$, with a given fixed rate R and a notional $N = 1$, can be computed as follows where we distinguish between the Cases (1) and (2) specified in Definition 5.2.*

Case (1) *We have*

$$P^{Swn}(0; T_0, T_n, R) = p(0, T_0)\left\{\sum_{k=1}^n e^{-A_{0,k}}\left[\int_{\bar{M}} D_{0,k}\exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22}y^2)\right.\right.$$

$$\cdot\left(\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3\theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3\tilde{C}_{0,k}^{33}}}\Phi(d_k^1(x, y)) - e^{-\tilde{C}_{0,k}^{33}(\bar{z}^1)^2}\Phi(d_k^2(x, y))\right.$$

$$\left.\left.\left. + \frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3\theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3\tilde{C}_{0,k}^{33}}}\Phi(-d_k^3(x, y)) - e^{-\tilde{C}_{0,k}^{33}(\bar{z}^2)^2}\Phi(-d_k^4(x, y))\right)\right.\right.$$

$$\times f_2(y)f_1(x)dydx + \int_{\bar{M}^c}\left(D_{0,k}e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2}\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}\right.$$

$$\left. - (R\gamma+1)e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}\Big)f_2(y)f_1(x)dydx\right]\Bigg\}. \tag{91}$$

Case (2) *We have*

$$P^{Swn}(0;T_0,T_n,R) = p(0,T_0)\left\{\sum_{k=1}^{n}e^{-A_{0,k}}\right.$$

$$\left[\int_{\bar{M}}D_{0,k}\exp(-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2)\left(\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}\right.\right.$$

$$\times\left[\Phi(d^3_k(x,y))-\Phi(d^1_k(x,y))\right]-e^{-\tilde{C}^{33}_{0,k}(\bar{z}^1)^2}\left[\Phi(d^4_k(x,y))\right.$$

$$\left.\left.-\Phi(d^2_k(x,y))\right]\right)f_2(y)f_1(x)dydx$$

$$+\int_{\bar{M}^c}\left(D_{0,k}e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2}\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}\right.$$

$$\left.\left.- (R\gamma+1)e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}\Big)f_2(y)f_1(x)dydx\right]\right\}. \tag{92}$$

The coefficients in these formulas are as specified in (67) *for* $t=T_0, f_1(x), f_2(x)$ *are the Gaussian densities corresponding to* (68) *for* $T=T_0$ *and the functions* $d^i_k(x,y)$, *for* $i=1,\ldots,4$ *and* $k=1,\ldots,n$, *are given by*

$$\begin{cases} d^1_k(x,y) := \dfrac{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\bar{z}^1(x,y)-(\bar{\alpha}^3_{T_0}-\theta_k\bar{\beta}^3_{T_0})}{\sqrt{\bar{\beta}^3_{T_0}}} \\[2ex] d^2_k(x,y) := \dfrac{\bar{z}^1(x,y)-\bar{\alpha}^3_{T_0}}{\sqrt{\bar{\beta}^3_{T_0}}} \\[2ex] d^3_k(x,y) := \dfrac{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\bar{z}^2(x,y)-(\bar{\alpha}^3_{T_0}-\theta_k\bar{\beta}^3_{T_0})}{\sqrt{\bar{\beta}^3_{T_0}}} \\[2ex] d^4_k(x,y) := \dfrac{\bar{z}^2(x,y)-\bar{\alpha}^3_{T_0}}{\sqrt{\bar{\beta}^3_{T_0}}} \end{cases} \tag{93}$$

with $\theta_k := \dfrac{\bar{\alpha}^3_{T_0}\left(1-1/\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\right)}{\bar{\beta}^3_{T_0}}$, *for* $k=1,\ldots,n$, *and where* $\bar{z}^1=\bar{z}^1(x,y)$, $\bar{z}^2 = \bar{z}^2(x,y)$ *are solutions in z of the equation* $g(x,y,z)=h(x,y)$.

In addition, the mean and variance values for the Gaussian factors $(\Psi^1_{T_0},\Psi^2_{T_0},\Psi^3_{T_0})$ *are here given by*

$$\begin{cases} \bar{\alpha}^1_{T_0} = e^{-b^1 T_0} \Psi^1_0 - \frac{(\sigma^1)^2}{2(b^1)^2} e^{-b^1 T_0} (1 - e^{2b^1 T_0}) - \frac{(\sigma^1)^2}{(b^1)^2}(1 - e^{b^1 T_0}) \Big] \\ \bar{\beta}^1_{T_0} = e^{-2b^1 T_0}(e^{2b^1 T_0} - 1)\frac{(\sigma^1)^2}{2(b^1)} \\ \bar{\alpha}^2_{T_0} = e^{-b^2 T_0} \Psi^2_0 \\ \bar{\beta}^2_{T_0} = e^{-2b^2 T_0} \int_0^{T_0} e^{2b^2 u + 4(\sigma^2)^2 \tilde{C}^{22}(u, T_0)} (\sigma^2)^2 du \\ \bar{\alpha}^3_{T_0} = e^{-b^3 T_0} \Psi^3_0 \\ \bar{\beta}^3_{T_0} = e^{-2b^3 T_0} \frac{(\sigma^3)^2}{2b^3}(e^{2b^3 T_0} - 1). \end{cases} \tag{94}$$

Remark 5.2 A remark analogous to Remark 5.1 applies here too concerning the sets $\bar{M}$ and $\bar{M}^c$.

Proof First of all notice that, when $h^3_k < 0$ or $h^3_k > \frac{1}{4(\sigma^3)^2 e^{-2b^3 T_k}}$ in Case (1), this implies $1 + 2\bar{\beta}^3_{T_0} \tilde{C}^{33}_{0,k} \ge 0$ (in Case (2) we always have $1 + 2\bar{\beta}^3_{T_0} \tilde{C}^{33}_{0,k} \ge 0$). Hence, the square-root of the latter expression in the various formulas of the statement of the proposition is well-defined. This can be checked, similarly as in the proof of Lemma 5.1, by direct computation taking into account the definitions of $\bar{\beta}^3_{T_0}$ in (94) and of $\tilde{C}^{33}_{0,k}$ in (67) and (65) for $t = T_0$.

We come now to the statement for:

Case 1. We distinguish between whether $(x, y) \in \bar{M}$ or $(x, y) \in \bar{M}^c$ and compute separately the two terms in the last equality in (90).

(i) For $(x, y) \in \bar{M}$ we have from Definition 5.2 that there exist $\bar{z}^1(x, y) \le 0$ and $\bar{z}^2(x, y) \ge 0$ so that, for $z \notin [\bar{z}^1, \bar{z}^2]$, one has $g(x, y, z) \ge g(x, y, \bar{z}^i)$. Taking into account that, under Q^{T_0}, the random variables $\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0}$ are independent, so that we shall write $f_{(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})}(x, y, z) = f_1(x)f_2(y)f_3(z)$ (see also (68) and the line following it), we obtain

$$P^1(0; T_0, T_n, R) = p(0, T_0) \Big[\sum_{k=1}^n D_{0,k} e^{-A_{0,k}} \int_M \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2)$$

$$\cdot \Big(\int_{-\infty}^{\bar{z}^1(x,y)} \exp(-\tilde{C}^{33}_{0,k} z^2) f_3(z) dz$$

$$- \int_{-\infty}^{\bar{z}^1(x,y)} \exp(-\tilde{C}^{33}_{0,k} (\bar{z}^1)^2) f_3(z) dz$$

$$+ \int_{\bar{z}^2(x,y)}^{+\infty} \exp(-\tilde{C}^{33}_{0,k} z^2) f_3(z) dz$$

$$- \int_{\bar{z}^2(x,y)}^{+\infty} \exp(-\tilde{C}^{33}_{0,k} (\bar{z}^2)^2) f_3(z) dz \Big) f_2(y) f_1(x) dy dx \Big]. \tag{95}$$

By means of calculations that are completely analogous to those in the proof of Proposition 5.1, we obtain, corresponding to (79)–(81) respectively and with the

same meaning of the symbols, the following explicit expressions for the integrals in the last four lines of (95), namely

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{-\tilde{C}_{0,k}^{33} z^2} f_3(z)dz = \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3 \theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3 \tilde{C}_{0,k}^{33}}} \Phi(d_k^1(x,y)), \tag{96}$$

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{-\tilde{C}_{0,k}^{33} (\bar{z}^1)^2} f_3(z)dz = e^{-\tilde{C}_{0,k}^{33} (\bar{z}^1)^2} \Phi(d_k^2(x,y)), \tag{97}$$

and, similarly,

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{-\tilde{C}_{0,k}^{33} z^2} f_3(z)dz = \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3 \theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3 \tilde{C}_{0,k}^{33}}} \Phi(-d_k^3(x,y)), \tag{98}$$

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{-\tilde{C}_{0,k}^{33} (\bar{z}^2)^2} f_3(z)dz = e^{-\tilde{C}_{0,k}^{33} (\bar{z}^2)^2} \Phi(-d_k^4(x,y)),$$

where the $d_k^i(x,y)$, for $i = 1, \ldots, 4$ and $k = 1, \ldots, n$, are as specified in (93).

(ii) If $(x, y) \in \bar{M}^c$ then, according to Definition 5.2 we have $g(x, y, z) \geq g(x, y, 0) > h(x, y)$ for all z. Noticing that, analogously to (96),

$$\int_{\mathbb{R}} e^{-\tilde{C}_{0,k}^{33} \zeta^2} f_3(\zeta)d\zeta = \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3 \theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3 \tilde{C}_{0,k}^{33}}}$$

we obtain the following expression

$$\begin{aligned}
P^2(0; T_0, T_n, R) &= p(0, T_0) \sum_{k=1}^n e^{-A_{0,k}} \Big[\int_{\bar{M}^c \times \mathbb{R}} \Big(D_{0,k} e^{-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2 - \tilde{C}_{0,k}^{33} z^2} \\
&\quad - (R\gamma + 1) e^{-B_{0,k}^1 x - C_{0,k}^{22} y^2} \Big) f_3(z) f_2(y) f_1(x) dz dy dx \Big] \\
&= p(0, T_0) \sum_{k=1}^n e^{-A_{0,k}} \Big[D_{0,k} \Big(\int_{M^c} e^{-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2} f_2(y) f_1(x) dy dx \Big) \\
&\quad \times \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3 \theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3 \tilde{C}_{0,k}^{33}}} - (R\gamma + 1) \Big(\int_{\bar{M}^c} e^{-B_{0,k}^1 x - C_{0,k}^{22} y^2} f_2(y) f_1(x) dy dx \Big) \Big].
\end{aligned} \tag{99}$$

Adding the two expressions in (i) and (ii) we obtain the statement for Case 1.

Case (2). Also for this case we distinguish between whether $(x, y) \in \bar{M}$ or $(x, y) \in \bar{M}^c$ and, again, compute separately the two terms in the last equality in (90).

(i) For $(x, y) \in \bar{M}$ we have that there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ so that, contrary to Case 1), one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$ when $z \in [\bar{z}^1, \bar{z}^2]$. It follows that

$$
\begin{aligned}
P^1(0; T_0, T_n, R) &= p(0, T_0)\Bigg[\sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} \int_{\bar{M}} \exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2) \\
&\quad \cdot \Bigg(\int_{\bar{z}^1(x,y)}^{\bar{z}^2(x,y)} \exp(-\tilde{C}_{0,k}^{33} z^2) f_3(z) dz \\
&\quad - \int_{\bar{z}^1(x,y)}^{\bar{z}^2(x,y)} \exp(-\tilde{C}_{0,k}^{33} (\bar{z}^1)^2) f_3(z) dz \Bigg) f_2(y) f_1(x) dy dx \Bigg] \\
&= p(0, T_0)\Bigg[\sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} \int_{\bar{M}} \exp(-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2) \\
&\quad \cdot \Bigg(\frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}_{T_0}^3 - \bar{\alpha}_{T_0}^3 \theta_k)}}{\sqrt{1 + 2\bar{\beta}_{T_0}^3 \tilde{C}_{0,k}^{33}}} \Big(\Phi(d_k^3(x, y)) - \Phi(d_k^1(x, y)) \Big) \\
&\quad - e^{-\tilde{C}_{0,k}^{33}(\bar{z}^1)^2} \Big(\Phi(d_k^4(x, y)) - \Phi(d_k^2(x, y)) \Big) \Bigg) f_2(y) f_1(x) dy dx \Bigg],
\end{aligned}
$$
(100)

where we have made use of (96) and (97), (98).

(ii) For $(x, y) \in \bar{M}^c$ we can conclude exactly as we did it for Case (1) and, by adding the two expressions in (i) and (ii), we obtain the statement also for Case (2).

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