Combinatorial Properties of Triangle-Free Rectangle Arrangements and the Squarability Problem

Jonathan Klawitter[1,2], Martin Nöllenburg[3(✉)], and Torsten Ueckerdt[2]

[1] Institut für Theoretische Informatik, Karlsruhe Institute of Technology, Karlsruhe, Germany

[2] Institut für Algebra und Geometrie, Karlsruhe Institute of Technology, Karlsruhe, Germany

torsten.ueckerdt@kit.edu

[3] Algorithms and Complexity Group, TU Wien, Vienna, Austria

noellenburg@ac.tuwien.ac.at

Abstract. We consider arrangements of axis-aligned rectangles in the plane. A geometric arrangement specifies the coordinates of all rectangles, while a combinatorial arrangement specifies only the respective intersection type in which each pair of rectangles intersects. First, we investigate combinatorial contact arrangements, i.e., arrangements of interior-disjoint rectangles, with a triangle-free intersection graph. We show that such rectangle arrangements are in bijection with the 4-orientations of an underlying planar multigraph and prove that there is a corresponding geometric rectangle contact arrangement. Using this, we give a new proof that every triangle-free planar graph is the contact graph of such an arrangement. Secondly, we introduce the question whether a given rectangle arrangement has a combinatorially equivalent square arrangement. In addition to some necessary conditions and counterexamples, we show that rectangle arrangements pierced by a horizontal line are squarable under certain sufficient conditions.

1 Introduction

We consider arrangements of axis-aligned rectangles and squares in the plane. Besides *geometric rectangle arrangements*, in which all rectangles are given with coordinates, we are also interested in *combinatorial rectangle arrangements*, i.e., equivalence classes of combinatorially equivalent arrangements. Our contribution is two-fold.

First we consider maximal (with a maximal number of contacts) combinatorial rectangle contact arrangements, in which no three rectangles share a point. For rectangle arrangements this is equivalent to the contact graph being *triangle-free*, unlike, e.g., for triangle contact arrangements. We prove a series of analogues to the well-known maximal combinatorial triangle contact arrangements and to Schnyder realizers. The contact graph G of a maximal triangle contact arrangement is a maximal planar graph. A 3-*orientation* is an orientation of the edges

E. Di Giacomo and A. Lubiw (Eds.): GD 2015, LNCS 9411, pp. 231–244, 2015.
DOI: 10.1007/978-3-319-27261-0_20

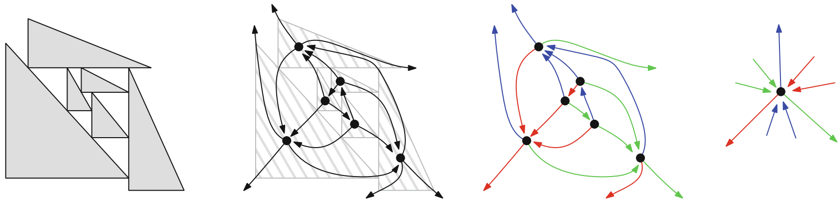

Fig. 1. Left to right: maximal combinatorial contact arrangement with axis-aligned triangles, no three sharing a point. 3-orientation of G'. Schnyder realizer of G'. Local coloring rules for Schnyder realizer (Color figure online).

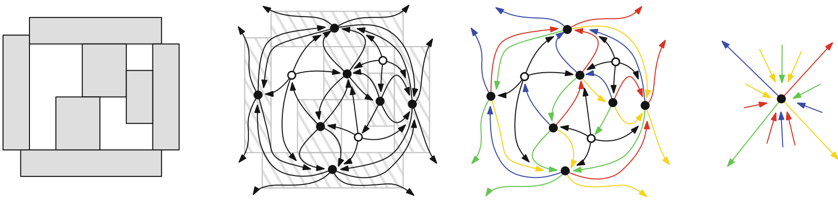

Fig. 2. Left to right: maximal combinatorial contact arrangement with axis-aligned rectangles, no three sharing a point. 4-orientation of underlying graph. Corner-edge-labeling of underlying graph. Local coloring rules for corner-edge-labeling (Color figure online).

of a graph G', obtained from G by adding six edges (two at each outer vertex), in which every vertex has exactly three outgoing edges. Each outer vertex has two outgoing edges that end in the outer face without having an endpoint there. A *Schnyder realizer* [10,11] is a 3-orientation of G' together with a coloring of its edges with colors $0, 1, 2$ such that every vertex has exactly one outgoing edge in each color and incoming edges are colored in the color of the "opposite" outgoing edge. The three outgoing edges represent the three corners of a triangle and the color specifies the corner, see Fig. 1. De Fraysseix *et al.* [3] proved that the maximal combinatorial triangle contact arrangements of G are in bijection with the 3-orientations of G' and the Schnyder realizers of G'. Schnyder proved that for every maximal planar graph G, G' admits a Schnyder realizer and hence G is a triangle contact graph.

In this paper we prove an analogous result, which, roughly speaking, is the following. We consider maximal triangle-free combinatorial rectangle contact arrangements. The corresponding contact graph G is planar with all faces of length 4 or 5. We define an underlying plane multigraph $\bar{G}$, whose vertex set also includes a vertex for each inner face of the contact graph, and define 4-*orientations* of $\bar{G}$. Here, every vertex has exactly four outgoing edges, where each outer vertex has two edges ending in the outer face. For a 4-orientation we introduce *corner-edge-labelings* of $\bar{G}$, which are, similar to Schnyder realizers, colorings of the outgoing edges at vertices of $\bar{G}$ corresponding to rectangles with colors $0, 1, 2, 3$ satisfying certain local rules. Each outgoing edge represents a corner of a rectangle and the color specifies which corner it is, see Fig. 2. We then prove that the combinatorial contact arrangements of G are in bijection with the 4-orientations of $\bar{G}$ and the corner-edge-labelings of $\bar{G}$.

Thomassen [12] proved that rectangle contact graphs are precisely the graphs admitting a planar embedding in which no triangle contains a vertex in its interior. We also prove here that for every maximal triangle-free planar graph G, $\bar{G}$ admits a 4-orientation, obtaining a new proof that G is a rectangle contact graph.

Our second result is concerned with the question whether a given geometric rectangle arrangement can be transformed into a combinatorially equivalent square arrangement. The similar question whether a pseudocircle arrangement can be transformed into a combinatorially equivalent circle arrangement has recently been studied by Kang and Müller [6], who showed that the problem is NP-hard. We say that a rectangle arrangement can be *squared* (or is *squarable*) if an equivalent square arrangement exists. Obviously, squares are a very restricted class of rectangles and not every rectangle arrangement can be squared. The natural open question is to characterize the squarable rectangle arrangements and to answer the complexity status of the corresponding decision problem. As a first step towards solving these questions, we show, on the one hand, some general necessary conditions and, on the other hand, sufficient conditions implying that certain subclasses of rectangle arrangements are always squarable.

Related Work. Intersection graphs and contact graphs of axis-aligned rectangles or squares in the plane are a popular, almost classic, topic in discrete mathematics and theoretical computer science with lots of applications in computational geometry, graph drawing and VLSI chip design. Most of the research for rectangle intersection graphs concerns their recognition [14], colorability [1] or the design of efficient algorithms such as for finding maximum cliques [5]. On the other hand, rectangle contact graphs are mainly investigated for their combinatorial and structural properties. Almost all the research here concerns edge-maximal 3-connected rectangle contact graphs, so called *rectangular duals*. These can be characterized by the absence of separating triangles [9,13] and the corresponding representations by touching rectangles can be seen as dissections of a rectangle into rectangles. Combinatorially equivalent dissections are in bijection with regular edge labelings [7] and transversal structures [4]. The question whether a rectangular dual has a rectangle dissection in which all rectangles are squares has been investigated by Felsner [2].

2 Preliminaries

In this paper a *rectangle* is an axis-aligned rectangle in the plane, i.e., the cross product $[x_1, x_2] \times [y_1, y_2]$ of two bounded closed intervals. A *geometric rectangle arrangement* is a finite set $\mathcal{R}$ of rectangles; it is a *contact arrangement* if any two rectangles have disjoint interiors. In a contact arrangement, any two non-disjoint rectangles R_1, R_2 have one of the two contact types *side contact* and *corner contact*, see Fig. 3 (left); we exclude the degenerate case of two rectangles

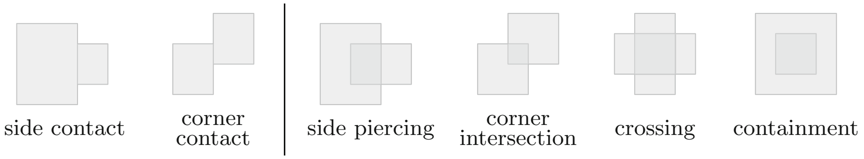

Fig. 3. Contact types (left) and intersection types (right) of rectangles.

sharing only one point. If $\mathcal{R}$ is not a contact arrangement, four intersection types are possible: *side piercing*, *corner intersection*, *crossing*, and *containment*, see Fig. 3 (right). Note that side contact and corner contact are degenerate cases of side piercing and corner intersection, whereas crossing and containment have no analogues in contact arrangements. If no two rectangles form a crossing, we say that $\mathcal{R}$ is *cross-free*. Moreover, in each type (except containment) it is further distinguished which sides of the rectangles touch or intersect.

Two rectangle arrangements $\mathcal{R}_1$ and $\mathcal{R}_2$ are *combinatorially equivalent* if $\mathcal{R}_1$ can be continuously deformed into $\mathcal{R}_2$ such that every intermediate state is a rectangle arrangement with the same intersection or contact type for every pair of rectangles. An equivalence class of combinatorially equivalent arrangements is called a *combinatorial rectangle arrangement*. So while a geometric arrangement specifies the coordinates of all rectangles, think of a combinatorial arrangement as specifying only the way in which any two rectangles touch or intersect. In particular, a combinatorial rectangle arrangement is defined by **(1)** for each rectangle R and each side of R the counterclockwise order of all intersecting (touching) rectangle edges, labeled by their rectangle R' and the respective side of R' (top, bottom, left, right), **(2)** for containments the respective component of the arrangement, in which a rectangle is contained.

In the *intersection graph* of a rectangle arrangement there is one vertex for each rectangle and two vertices are adjacent if and only if the corresponding rectangles intersect. As combinatorially equivalent arrangements have the same intersection graph, combinatorial arrangements themselves have a well-defined intersection graph. For rectangle contact arrangements (combinatorial or geometric) the intersection graph is also called the *contact graph*. Note that such contact graphs are planar, as we excluded the case of four rectangles meeting in a corner.

3 Statement of Results

3.1 Maximal Triangle-Free Planar Graphs and Rectangle Contact Arrangements

We consider so-called *MTP-graphs*, that is, (M)aximal (T)riangle-free (P)lane graphs with a quadrangular outer face. Note that each face in such an MTP-graph is a 4-cycle or 5-cycle, and that every plane triangle-free graph is an induced subgraph of some MTP-graph. Given an MTP-graph G a rectangle contact arrangement of G is one whose contact graph is G, where the embedding

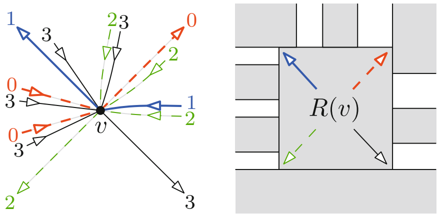

Fig. 4. Local color patterns in corner-edge-labelings of an MTP-graph at a vertex v, together with the corresponding part in a rectangle contact arrangement (Color figure online).

inherited from the arrangement is the given embedding of G, and where each outer rectangle has two corners in the unbounded region[1]. We define the closure, 4-orientations and corner-edge-labelings:

The *closure* $\bar{G}$ of G is derived from G by replacing each edge of G with a pair of parallel edges, called an *edge pair*, and adding into each inner face f of G a new vertex, also denoted by f, connected by an edge, called a *loose edge*, to each vertex incident to that face. At each outer vertex we add two loose edges pointing into the outer face, although we do not add a vertex for the outer face. Note that $\bar{G}$ inherits a unique plane embedding with each inner face being a triangle or a 2-gon.

A 4-orientation of $\bar{G}$ is an orientation of the edges and half-edges of $\bar{G}$ such that every vertex has outdegree exactly 4. An edge pair is called *uni-directed* if it is oriented consistently and *bi-directed* otherwise.

A corner-edge-labeling of $\bar{G}$ is a 4-orientation of $\bar{G}$ together with a coloring of the outgoing edges of $\bar{G}$ at each vertex of G with colors $0, 1, 2, 3$ (see Fig. 4) such that

(i) around each vertex v of G we have outgoing edges in color $0, 1, 2, 3$ in this counterclockwise order and

(ii) in the wedge, called *incoming wedge*, at v counterclockwise between the outgoing edges of color i and $i{+}1$ there are some (possibly none) incoming edges colored $i + 2$ or $i + 3$, $i = 0, 1, 2, 3$, all indices modulo 4.

In a corner-edge-labeling the four outgoing edges at a vertex of $\bar{G}$ corresponding to a face of G are not colored. Further we remark that (i) implies that uni-directed pairs are colored i and $i - 1$, while (ii) implies that bi-directed pairs are colored i and $i + 2$, for some $i \in \{0, 1, 2, 3\}$, where all indices are considered modulo 4. The following theorem is proved in Sect. 4.

Theorem 1. *Let G be an MTP-graph, then each of the following are in bijection:*

- *the combinatorial rectangle contact arrangements of G*
- *the corner-edge-labelings of $\bar{G}$*
- *the 4-orientations of $\bar{G}$.*

[1] Other configurations of the outer four rectangles can be easily derived from this.

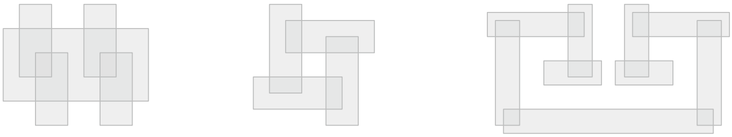

Fig. 5. Three cross-free unsquarable rectangle arrangements.

Using the bijection between 4-orientations of $\bar{G}$ and combinatorial rectangle contact arrangements of G given in Theorem 1, we can give a new proof that every MTP-graph G is a rectangle contact graph, which is the statement of the next theorem; its proof is given in the full paper [8] and sketched in Sect. 5.

Theorem 2. *For every MTP-graph G, $\bar{G}$ has a 4-orientation and it can be computed in linear time. In particular, G has a rectangle contact arrangement.*

We remark that our technique in the proof of Theorem 1 constructs from a given 4-orientation of $\bar{G}$ in linear time a geometric rectangle contact arrangement of G in the $2n \times 2n$ square grid, where n is the number of vertices in G. Thus also the rectangle contact arrangement in Theorem 2 can be computed in linear time and uses only a linear-size grid.

3.2 Squarability and Line-Pierced Rectangle Arrangements

In the squarability problem, we are given a rectangle arrangement $\mathcal{R}$ and want to decide whether $\mathcal{R}$ can be squared. The first observation is that there are obvious obstructions to the squarability of a rectangle arrangement. If any two rectangles in $\mathcal{R}$ are crossing (see Fig. 3) then there are obviously no two combinatorially equivalent squares.

But even if we restrict ourselves to cross-free rectangle arrangements, we can find unsquarable configurations. One such arrangement is depicted in Fig. 5 (left). To get an unsquarable arrangement with a triangle-free intersection graph, we can use the fact that two side-piercing rectangles translate immediately into a smaller-than relation for the corresponding squares: the side length of the square to pierce into the side of another square needs to be strictly smaller. Hence any rectangle arrangement that contains a cycle of side-piercing rectangles cannot be squarable, see Fig. 5 (middle). Moreover, we may even create a counterexample of a rectangle arrangement whose intersection graph is a path and that causes a geometrically infeasible configuration for squares, see Fig. 5 (right).

Proposition 1. *Some cross-free rectangle arrangements are unsquarable, even if the intersection graph is a path.*

Therefore we focus on a non-trivial subclass of rectangle arrangements that we call line-pierced. A rectangle arrangement $\mathcal{R}$ is *line-pierced* if there exists a horizontal line ℓ such that $\ell \cap R \neq \emptyset$ for all $R \in \mathcal{R}$. The line-piercing strongly restricts the possible vertical positions of the rectangles in $\mathcal{R}$, which lets us prove two sufficient conditions for squarability in the following theorem.

Theorem 3. *Let $\mathcal{R}$ be a cross-free, line-pierced rectangle arrangement.*

- *If $\mathcal{R}$ is triangle-free, then $\mathcal{R}$ is squarable.*
- *If $\mathcal{R}$ has only corner intersections, then $\mathcal{R}$ is squarable, even using line-pierced unit squares.*

On the other hand, cross-free, line-pierced rectangle arrangements in general may have forbidden cycles or other geometric obstructions to squarability. We give two examples in Sect. 6, together with a sketch of the proof of Theorem 3.

4 Bijections Between 4-Orientations, Corner-Edge-Labelings and Rectangle Contact Arrangements – Proof of Theorem 1

Throughout this section let $G = (V, E)$ be a fixed MTP-graph and $\bar{G}$ be its closure. By definition, every corner-edge-labeling of $\bar{G}$ induces a 4-orientation of $\bar{G}$. We prove Theorem 1, i.e., that combinatorial rectangle contact arrangements of G, 4-orientations of $\bar{G}$ and corner-edge-labelings of $\bar{G}$ are in bijection, in three steps:

- Every rectangle contact arrangement of G induces a 4-orientation of $\bar{G}$. (Lemma 1)
- Every 4-orientation of $\bar{G}$ induces a corner-edge-labeling of $\bar{G}$. (Lemma 3)
- Every corner-edge-labeling of $\bar{G}$ induces a rectangle contact arrangement of G. (Lemma 4)

Omitted proofs are provided in the full version of this paper [8].

4.1 From Rectangle Arrangements to 4-Orientations

Lemma 1. *Every rectangle contact arrangement of G induces a 4-orientation of $\bar{G}$.*

The proof idea is already given in Fig. 2: For every rectangle draw an outgoing edge through each of the four corners and for every inner face draw an outgoing edge through each of the four extremal sides.

We continue with a crucial property of 4-orientations. For a simple cycle C of G, consider the corresponding cycle $\bar{C}$ of edge pairs in $\bar{G}$. The *interior* of $\bar{C}$ is the bounded component of $\mathbb{R}^2$ incident to all vertices in C after the removal of all vertices and edges of $\bar{C}$. In a fixed 4-orientation of $\bar{G}$ a directed edge $e = (u, v)$ *points inside* C if $u \in V(C)$ and e lies in the interior of $\bar{C}$, i.e., either v lies in the interior of C, or e is a chord of $\bar{C}$ in the interior of $\bar{C}$.

Lemma 2. *For every 4-orientation of $\bar{G}$ and every simple cycle C of G the number of edges pointing inside C is exactly $|V(C)| - 4$.*

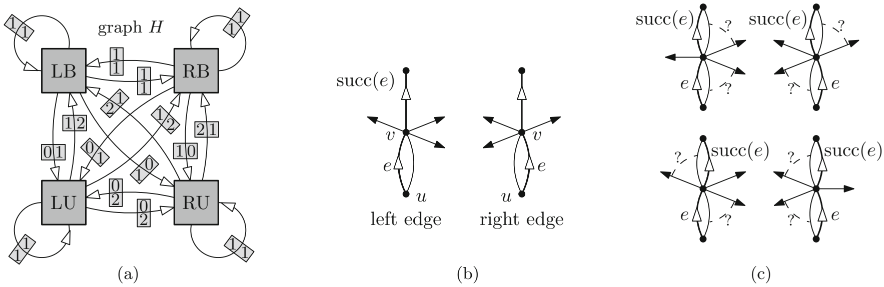

Fig. 6. (a) The graph H. **L**, **R**, **U**, **B** stands for left edge, right edge, uni-directed and bi-directed edge pair, respectively. The number of outgoing edges in the left and right wedge are shown on the left and right of the corresponding arrow. (b) Illustration of the definition of succ(e). (c) Summarizing the 16 possible cases for e and succ(e). Edges connected by a dashed arc may or may not coincide.

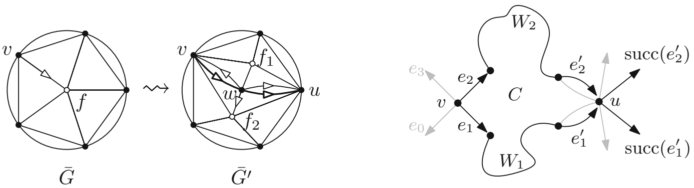

Fig. 7. Left: Stacking a new vertex w into a 5-face f of G. The orientation of edges on the boundary of f, as well as outgoing edges at f, f_1, f_2 is omitted. The directed edge (v, w) and its successor (w, u) are highlighted. Right: Illustration of the proof of the Claim in the proof of Lemma 3.

4.2 From 4-Orientations to Corner-Edge-Labelings

Next we shall show how a 4-orientation of $\bar{G}$ can be augmented (by choosing colors for the edges) into a corner-edge-labeling. Fix a 4-orientation. If e is a directed edge in an edge pair, then e is called a *left edge*, respectively *right edge*, when the 2-gon enclosed by the edge pair lies on the right, respectively on the left, when going along e in its direction. Thus, a uni-directed edge pair consists of one left edge and one right edge, while a bi-directed edge pair either consists of two left edges (clockwise oriented 2-gon) or two right edges (counterclockwise oriented 2-gon).

If $e = (u, v)$ is an edge in an edge pair, let e_2 and e_3 be the second and third outgoing edge at v when going counterclockwise around v starting with e. We define the *successor* of e as $\text{succ}(e) = e_2$ if e is a right edge, and $\text{succ}(e) = e_3$ if e is a left edge, see Fig. 6 (b,c). Note that in a corner-edge-labeling succ(e) is exactly the outgoing edge at v that has the same color as e, see Fig. 4.

Note that $e' = \text{succ}(e)$ may be a loose edge in $\bar{G}$ at the concave vertex for some 5-face in G. For the sake of shorter proofs below, we shall avoid the treatment of this case. To do so, we augment G to a supergraph G' such that starting with any edge in any edge pair and repeatedly taking the successor, we never run into a loose edge pointing to an inner face.

The graph G' is formally obtained from G by stacking a new vertex w into each 5-face f, with an edge to the incoming neighbor v of f in $\bar{G}$ and the vertex u at f that comes second after v in the clockwise order around f in $\bar{G}$. (Indeed, the second vertex in counterclockwise order would be equally good for our purposes.) Let f_1 and f_2 be the resulting 4-face and 5-face incident to w, respectively. We obtain a 4-orientation of the closure $\bar{G}'$ of G' by orienting all edges at f_1 as outgoing, both edges between v and w as right edges (counterclockwise), the remaining three edges at w as outgoing, and the remaining four edges at f_2 as outgoing. See Fig. 7 (left) for an illustration.

Before we augment the 4-orientation of $\bar{G}'$ into a corner-edge-labeling, we need one last observation. Let e and $\mathrm{succ}(e)$ be two edges in edge pairs of $\bar{G}'$ with common vertex v. Consider the wedges at v between e and $\mathrm{succ}(e)$ when going clockwise (left wedge) and counterclockwise (right wedge) around v. Each of e, $\mathrm{succ}(e)$ can be a left edge or right edge, and in a uni-directed pair or a bi-directed pair. This gives us four types of edges and 16 possibilities for the types of e and $\mathrm{succ}(e)$. The graph H in Fig. 6(a) shows for each of these 16 possibilities the number of outgoing edges at v in the left and right wedge at v.

Observation 4. *For every directed closed walk on k edges in the graph H in Fig. 6(a) we have*

$$\#\textit{edges in left wedges} = \#\textit{edges in right wedges} = k.$$

Proof. It suffices to check each directed cycle on k edges, $k = 1, 2, 3, 4$. □

Lemma 3. *Every 4-orientation of $\bar{G}$ induces a corner-edge-labeling of $\bar{G}$.*

A detailed proof of Lemma 3 is given in the full version of this paper [8].

Proof (Sketch). Consider the augmented graph G', its closure $\bar{G}'$ and 4-orientation as defined above. For any edge e in an edge pair in $\bar{G}'$ (and hence every edge of $\bar{G}$ outgoing at some vertex of G) consider the directed walk W_e in $\bar{G}'$ starting with e by repeatedly taking the successor as long as it exists (namely the current edge is in an edge pair).

First we show that W_e is a simple path ending at one of the eight loose edges in the outer face. Indeed, otherwise W_e would contain a simple cycle C where every edge on C, except the first, is the successor of its preceding edge on C. From the graph H of Fig. 6(a) we see that every wedge of C contains at most two outgoing edges. With Observation 4 the number of edges pointing inside C is at least $|V(C)| - 2$ and at most $|V(C)| + 2$, which is a contradiction to Lemma 2.

Now let v_0, v_1, v_2, v_3 be the outer vertices in this counterclockwise order. Define the color of e to be i if W_e ends with the right loose edge at v_i or the left loose edge at v_{i-1}, indices modulo 4. By definition every edge has the same color as its successor in $\bar{G}'$ (if it exists). Thus this coloring is a corner-edge-labeling of $\bar{G}'$ if at every vertex v of G the four outgoing edges are colored 0, 1, 2, and 3, in this counterclockwise order around v.

Claim. Let e_1, e_2 be two outgoing edges at v for which $W_{e_1} \cap W_{e_2}$ consists of more than just v. Then e_1 and e_2 appear consecutively among the outgoing edges around v, say e_1 clockwise after e_2.

Moreover, if $u \neq v$ is a vertex in $W_{e_1} \cap W_{e_2}$ for which the subpaths W_1 of W_{e_1} and W_2 of W_{e_2} between v and u do not share inner vertices, then the last edge e_1' of W_1 is a right edge and the last edge e_2' of W_2 is a left edge, e_1' and e_2' are part of (possibly the same) uni-directed pairs and these pairs sit in the same incoming wedge at u.

To prove this claim, we consider the cycle $C = W_1 \cup W_2$, count the edges pointing inside with the graph H and conclude that neither u nor v may have edges pointing inside C. See Fig. 7 (right) for an illustration.

The claim implies that the two walks W_{e_1} and W_{e_2} can neither cross, nor have an edge in common. Considering the four walks starting in a given vertex, we can argue (with the second part of the claim) that our coloring is a corner-edge-labeling of $\bar{G}'$. Finally, we inherit a corner-edge-labeling of $\bar{G}$ by reverting the stacking of artificial vertices in 5-faces. □

4.3 From Corner-Edge-Labelings to Rectangle Contact Arrangements

It remains to compute a rectangle arrangement of G based on a given corner-edge-labeling of $\bar{G}$. That is, we shall prove the following lemma.

Lemma 4. *Every corner-edge-labeling of $\bar{G}$ induces a rectangle contact arrangement of G.*

A detailed proof of Lemma 4 is given in the full version of this paper [8].

Proof (Sketch). Fix a corner-edge-labeling of $\bar{G}$. For every vertex v of G we introduce two pairs of variables $x_1(v), x_2(v)$ and $y_1(v), y_2(v)$ and set up a system of inequalities and equalities such that any solution defines a rectangle contact arrangement $\{R(v) \mid v \in V\}$ of G with $R(v) = [x_1(v), x_2(v)] \times [y_1(v), y_2(v)]$, which is compatible with the given corner-edge-labeling.

For every edge vw of G the way in which $R(v)$ and $R(w)$ are supposed to touch is encoded in the given corner-edge-labeling and this can be described by the inequalities and equalities in Table 1. Here we list the constraint and the conditions (color and orientation) of a single directed edge between v and w or a uni-directed edge pair outgoing at v and incoming at w in $\bar{G}$ under which we have this constraint.

Instead of showing that the system in Table 1 has a solution, we define another set of constraints implying all constraints in Table 1, for which it is easier to prove feasibility.

It suffices to define a system $\mathcal{I}_x$ for x-coordinates and treat the y-coordinates analogously. In $\mathcal{I}_x$ we have $x_1(v) < x_2(v)$ for every vertex v together with all equalities in the left of Table 1, but only those inequalities in the left of Table 1 that arise from edges in bi-directed edge pairs. The inequalities arising from uni-directed edge pairs are implied by the following set of inequalities. For a vertex

Table 1. Constraints encoding the type of contact between $R(v)$ and $R(w)$, defined based on the orientation and color(s) of the edge pair between v and w in $\bar{G}$.

constraint	edge	color	out
$x_1(w) < x_1(v) < x_2(w)$	right	2	v
	left	1	v
$x_1(w) < x_2(v) < x_2(w)$	right	0	v
	left	3	v
$x_1(w) = x_2(v)$	right	1	w
	left	2	w
	uni	0, 3	v

constraint	edge	color	out
$y_1(w) < y_1(v) < y_2(w)$	right	3	v
	left	2	v
$y_1(w) < y_2(v) < y_2(w)$	right	1	v
	left	0	v
$y_1(w) = y_2(v)$	right	2	w
	left	3	w
	uni	1, 0	v

v in G let $S_1(v) = a_1, \ldots, a_k$ and $S_2(v) = b_1, \ldots, b_\ell$ be the counterclockwise sequences of neighbors of v in the incoming wedges at v bounded by its outgoing edges of color 0 and 1, and color 2 and 3, respectively. See the left of Fig. 8. Then we have in $\mathcal{I}_x$ the inequalities

$$x_1(a_i) > x_2(a_{i+1}) \text{ for } i = 1, \ldots, k-1 \text{ and } x_2(b_i) < x_1(b_{i+1}) \text{ for } i = 1, \ldots, \ell - 1. \quad (1)$$

If $k = 1$ we have no constraint for $S_1(v)$ and if $\ell = 1$ we have no constraint for $S_2(v)$.

We associate the system $\mathcal{I}_x$ with a partially oriented graph I_x whose vertex set is $\{x_1(v), x_2(v) \mid v \in V\}$. For each inequality $a > b$ we have an oriented edge (a, b) in I_x, while for each equality $a = b$ we have an undirected edge ab in I_x, see Fig. 8.

We observe that I_x is planar and prove that I_x has no cycle C in which all directed edges are oriented consistently, which clearly implies that $\mathcal{I}_x$ has a solution. This is done by showing that no inner face is such a cycle, and that for every inner vertex u, vertex $x_1(u)$ has an incident undirected edge or incident outgoing edge and vertex $x_2(u)$ has an incident undirected edge or incident incoming edge. □

5 MTP Graphs Are Rectangle Contact Graphs – Proofsketch of Theorem 2

Theorem 2 is formally proven in the full version of this paper [8]. The idea is to prove by induction on the number of vertices that for an MTP-graph G we find a 4-orientation of $\bar{G}$. In the inductive step we either have (Case 1) that G has an inner 4-face, or (Case 2) that one can contract an inner edge e, keeping it an MTP-graph. Figures 9 and 10 illustrate how to find a 4-orientation in Cases 1 and 2, respectively.

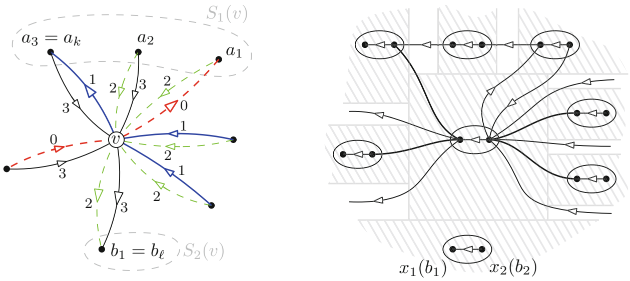

Fig. 8. Illustrating the definition of I_x around a vertex v. On the right a hypothetical rectangle contact arrangement is indicated (Color figure online).

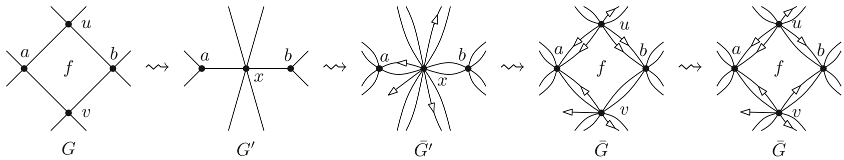

Fig. 9. Collapsing an inner 4-face and inheriting a 4-orientation when uncollapsing.

6 Line-Pierced Rectangle Arrangements and Squarability – Proofsketch of Theorem 3

Recall that a rectangle arrangement $\mathcal{R}$ is line-pierced if there is a horizontal line ℓ that intersects every rectangle in $\mathcal{R}$. Note that by the line-piercing property of $\mathcal{R}$ the intersection graph remains the same if we project each rectangle $R = [a, b] \times [c, d] \in \mathcal{R}$ onto the interval $[a, b] \subseteq \mathbb{R}$. In particular, the intersection graph $G_{\mathcal{R}}$ of a line-pierced rectangle arrangement $\mathcal{R}$ is an *interval graph*, i.e., intersection graph of intervals on the real line.

Line-pierced rectangle arrangements, however, carry more information than one-dimensional interval graphs since the vertical positions of intersection points between rectangles do influence the combinatorial properties of the arrangement. We obtain two squarability results for line-pierced arrangements in Propositions 2 and 3, which yield Theorem 3.

Proposition 2. *Every line-pierced, triangle-free, and cross-free rectangle arrangement $\mathcal{R}$ is squarable.*

There are instances, however, that satisfy the conditions of Proposition 2 and thus have a squaring, but not a line-pierced one. An example is given in Fig. 12.

Proposition 3. *Every line-pierced rectangle arrangement $\mathcal{R}$ restricted to corner intersections is squarable. There even exists a corresponding squaring with unit squares that remains line-pierced.*

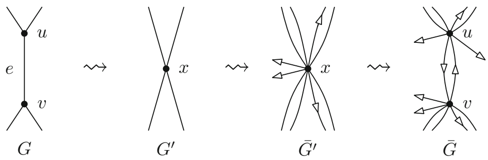

Fig. 10. Contracting an edge and keeping a 4-orientation when uncontracting.

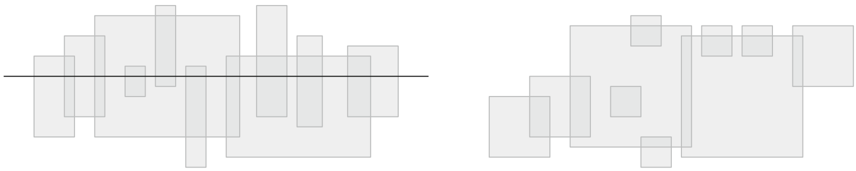

Fig. 11. Constructing a combinatorially equivalent squaring from a line-pierce, triangle-free, and cross-free rectangle arrangement.

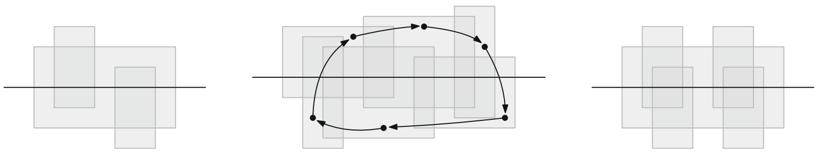

Fig. 12. Left: A line-pierced, triangle-free rectangle arrangement that has no line-pierced squaring. Middle: An unsquarable line-pierced rectangle arrangement due to a forbidden cycle of side-piercing intersections. Right: Squaring the two vertical pairs of rectangles on the right implies that the central square would need to be wider than tall.

Propositions 2 and 3 are proved in the full version of this paper [8]. The crucial observation is that the intersection graph of $\mathcal{R}$ is a caterpillar in the former case (Fig. 11) and a unit-interval graph in the latter case. The results can then be proven by induction on the number of vertices by iteratively removing the "rightmost" rectangle in the representation.

If we drop the restrictions to corner intersections and triangle-free arrangements, we can immediately find unsquarable instances, either by creating cyclic “‘smaller than”’ relations or by introducing intersection patterns that become geometrically infeasible for squares. Two examples are given in Fig. 12.

7 Conclusions

We have introduced corner-edge-labelings, a new combinatorial structure similar to Schnyder realizers, which captures the combinatorially equivalent maximal rectangle arrangements with no three rectangles sharing a point. Using this, we gave a new proof that every triangle-free planar graph is a rectangle contact graph. We also introduced the squarability problem, which asks for a given rectangle arrangement whether there is a combinatorially equivalent arrangement using only squares. We provide some forbidden configuration for the squarability of an arrangement and show that certain subclasses of line-pierced arrangements are always squarable. It remains open whether the decision problem for general arrangements is NP-complete.

Surprisingly, every unsquarable arrangement that we know has a crossing or a side-piercing. Hence we would like to ask whether every rectangle arrangement with only corner intersections is squarable. Another natural question is whether every triangle-free planar graph is a square contact graph.

References

1. Asplund, E., Grünbaum, B.: On a coloring problem. Mathematica Scandinavica **8**, 181–188 (1960)
2. Felsner, S.: Rectangle and square representations of planar graphs. In: Pach, J. (ed.) Thirty Essays in Geometric Graph Theory, pp. 213–248. Springer, New York (2012)
3. de Fraysseix, H., de Mendez, P.O., Rosenstiehl, P.: On triangle contact graphs. Comb. Probab. Comput. **3**, 233–246 (1994)
4. Fusy, E.: Transversal structures on triangulations: a combinatorial study and straight-line drawings. Discrete Math. **309**(7), 1870–1894 (2009)
5. Imai, H., Asano, T.: Finding the connected components and a maximum clique of an intersection graph of rectangles in the plane. J. Algorithms **4**(4), 310–323 (1983)
6. Kang, R.J., Müller, T.: Arrangements of pseudocircles and circles. Discrete Comput. Geom. **51**, 896–925 (2014)
7. Kant, G., He, X.: Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems. Theor. Comput. Sci. **172**(1–2), 175–193 (1997)
8. Klawitter, J., Nöllenburg, M., Ueckerdt, T.: Combinatorial properties of triangle-free rectangle arrangements and the squarability problem. CoRR, arXiv:1509.00835, September 2015
9. Koźmiński, K., Kinnen, E.: Rectangular duals of planar graphs. Networks **15**, 145–157 (1985)
10. Schnyder, W.: Planar graphs and poset dimension. Order **5**(4), 323–343 (1989)
11. Schnyder, W.: Embedding planar graphs on the grid. In: 1st ACM-SIAM Symposium on Discrete Algorithms, SODA 1990, pp. 138–148 (1990)
12. Thomassen, C.: Interval representations of planar graphs. J. Comb. Theor. Ser. B **40**(1), 9–20 (1986)
13. Ungar, P.: On diagrams representing graphs. J. London Math. Soc. **28**, 336–342 (1953)
14. Yannakakis, M.: The complexity of the partial order dimension problem. SIAM J. Algebraic Discrete Methods **3**(3), 351–358 (1982)