Small-Area Orthogonal Drawings of 3-Connected Graphs

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Abstract. It is well-known that every graph with maximum degree 4 has an orthogonal drawing with area at most $\frac{49}{64}n^2 + O(n) \approx 0.76n^2$. In this paper, we show that if the graph is 3-connected, then the area can be reduced even further to $\frac{9}{16}n^2 + O(n) \approx 0.56n^2$. The drawing uses the 3-canonical order for (not necessarily planar) 3-connected graphs, which is a special Mondshein sequence and can hence be computed in linear time. To our knowledge, this is the first application of a Mondshein sequence in graph drawing.

1 Introduction

An orthogonal drawing of a graph $G = (V, E)$ is an assignment of vertices to *points* and edges to *polygonal lines* connecting their endpoints such that all edge-segments are horizontal or vertical. Edges are allowed to intersect, but only in single points that are not bends of the polygonal lines. Such an orthogonal drawing can exist only if every vertex has degree at most 4; we call such a graph a *4-graph*. It is easy to see that every 4-graph has an orthogonal drawing with area $O(n^2)$, and this is asymptotically optimal [17].

For planar 2-connected graphs, several authors showed independently [10,15] how to achieve area $n \times n$, and this is optimal [16]. We measure the drawing-size as follows. Assume (as we do throughout the paper) that all vertices and bends are at points with integral coordinates. If H rows and W columns of the integer grid intersect the drawing, then we say that the drawing occupies a $W \times H$*-grid* with *width* W, *height* H, *half-perimeter* $H + W$ and *area* $H \cdot W$. Some papers count as width/height the width/height of the smallest enclosing axis-aligned box. This is one unit less than with our measure.

For arbitrary graphs (i.e., graphs that are not necessarily planar), improved bounds on the area of orthogonal drawings were developed much later, decreasing from $4n^2$ [11] to n^2 [1] to $0.76n^2$ [9]. (In all these statements, we omit lower-order terms for ease of notation.)

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Our Results: In this paper, we decrease the area-bound for orthogonal drawings further to $0.56n^2 + O(n)$ under the assumption that the graph is 3-connected. The approach is similar to the one by Papakostas and Tollis [9]: add vertices to the drawing in a specific order, and pair some of these vertices so that in each pair one vertex re-uses a row or column that was used by the other. The main difference in our paper is that 3-connectivity allows the use of a different, stronger, vertex order.

It has been known for a long time that any *planar* 3-connected graph has a so-called canonical order [7], which is useful for planar graph drawing algorithms. It was mentioned that such a canonical order also exists in non-planar graphs (e.g. in [4, Remark on p.113]), but it was not clear how to find it efficiently, and it has to our knowledge not been used for graph drawing algorithms. Recently, the second author studied the so-called Mondshein sequence, which is an edge partition of a 3-connected graph with special properties [8], and showed that it can be computed in linear time [13]. A Mondshein sequence is the appropriate generalization of the canonical order to (not necessarily planar) 3-connected graphs [13] and is most naturally defined by ear decompositions. However, in order to highlight its relation to canonical orders, we define a Mondshein sequence here as a special vertex partition and call it a 3-canonical order.

We use this 3-canonical order to add vertices to the orthogonal drawing. This almost immediately lowers the resulting area, because vertices with one incoming edge can only occur in chains. We then mimic the pairing-technique of Papakostas and Tollis, and pair groups of the 3-canonical order in such a way that even more rows and columns can be saved, resulting in a half-perimeter of $\frac{3}{2}n + O(1)$ and the area-bound follows.

No previous algorithms were known that achieve smaller area for 3-connected 4-graphs than for 2-connected 4-graphs. For *planar* graphs, the orthogonal drawing algorithm by Kant [7] draws 3-connected planar 4-graphs with area $(\frac{2}{3}n)^2 + O(n)$ [14], while the best-possible area for planar 2-connected graphs is n^2 [16].

2 Preliminaries

Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges. The *degree* of a vertex v is the number of incident edges. In this paper, all graphs are assumed to be *4-graphs*, i.e., all vertex degrees are at most 4. A graph is called *4-regular* if every vertex has degree exactly 4; such a graph has $m = 2n$ edges.

A graph G is called *connected* if, for any two vertices u, v, there is a path in G connecting u and v. It is called 3-connected if $n > 3$ and, for any two vertices u, v, the graph $G - \{u, v\}$ is connected.

A *loop* is an edge (v, v) that connects an endpoint with itself. A *multi-edge* is an edge (u, v) for which another copy of edge (u, v) exists. When not otherwise stated, the graph G that we want to draw is *simple*, i.e., it has neither loops nor multi-edges. While modifying G, we will sometimes temporarily add a *double edge*, i.e., an edge for which exactly one other copy exists (we refer always to the added edge as double edge, the copy is not a double edge).

2.1 The 3-Canonical Order

Definition 1. Let G be a 3-connected graph. A *3-canonical order* (or *Mondshein sequence*) is a partition of V into groups $V = V_1 \cup \cdots \cup V_k$ such that

- $V_1 = \{v_1, v_2\}$, where (v_1, v_2) is an edge.
- $V_k = \{v_n\}$, where (v_1, v_n) is an edge.
- For any $1 < i < k$, one of the following holds:
  - $V_i = \{z\}$, where z has at least two predecessors and at least one successor.
  - $V_i = \{z_1, \ldots, z_\ell\}$ for some $\ell \geq 2$, where
    - $z_1, \ldots, z_\ell$ is an induced path in G (i.e. edges $z_1 - z_2 - \cdots - z_\ell$ exist, and there are no edges (z_i, z_j) with $i < j - 1$),
    - z_1 and z_ℓ have exactly one predecessor each, and these predecessors are different,
    - z_j for $1 < j < \ell$ has no predecessor,
    - $z_j \in V_i$ for $1 \leq j \leq \ell$ has at least one successor.

Here, a *predecessor* [*successor*] of a vertex in V_i is a neighbor that occurs in a group V_h with $h < i$ [$h > i$]. See Fig. 1 for a 3-canonical order.

We call a vertex group V_i a *singleton* if $|V_i| = 1$, and a *chain* if $|V_i| \geq 2$ and $i \geq 2$. We distinguish chains further into *short chains* with $|V_i| = 2$ and *long chains* with $|V_i| \geq 3$. A 3-canonical order imposes a natural orientation on the edges of the graph from lower-indexed groups to higher-indexed groups and, for edges within a chain, from one (arbitrary) end of the path to the other. This implies in-degree $indeg(v) \geq 2$ for any singleton, $indeg(v) = 2$ for exactly one vertex of each chain, and $indeg(v) = 1$ for all other vertices of a chain.

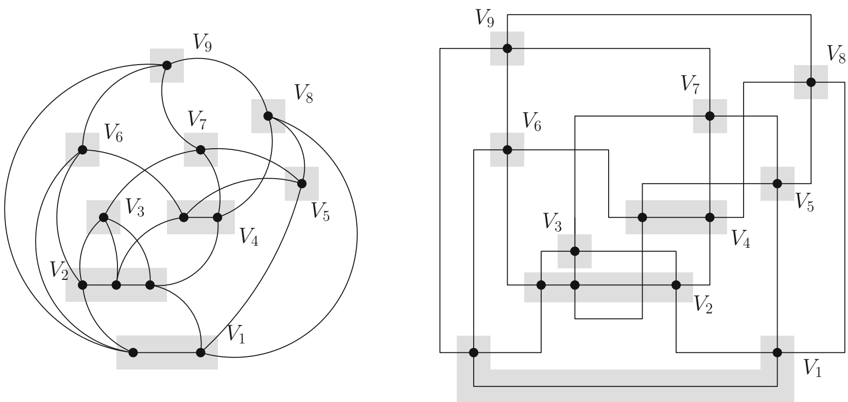

Fig. 1. A 4-regular 3-connected graph with a 3-canonical order, and the drawing created with our algorithm. For illustrative purposes, we show the drawing exactly as created, even though many more grid lines and bends could be saved with straightforward compaction steps. V_2 is a long chain, V_4 is a short chain, V_5 is a 2-2-singleton, V_3, V_6, V_7 and V_8 are 3-1-singletons.

Numerous related methods of ordering vertices of 3-connected graphs exist, e.g. *(2,1)-sequences* [8], *non-separating ear decompositions* [2,13], and, limited to planar graphs, *canonical orders for maximal planar graphs* [6], *canonical orders for 3-connected planar graphs* [7] and *orderly spanning trees* [3]. A Mondshein sequence (i.e. a 3-canonical order) of a 3-connected graph implies all these orders, up to minor subtleties.

The most efficient way known to compute a Mondshein sequence (proving in particular that one exists) uses *non-separating ear decompositions* [2,13]. This is a partition of the edges into *ears* $P_1 \cup \dots \cup P_k = E$ such that P_1 is an induced cycle, P_i for $i > 1$ is a non-empty induced path that intersects $P_1 \cup \dots \cup P_{i-1}$ in exactly its endpoints, and $G - \bigcup_{j=1}^{i} P_j$ is connected for every $i < k$. Such a non-separating ear decomposition exists for any 3-connected graph [2], and we can even fix two edges v_1v_2 and v_2v_n and require that v_1v_2 is in the cycle P_1 and that v_n is the only vertex in P_k; hence, P_k will be a singleton.

Further, such a non-separating ear decomposition can be computed in linear time [12,13]. The sets of newly added vertices for each P_i will be the vertex groups of a 3-canonical order (additionally, P_1 is split into the groups $V_1 := \{v_1, v_2\}$ and $V_2 := V(P_1) - \{v_1, v_2\}$). Although vertices in V_i may have arbitrarily many incident edges in a non-separating ear decomposition, we can easily get rid of these extra edges by a simple short-cutting routine in linear time (see Lemmas 8 and 12 in [12]). This gives a 3-canonical order. Thus, a linear-time algorithm for computing a 3-canonical order follows immediately from the one for non-separating ear decompositions.

2.2 Making 3-Connected 4-Graphs 4-Regular

It will greatly simplify the description of the algorithm if we only give it for 4-regular graphs. Thus, we want to modify a 3-connected 4-graph G such that the resulting graph G' is 4-regular, draw G', and then delete added edges to obtain a drawing of G. However, we must maintain a simple graph since the existence of 3-canonical orders depends on simplicity. This turns out to be impossible (e.g. for the graph obtained from the octahedron by subdividing two distinct edges with a new vertex and joining the new vertices by an edge), but allowing one double edge is sufficient.

Lemma 2. *Let G be a simple 3-connected 4-graph with $n \geq 5$. Then we can add edges to G' such that the resulting graph G' is 3-connected, 4-regular, and has at most one double edge.*

Proof. Since G is 3-connected, any vertex has degree 3 or 4. If there are four or more vertices of degree 3, then they cannot be mutually adjacent (otherwise $G = K_4$, which contradicts $n \geq 5$). Hence, we can add an edge between two non-adjacent vertices of degree 3; this maintains simplicity and 3-connectivity.

We repeat until only two vertices of degree 3 are left (recall that the number of vertices of odd degree is even). Now an edge between these two vertices is added, even if one existed already; this edge is the only one that may become a double edge. The resulting graph is 4-regular and satisfies all conditions. □

3 Creating Orthogonal Drawings

From now on, let G be a 3-connected 4-regular graph that has no loops and at most one double edge. Compute a 3-canonical order $V = V_1 \cup \cdots \cup V_k$ of G with $V_k = \{v_n\}$, choosing $v_1 v_n$ to be the double edge if there is one. Let x^{short} and x^{long} be the number of short and long chains. Let $x^{j\text{-}\ell}$ be the number of vertices with in-degree j and out-degree ℓ. Since G is 4-regular, we must have $j + \ell = 4$. A *j-ℓ-singleton* is a vertex z that constitutes a singleton group V_i for $1 < i \leq k$ and that has in-degree j and out-degree ℓ.

Observation 3. *Let G be a 4-regular graph with a 3-canonical order. Then*

1. $x^{0\text{-}4} = x^{4\text{-}0} = 1$
2. $x^{1\text{-}3} = x^{3\text{-}1}$
3. *Every chain V_i contributes one to $x^{2\text{-}2}$ and $|V_i| - 1$ to $x^{1\text{-}3}$.*

Proof. (1) holds, since every vertex that is different from v_n has a successor, and every vertex that is different from v_1 has an incoming edge from either a predecessor or within its chain. For (2), observe that $2n = m = \sum_v indeg(v) = x^{1\text{-}3} + 2x^{2\text{-}2} + 3x^{3\text{-}1} + 4x^{4\text{-}0}$ and $n = x^{0\text{-}4} + x^{1\text{-}3} + x^{2\text{-}2} + x^{3\text{-}1} + x^{4\text{-}0}$, and rearrange. For (3), say $V_i = \{z_1, \ldots, z_\ell\}$ is directed from z_1 to z_ℓ. Then $indeg(z_\ell) = 2$ and $indeg(z_j) = 1$ for all $j < \ell$. □

3.1 A Simple Algorithm

As in many previous orthogonal drawing papers [1,7,9], the idea is to draw the graph G_i induced by $V_1 \cup \cdots \cup V_i$ in such a way that all *unfinished edges* (edges with one end in G_i and the other in $G - G_i$) end in a column that is unused above the point where the drawing ends.

Embedding the First Two Vertices: If (v_1, v_n) is a single edge, then v_1 and v_2 are embedded exactly as in [1]: refer to Fig. 2. If (v_1, v_n) is a double edge, then it was added only for the purpose of making the graph 4-regular and need not be drawn. In that case, we omit one of the outgoing edges of v_1 that has a bend.

Embedding a Singleton: If V_i is a singleton $\{z\}$, we embed z exactly as in [1]: refer to Fig. 2. For $indeg(z) \in \{2, 3\}$, this adds one new row and $outdeg(z) - 1 = 3 - indeg(z)$ many new columns. For $indeg(z) = 4$, $z = v_n$; if (v_1, v_n) is a double edge, we omit the edge having two bends.

Embedding Chains: Let V_i be a chain, say $V_i = \{z_1, \ldots, z_\ell\}$ with $\ell \geq 2$. For chains, our algorithm is substantially different from [1]. Only z_1 and z_ℓ have predecessors. We place the chain-vertices on a new horizontal row above the previous drawing, between the edges from the predecessors; see Fig. 3. We add new columns as needed to have space for new vertices and outgoing edges without using columns that are in use for other unfinished edges. We also use a second new row if the chain is a long chain.

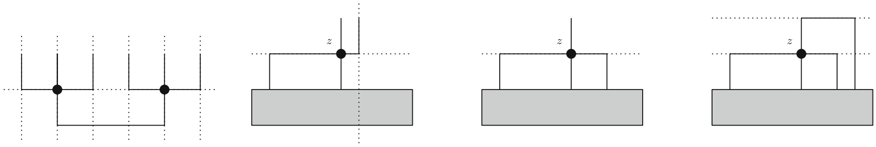

Fig. 2. Embedding the first two vertices, and a singleton with in-degree 2, 3, 4. Newly added grid-lines are dotted.

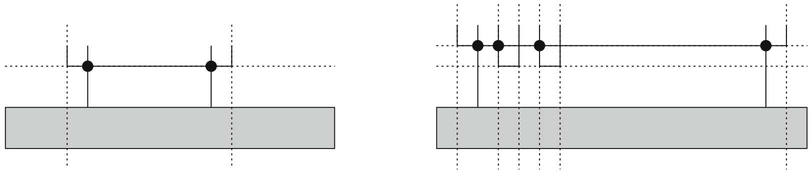

Fig. 3. Embedding short and long chains.

Observation 4. *The increase in the half-perimeter is as follows:*

- *For the first and last vertex-group:* $O(1)$
- *For a 3-1-singleton:* $+1$ *(we add one row)*
- *For a 2-2-singleton:* $+2$ *(we add one row and one column)*
- *For a short chain:* $+3$ *(we add one row and two columns)*
- *For a long chain* V_i*:* $+2|V_i|$ *(we add two rows and* $2|V_i| - 2$ *columns)*

Corollary 5. *The half-perimeter is at most* $\frac{3}{2}n + \frac{1}{2}x^{2\text{-}2} - x^{short} + O(1)$.

Proof. From Observation 4 and using Observation 3.3 the half-perimeter is at most $x^{3\text{-}1} + 2x^{2\text{-}2} + 2x^{1\text{-}3} - x^{\text{short}} + O(1)$. By Observation 3.2 this is at most $\frac{3}{2}x^{3\text{-}1} + 2x^{2\text{-}2} + \frac{3}{2}x^{1\text{-}3} - x^{\text{short}} + O(1)$, which gives the result. □

Theorem 6. *Every simple 3-connected 4-graph has an orthogonal drawing of area at most* $\frac{25}{36}n^2 + O(n) \approx 0.69n^2$.

Proof. First, make the graph 4-regular, compute the 3-canonical order, and consider the number $x^{2\text{-}2}$ of 2-2-vertices.

1. If $x^{2\text{-}2} \leq n/3$, apply the above algorithm. By Corollary 5, the half-perimeter is at most $\frac{3}{2}n + \frac{1}{6}n + O(1) \leq \frac{5}{3}n + O(1)$.
2. If $x^{2\text{-}2} \geq n/3$, apply the algorithm from [9]. They state their area bound as $0.76n^2 + O(1)$, but one can observe (see [9, Theorem 3.1, ll.2–5]) that their half-perimeter is at most $2n - \frac{1}{2}(x^{1\text{-}3} + x^{2\text{-}2}) + O(1)$, since they pair at least $x^{1\text{-}3} + x^{2\text{-}2}$ vertices. Using Observation 3.2 and ignoring $O(1)$ terms, we have $x^{1\text{-}3} + x^{2\text{-}2} = \frac{1}{2}x^{1\text{-}3} + x^{2\text{-}2} + \frac{1}{2}x^{3\text{-}1} = \frac{1}{2}n + \frac{1}{2}x^{2\text{-}2}$. Hence, the half-perimeter of their algorithm is at most $\frac{7}{4}n - \frac{1}{4}x^{2\text{-}2} + O(1) \leq (\frac{7}{4} - \frac{1}{12})n + O(1) = \frac{5}{3}n + O(1)$.

In both cases, we get a drawing with half-perimeter $\frac{5}{3}n + O(1)$. The area of it is maximal if the two sides are equally large and thus at most $(\frac{5}{6}n + O(1))^2$. □

3.2 Improvement via Pairing

We already know a bound of $\frac{3}{2}n+\frac{1}{2}x^{2\text{-}2}-x^{\text{short}}+O(1)$ on the half-perimeter. This section improves this further to half-perimeter $\frac{3}{2}n+O(1)$. The idea is strongly inspired by the pairing technique of Papakostas and Tollis [9]. They created pairs of vertices with special properties such that at least $\frac{1}{2}(x^{2\text{-}2}+x^{1\text{-}3})$ such pairs must exist. For each pair, they can save at least one grid-line, compared to the $2n+O(1)$ grid-lines created with [1].

Our approach is similar, but instead of pairing vertices, we pair groups of the canonical order by scanning them in backward order as follows:

1. Initialize $i := k-1$. (We ignore the last group, which is a 4-0-singleton.)
2. While V_i is a 3-1-singleton and $i > 2$, set $i := i-1$.
3. If $i = 2$, break. Else, V_i is a chain or a 2-2-singleton and we choose the partner of V_i as follows: Initialize $j := i-1$. While V_j is a 3-1-singleton whose successor is not in V_i, set $j := j-1$. Now, pair V_i with V_j. Observe that such a V_j with $j \geq 2$ always exists, since $i > 2$ and V_2 is not a 3-1-singleton.
4. Update $i := j-1$ and repeat from Step (2) onwards.

In the small example in Fig. 1, the 2-2-singleton V_5 gets paired with the short chain V_4, and all other groups are not paired.

Observe that, with the possible exception of V_2, every chain is paired and every 2-2-vertex is in a paired group (either as 2-2-singleton or as part of a chain). Hence there are at least $\frac{1}{2}(x^{2\text{-}2}-1)$ pairs. The key observation is the following:

Lemma 7. *Let V_i, V_j be two vertex groups that are paired. Then there exists a method of drawing V_i and V_j (without affecting the layout of any other vertices) such that the increase to rows and columns is at most $2|V_i \cup V_j|-1$.*

We defer the (lengthy) proof of Lemma 7 to the next section, and study here first its consequences. We can draw V_1 and V_k using $O(1)$ grid-lines. We can draw V_2 using $2|V_2| = 2x^{2\text{-}2}_{V_2} + 2x^{1\text{-}3}_{V_2}$ new grid-lines, where $x^{\ell\text{-}k}_W$ denotes the number of vertices of in-degree ℓ and out-degree k in vertex set W. We can draw any unpaired 3-1-singleton using one new grid-line. Finally, we can draw each pair using $2|V_i \cup V_j|-1 = 2x^{2\text{-}2}_{V_i\cup V_j}+2x^{1\text{-}3}_{V_i\cup V_j}-1$ new grid-lines. This covers all vertices, since all 2-2-singletons and all chains belong to pairs or are V_2, and since there are no 1-3-singletons.

Putting it all together and using Observation 3.2, the number of grid-lines hence is $2x^{1\text{-}3}+2x^{2\text{-}2}+x^{3\text{-}1}-\#\text{pairs}+O(1) \leq 2x^{1\text{-}3}+\frac{3}{2}x^{2\text{-}2}+x^{3\text{-}1}+O(1) = \frac{3}{2}n+O(1)$ as desired. Since a drawing with half-perimeter $\frac{3}{2}n$ has area at most $(\frac{3}{4}n)^2 = \frac{9}{16}n^2$, we can conclude:

Theorem 8. *Every simple 3-connected 4-graph has an orthogonal drawing of area at most $\frac{9}{16}n^2+O(n) \approx 0.56n^2$.*

We briefly discuss the run-time. The 3-canonical order can be found in linear time. Most steps of the drawing algorithm work in constant time per vertex, hence $O(n)$ time total. One difficulty is that to place a group we must know the relative order of the columns of the edges from the predecessors. As discussed extensively in [1], we can do this either by storing columns as a balanced binary search tree (which uses $O(\log n)$ time per vertex-addition), or using the data structure by Dietz and Sleator [5] which allows to find the order in $O(1)$ time per vertex-addition. Thus, the worst-case run-time to find the drawing is $O(n)$.

4 Proof of Lemma 7

Recall that we must show that two paired vertex groups V_i and V_j, with $j < i$, can be embedded such that we use at most $2|V_i| + 2|V_j| - 1$ new grid-lines. The proof of this is a massive case analysis, depending on which type of group V_i and V_j are, and whether there are edges between them or not.[1] We first observe some properties of pairs.

Observation 9. *By choice of the pairing, the following holds:*

1. *For any pair (V_i, V_j) such that $j < i$, V_i is either a 2-2-singleton or a chain.*
2. *If V_i is paired with V_j such that $j < i$, then all predecessors of V_i are in V_j or occurred in a group before V_j.*

The following notation will cut down the number of cases a bit. We say that groups V_i and V_j are *adjacent* if there is an edge from a vertex in one to a vertex in the other group. If two paired groups V_i, V_j are not adjacent, then by Observation 9.2 all predecessors of V_i occur before group V_j. We hence can safely draw V_i first, and then draw V_j, thereby effectively exchanging the roles of V_i and V_j in the pair. Now, we distinguish five cases:

1. At least one of V_i and V_j is a short chain. Say V_i is the short chain, the other case is similar. Recall that the standard layout for a short chain uses 3 new grid-lines, but $x^{2\text{-}2}_{V_i} + x^{1\text{-}3}_{V_i} = 2$. So the layout of a short chain automatically saves one grid-line. We do not change the algorithm at all in this case; laying out V_i and V_j exactly as before results in at most $2x^{2\text{-}2}_{V_i \cup V_j} + 2x^{1\text{-}3}_{V_i \cup V_j} - 1$ new grid-lines. (This is what happens in the example of Fig. 1.)

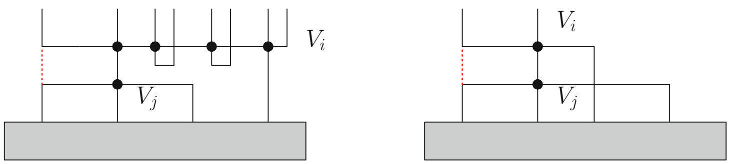

Fig. 4. Reusing the column freed by a 3-1-singleton with a later chain or singleton. In this and the following figures, the re-used grid-line is dotted and red.

[1] The constructions we give have been designed as to keep the description simple; often even more grid-lines could be saved by doing more complicated constructions.

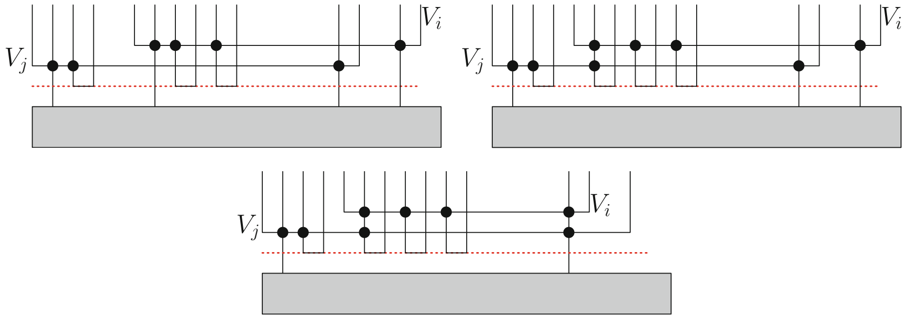

Fig. 5. Sharing the extra row between two long chains when there are 0, 1 or 2 predecessors of V_i in V_j.

2. One of V_i and V_j is a 3-1-singleton. By Observation 9, the 3-1-singleton must be V_j. By the pairing algorithm, the unique outgoing edge of the 3-1-singleton must lead to V_i. Draw V_j as before. We can then draw V_i such that it re-uses one of the columns that were freed by V_j; see Fig. 4.
3. V_i and V_j are both long chains. In this case, both V_i and V_j can use the same extra row for the "detours" that their *middle* vertices (by which we mean vertices that are neither the first nor the last vertex of the chain) use. Since we can freely choose into which columns these middle vertices are placed, we can ensure that none of these "detours" overlap and, hence, one row suffices for both chains. This holds even if one or both of the predecessors of V_i are in V_j, as these are distinct and the two corresponding incoming edges of V_i extend the edges that were already drawn for V_j; see Fig. 5.
4. None of the previous cases applies and V_j is a 2-2-singleton. By Observation 9.1 and since Case (1) does not apply, V_i is either a 2-2-singleton or a long chain. There are two columns reserved for edges from predecessors of V_j. Since predecessors of V_i are distinct, at most one of them can be the 2-2-singleton in V_j. Thus, there also is at least one column reserved for an edge from a predecessor of V_i not in V_j. We call these three or four columns the *predecessor-columns*. We have three sub-cases depending on the relative location of these columns:
   (a) The leftmost predecessor-column leads to V_j. In this case, we save a column almost exactly as in [9]. Place V_j as before, in the right one of its predecessor-columns. This leaves the leftmost predecessor-column free to be reused. Now no matter whether V_i is a 2-2-singleton or a long chain, or whether V_i is adjacent to V_j or not, we can re-use this leftmost column for one outgoing edge of V_i with a suitable placement; see Fig. 6.
   (b) The rightmost predecessor-column leads to V_j. This case is symmetric to the previous one.
   (c) The leftmost and rightmost predecessor-columns lead to V_i. This implies that V_i has two predecessors not in V_j. Hence, V_i cannot be adjacent to V_j. If V_i is a 2-2-singleton, then (as discussed earlier) we can exchange

the roles of V_i and V_j, which brings us to Case 4(a). If V_i is a long chain, then place V_j in the standard fashion. We then place the long chain V_i such that the "detours" of its middle vertices re-use the row of V_j. See Fig. 7.

5. None of the previous cases applies. Then V_j is a chain, say $V_j = \{z_1, \ldots, z_\ell\}$, and $\ell \geq 3$ since Case (1) does not apply. We assume the naming is such that the predecessor column of z_1 is left of the predecessor column of z_ℓ.

   Since we are not in a previous case, V_i must be a 2-2-singleton, say z. If V_i is not adjacent to V_j, then we can again exchange the roles of V_i and V_j, which brings us to Case (4). Hence, we may assume that there are edges between V_j and V_i. We distinguish the following sub-cases depending on how many such edges there are and whether their ends are middle vertices.

   (a) z has exactly one neighbor in V_j, and it is either z_1 or z_ℓ. We rearrange $V_i \cup V_j$ into two different chains. Let z be adjacent to z_1 (the other case is symmetric). Then $\{z, z_1\}$ forms one chain and $\{z_2, \ldots, z_\ell\}$ forms another. Embed these two chains as usual. Since $\{z, z_1\}$ forms a short chain, this saves one grid-line; see Fig. 8(left).

   (b) z has exactly one neighbor in V_j, and it is z_h for some $1 < h < \ell$. Embed the chain V_j as usual, but omit the new column next to z_h. For embedding z, we place a new row *below* the rows for the chain. Using this new row, we can connect the bottom outgoing edge of z_h to the horizontal incoming edge of z; see Fig. 8(right).

   (c) z has two neighbors in V_j, and both of them are middle vertices z_g, z_h for $1 < g < h < \ell$. Embed the chain V_j as usual, but omit the new columns next to z_g and z_h. Place a new row *between* the two rows for the chain and use it to connect the two bottom outgoing edges of z_g and z_h to place z, re-using the row for the detours to place the bottom outgoing edge of z. This uses an extra column for z, but saved two columns at z_g and z_h, so overall one grid-line has been saved; see Fig. 9(top left).

   (d) z is adjacent to z_1 and z_2 (the case of adjacency to $z_{\ell-1}$ and z_ℓ is symmetric). Embed $z_2, \ldots, z_\ell$ as usual for a chain, then place z_1 below z_2. The horizontally outgoing edge of z_2 intersects one outgoing edge of z_1. Put z at this place to save a row and a column; see Fig. 9(top right).

   (e) z is adjacent to z_1 and z_h with $h > 2$ (the case of adjacency to z_ℓ and z_h with $h < \ell - 1$ is symmetric). Draw the chain V_j with the modification that z_h is *below* z_{h-1}, but still all middle vertices use the same extra row for their downward outgoing edges. This uses 3 rows, but now z can be placed using the two left outgoing edges of z_1 and z_h, saving a row for z and a column for the left outgoing edge of z_h; see Fig. 9(bottom), both for $h < \ell$ and $h = \ell$.

This ends the proof of Lemma 7 and hence shows Theorem 8.

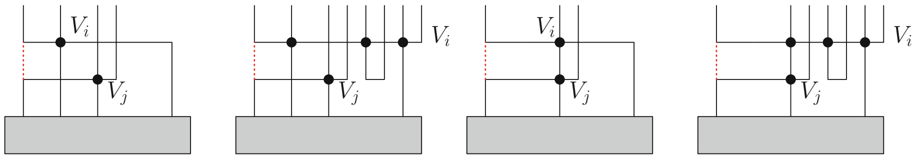

Fig. 6. Reusing the predecessor-column freed by a 2-2-singleton V_j in Case 4(a). Left two pictures: V_i is not adjacent to V_j. Right two: V_i is adjacent to V_j.

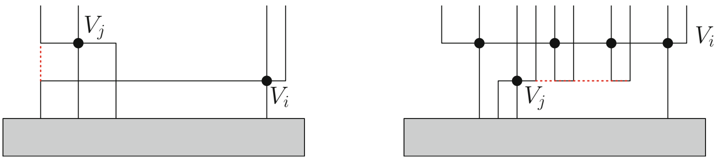

Fig. 7. If the predecessor-columns of V_j are between the ones of V_i, then we can either revert to Case 4(a) or the long chain V_i can re-use the row of V_j.

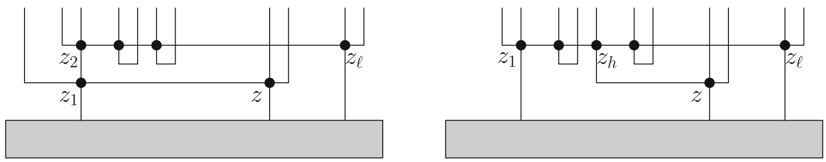

Fig. 8. V_j is a long chain, V_i is a 2-2-singleton with one predecessor in V_j.

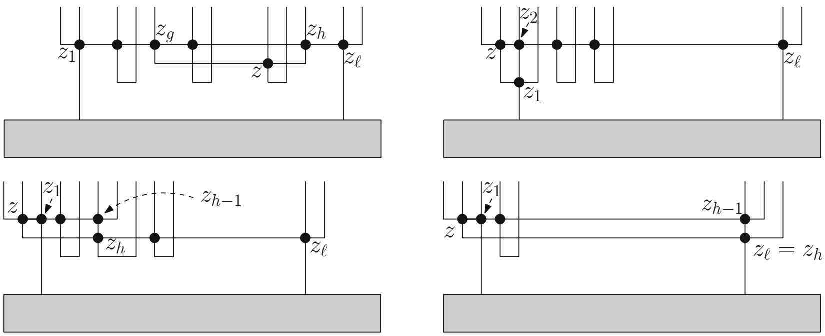

Fig. 9. V_j is a long chain, V_i is a 2-2-singleton, and there are exactly two edges between them. (Top) Cases 5(c) and (d). (Bottom) Case 5(e).

5 Conclusion

In this paper, we gave an algorithm to create an orthogonal drawing of a 3-connected 4-graph that has area at most $\frac{9}{16}n^2+O(n) \approx 0.56n^2$. As a main tool, we used the 3-canonical order / Mondshein sequence for non-planar 3-connected graphs, whose existence was long known but only recently efficient algorithms for it were found. To our knowledge, this is the first application of the 3-canonical order on non-planar graphs in graph-drawing. Among the many remaining open problems are the following:

- Can we draw 2-connected 4-graphs with area less than $0.76n^2$? A natural approach would be to draw each 3-connected component with area $0.56n^2$ and to merge them suitably, but there are many cases depending on how the cut-vertices and virtual edges are drawn, and so this is far from trivial.
- Can we draw 3-connected 4-graphs with $(2-\varepsilon)n$ bends, for some $\varepsilon > 0$? With an entirely different algorithm (not given here), we have been able to prove a bound of $2n - x^{2-2} + O(1)$ bends, so an improved bound seems likely.
- Our algorithm was strongly inspired by the one of Kant [7] for 3-connected planar graphs. Are there other graph drawing algorithms for planar 3-connected graphs that can be transferred to non-planar 3-connected graphs by using the 3-canonical order?

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