Chapter 3
Stability Problems

3.1 Phenomena

A beam is loaded in its axial direction by a compressive force. The force is increased. Suddenly the beam moves perpendicular to its axis: it buckles (Fig. 3.1).

Other instability phenomena of a beam are drilling under a compressive load and drilling under a bending load as well as combinations.

A similar effect, namely deflection, i.e. transverse displacement, under in-plane loading, can be seen considering a plate. Again it buckles (Fig. 3.2).

These phenomena have in common that the displacements occur perpendicular to the load direction when a certain load level is exceeded and that a theoretical equilibrium is possible for higher loads on the ideal system. However, a minimal disturbance—in practice always existing—will lead to buckling. This effect is called a *bifurcation* problem because of the two equilibrium paths (ideal and buckled), see Fig. 3.3.

In case of the two-legged truss from Fig. 2.12 the displacement starts being nearly proportional to the load but later the displacement more and more increases until the load cannot be enlarged any more. At that stage the loaded point is still above the line connecting the two foot points (Fig. 3.4). In a force-controlled test the system will comply suddenly and—provided that it is not destroyed—reach equilibrium not before the former top is now down (Fig. 3.5).

As in the cases above the load cannot exceed a critical one. Unlike in buckling the system moves in the direction predicted by the load. This type is call a snap-through problem. Snap-through needs not to be a system failure but can be desired like in case of a switch where the dynamic snap-through should limit the danger of an electric spark.

Common characteristic of these two phenomena is that there is a point where two neighboured equilibrium states with the same load level but slightly resp. infinitesimal different displacement states exist and thus a transition from one state to the other can occur without changing the load (see Fig. 3.11).

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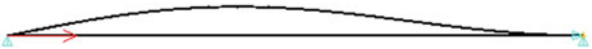

Fig. 3.1 Buckling of a beam, third Euler case

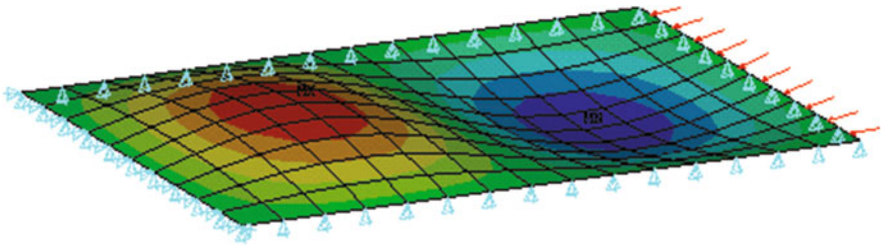

Fig. 3.2 Plate buckling

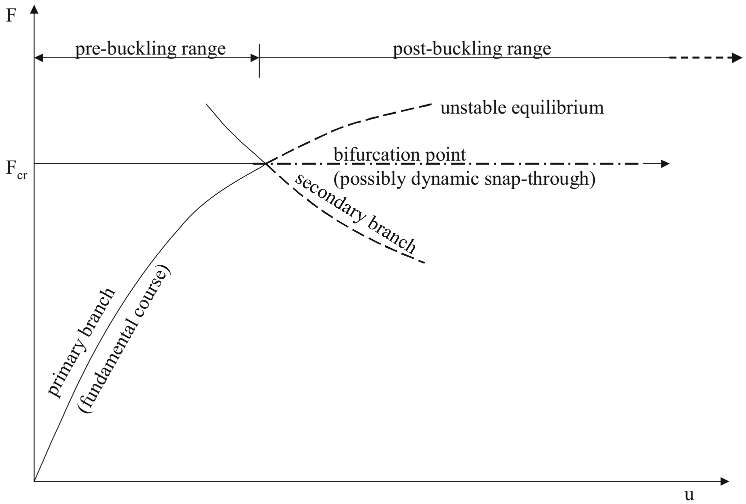

Fig. 3.3 Load–displacement diagram of a bifurcation problem

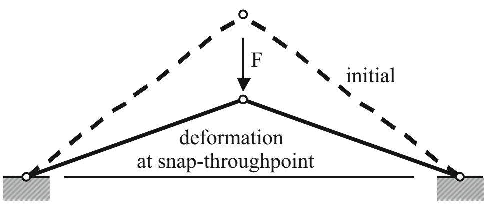

Fig. 3.4 Deformation at snap-through point

Bifurcation problems are classified by the post-critical behaviour (Fig. 3.6). If a load-increase—even a small one only—becomes possible after the bifurcation the *post-critical behaviour* is called *stable* otherwise *unstable*. The latter is very

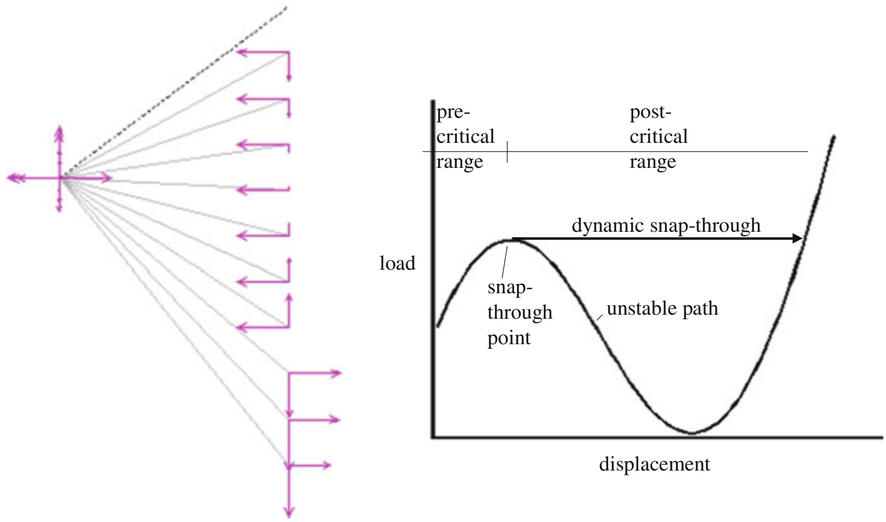

Fig. 3.5 Snap-through problem: displacement states, reaction forces and load–displacement curve

dangerous because the load level at the bifurcation point cannot be sustained which can denote the complete destruction. Therefore a higher safety factor must be chosen.

The post-critical behaviour can depend on the direction of the sudden motion, stable in the one, unstable in the other direction. This is called *asymmetric*.

Even if the post-critical behaviour is classified as stable such large displacements can occur that the system cannot be used any longer. Before buckling, however, the behaviour is stable even if a certain imperfection (see Sect. 3.4) leading to bending exists. Thus, it makes sense to determine a safety distance between the system in use and the ideal critical load.

If the post-critical behaviour is unstable bending or an imperfection will reduce the maximum load-carrying capacity significantly so that the ideal critical load is of limited meaning for the safety of the system. Thus it is of particular importance to take imperfections (see Sect. 3.4) into account.

In the load–displacement diagram (Fig. 3.3) the connection of the equilibrium states of the ideal system forms the primary path becoming unstable after the bifurcation point and thus existing theoretically only.

The equilibrium states after the bifurcation form the secondary path. However, further bifurcations (called secondary) can occur when the system jumps from one buckling mode to the other (Fig. 3.7). Some of these modes can be reached directly from the primary path but at load levels higher than the first critical one.

The danger of buckling and bifurcation can exist within the same system. Consider the two-legged truss. Before the snap-through the leg can buckle when its critical load is reached (Fig. 3.8). This will also result in an earlier snap-through (Fig. 3.9).

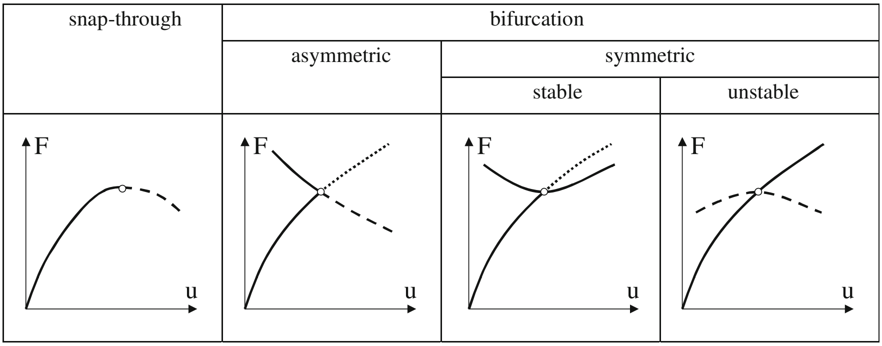

Fig. 3.6 Classification of instability phenomena after Koiter (instable branches dashed)

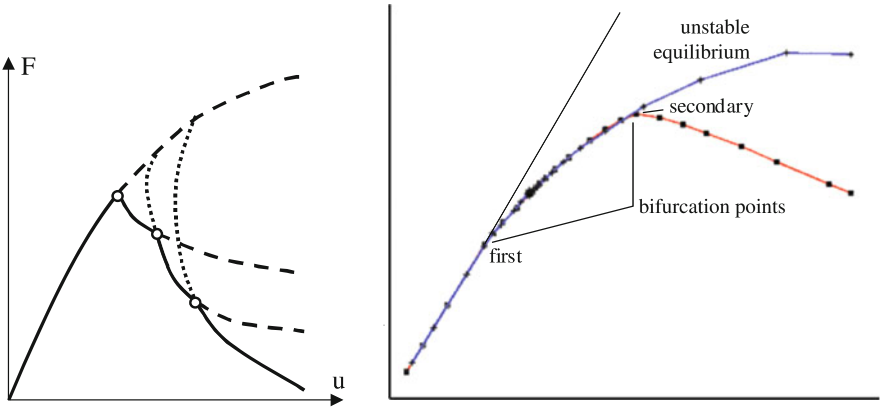

Fig. 3.7 Secondary bifurcation, schematic (*left*), stiffened shell (*right*)

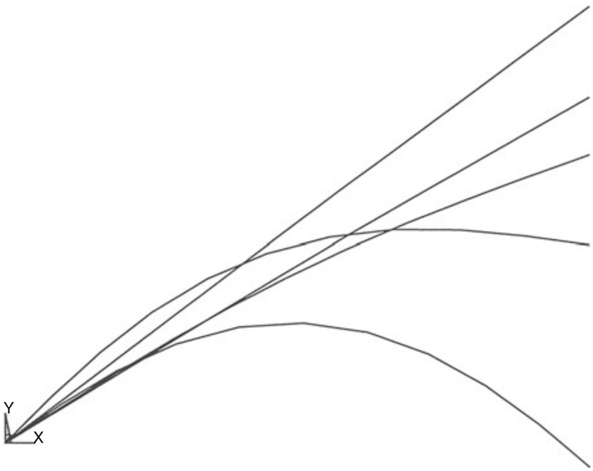

Fig. 3.8 Half model of the two-legged truss, deformed system before and after the bifurcation

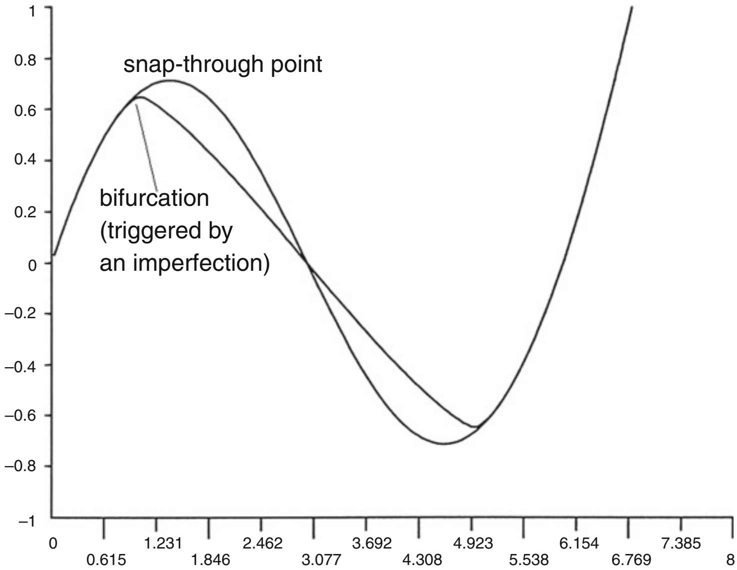

Fig. 3.9 Two-legged truss, snap-through and bifurcation problem

3.2 Conditions for Critical Points, Indifference Criterion

3.2.1 General

The equilibrium can be classified into stable, indifferent and unstable. If the equilibrium is stable an applied load will lead to a displacement but the system comes back to its previous state if the load is removed; in case of an unstable equilibrium the system will never come back but will move away from its previous configuration. In between is the indifferent equilibrium where the system will remain in its new configuration when the load is removed (Fig. 3.10).

At the critical point, be it a snap-through or a bifurcation point, an—at least infinitesimal—motion without a load increment is possible. This means indifferent equilibrium (Fig. 3.11).

Usually the displacement due to a load increment is calculated in the Newton-Raphson scheme by

$$\mathbf{K}_T \Delta\, \hat{\mathbf{u}} = \Delta \mathbf{f} \tag{3.1}$$

At the critical point, however,

$$\mathbf{K}_T \Delta\, \hat{\mathbf{u}} = \mathbf{0} \tag{3.2}$$

holds due to $\Delta \mathbf{f} = 0$.

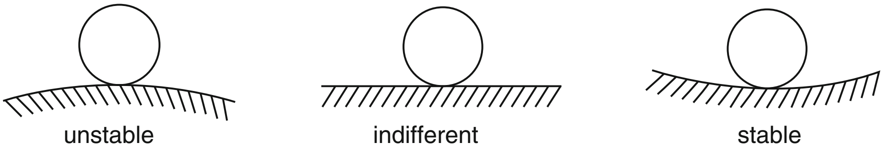

Fig. 3.10 Equilibrium states

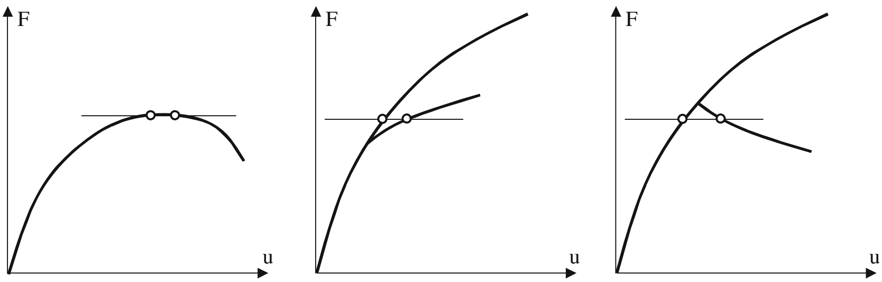

Fig. 3.11 Load–displacement curves for snap-through (*left*) and bifurcation (*right*) with two neighbouring equilibrium states at the same load level

This system of equations has a non-trivial solution only if the matrix $\mathbf{K}_T$ is singular. The trivial solution is that no displacement increment occurs if no load increment is applied which would be calculated if the system of equations was uniquely solvable.

Indicators for the singular matrix are:

1. the determinant $\det\mathbf{K}_T = 0$ or
2. at least one eigenvalue ω of $\mathbf{K}_T$ is zero, where ω is the solution of $(\mathbf{K}_T - \omega\mathbf{I})\boldsymbol{\varphi} = \mathbf{0}$ or
3. at least one zero main diagonal element (pivot) occurs in the matrix triangularised in the Gaussian algorithm.

These three conditions are equivalent. It has to be assumed that this holds for a converged state.

Following these criteria a solution is on an unstable path if

1. the determinant $\det\mathbf{K}_T < 0$ or
2. there is at least one negative eigenvalue ω or
3. at least one negative main diagonal element of the triangularised matrix occurs.

Increased loads can lead to more negative eigenvalues or main diagonal elements each indicating a possible bifurcation point.

Condition 1 (determinant) has some limitations:

- An even number of negative eigenvalues lead to a positive determinant although the actual load path is unstable (example in Fig. 3.12).

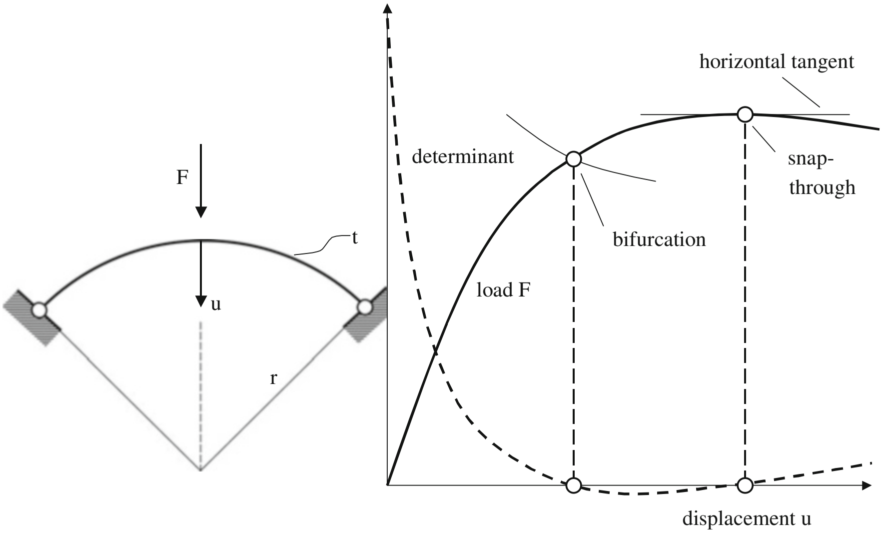

Fig. 3.12 Shallow circular arc, load–displacement curve and determinant

- The easiest way to calculate the determinant is to multiply the main diagonal elements after a Gaussian elimination process. That means criterion 2 (pivots) can be evaluated earlier.
- The determinant can be a very large number so that 10^{990} can indicate instability when it had been $10^{1,000}$ before.

3.2.2 Formulations of the Instability Condition

As shown in Sect. 2.3.4.1 the tangential stiffness matrix has at least two parts, the initial displacement and the initial stress matrix:

$$\mathbf{K}_T = \mathbf{K}_u + \mathbf{K}_\sigma \tag{3.3}$$

Some authors use a split of the initial displacement matrix $\mathbf{K}_u$—which makes sense in a certain context only—, a split into the constant part from linear theory $\mathbf{K}_0$ and a non-linear part $\mathbf{K}_n$:

$$\mathbf{K}_T = \mathbf{K}_0 + \mathbf{K}_n + \mathbf{K}_\sigma \tag{3.4}$$

In this way different eigenvalue problems (EVPs) can be formulated:

1. the above mentioned one $(\mathbf{K}_T - \omega\mathbf{I})\,\boldsymbol{\varphi} = \mathbf{0}$, where the critical eigenvalue is $\omega = 0$ (mentioned above as indicator)
2. $(\mathbf{K}_u + \Lambda_2\mathbf{K}_\sigma)\,\boldsymbol{\varphi} = (\mathbf{K}_0 + \mathbf{K}_n + \Lambda_2\mathbf{K}_\sigma)\,\boldsymbol{\varphi} = \mathbf{0}$ where $\Lambda_2 = 1$ is critical

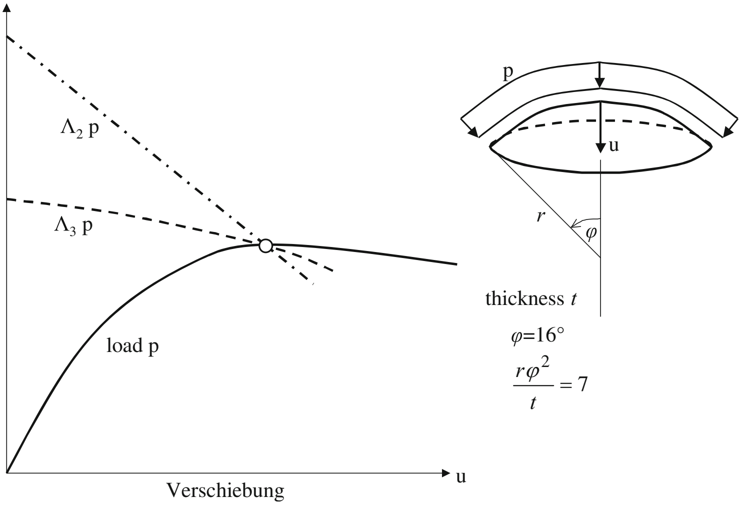

Fig. 3.13 Evolution of the eigenvalues Λ_2 and Λ_3 for a spherical cap under external pressure

3. $[\mathbf{K}_0 + \Lambda_3(\mathbf{K}_n + \mathbf{K}_\sigma)]\,\boldsymbol{\varphi} = \mathbf{0}$ where $\Lambda_3 = 1$ is critical.

$\omega = 0$ in the first case as well as $\Lambda_i = 1$ in the other two cases means, that the total matrix (in the brackets in front of $\boldsymbol{\varphi}$) yields $\mathbf{K}_T$, i.e. the solutions will match at the critical point. The evolution of the eigenvalues with the load level, however, can be different (see Fig. 3.13 for an example).

One disadvantage of eigenvalue problem 1 can be that some eigenvalue solvers have difficulties with negative eigenvalues (at over-critical load levels), a further advantage of the formulations 2 and 3 is that

$$\mathbf{f}^* = \Lambda_i \mathbf{f}^{ext} \tag{3.5}$$

can be taken as the next estimate for the critical load *during the load incrementation process*. It approaches the critical load from the linear buckling analysis (LBA) (Sect. 2.2.3) if the applied load is small. In all cases the load must be applied incrementally until one of the instability criteria is fulfilled. At least in the vicinity of the critical load an extrapolation of the relation between eigenvalue and load level can become meaningful.

Figure 3.14 shows the load–deflection curve of the two-legged truss together with the estimated critical load $\mathbf{f}^*$ from eigenvalue analysis of type 2.

The most important application of these type of eigenvalue buckling analysis parallel to a non-linear calculation (eigenvalue tracking) is not to determine the critical load but

- to decide whether non-convergence occurs due to a physical stability problem ($\omega \approx 0$ or $\Lambda \approx 1$) or to numerical reasons
- to detect if a solution state is on an unstable path ($\omega < 0$ or $\Lambda < 1$)

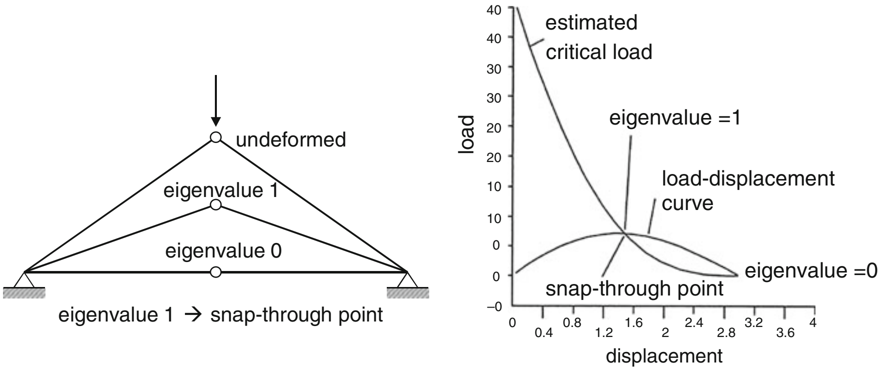

Fig. 3.14 Evolution of the eigenvalue Λ_2 with the load level over the related displacement

3.2.3 Modal Analysis (Natural-Frequencies Analysis) and Stability Problems

As it is well known pre-stressing influences the natural frequencies of a system. The best examples are strings of music instruments. If a system with a certain bending stiffness shows compressive stresses the natural frequencies decrease. In case of a stability problem the system can be deflected without returning after removing the perturbation. For an oscillation that means that the period becomes infinite, i.e. the natural frequency tends to zero. Therefore, a pre-stressed modal analysis can be used instead of the buckling analyses described above. At the critical point the eigenvectors (modes) from buckling and modal analysis match.

The eigenvalue problem reads:

$$\left(\mathbf{K}_T - \omega^2\mathbf{M}\right)\boldsymbol{\varphi} = \mathbf{0} \tag{3.6}$$

with $\mathbf{M}$ the mass matrix and ω the eigen angular frequency, the natural frequency times 2π.

In Fig. 3.16 the first five natural frequencies of a spherical cap under concentrated pressure from Fig. 3.15 are plotted against a characteristic displacement. Furthermore, the pressure level is shown (light blue curve). Since the arc-length method (see Sect. 4.4) is used to control the analysis the pressure increases, reaches a maximum, decreases, reaches a local minimum and increases again. Since the purpose of the eigensolver is the determination of frequencies they and the related modes are suppressed if a negative square of ω occurs. Thus the first mode vanishes when the critical point, here a snap-through point, the maximum load, is passed. The previous second mode becomes the new first one and a further mode is added. Furthermore, mode jumping occurs, i.e. the order of the eigenvalues related to certain modes changes. For these two reasons the curves connecting the eigenvalues in their current order show jumps. That is why in Fig. 3.16 the eigenvalues related to the modes later forming the three lowest eigenvalues are marked in black. It is

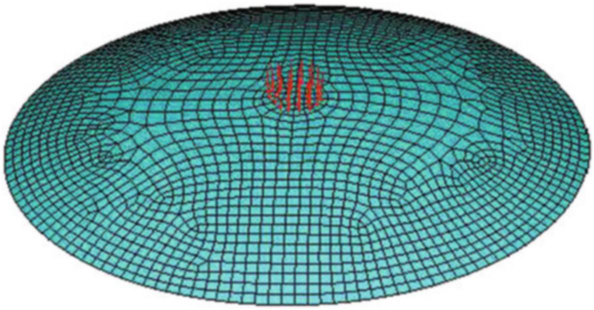

Fig. 3.15 Spherical cap under concentrated external pressure

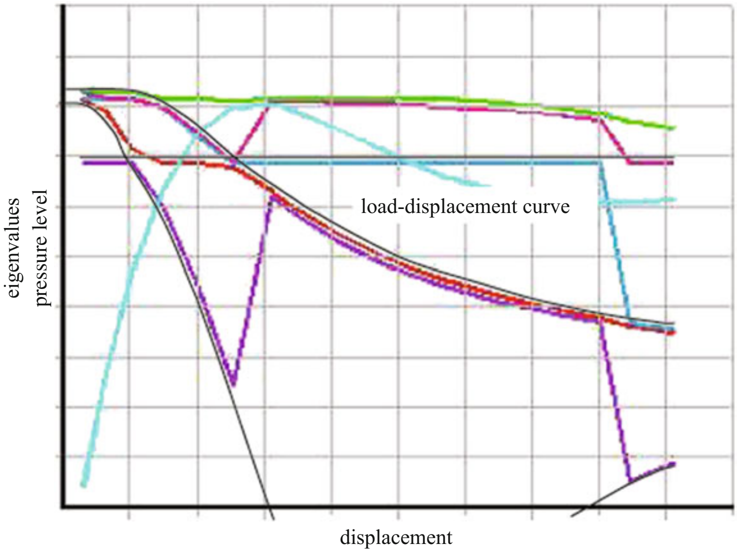

Fig. 3.16 Natural frequencies depending on the equilibrium state

visible that they form a smooth curve. The vanished mode reappears when the carryable load increases again (positive slope–positive eigenvalue).

The use of this eigenvalue tracking can be seen in the example from Fig. 3.17, an ultimate load analysis of a stiffened sector of a cylindrical shell (cf. [21]). Depending on the load incrementation three paths can be distinguished, one leading to a much too high load, one to a slightly too high load which is obtained in the numerical analysis but cannot be carried by the system in practice, i.e. the equilibrium becomes unstable. Then negative eigenvalues are indicators for this fact, the related modes give ideas for imperfections (see Sect. 3.4) to reach the lowest path.

In a modal analysis of such cases a certain mode may vanish (be suppressed) because the first eigenvalue ω^2 becomes negative, thus the natural frequency imaginary. This happens above a bifurcation point if no bifurcation has happened.

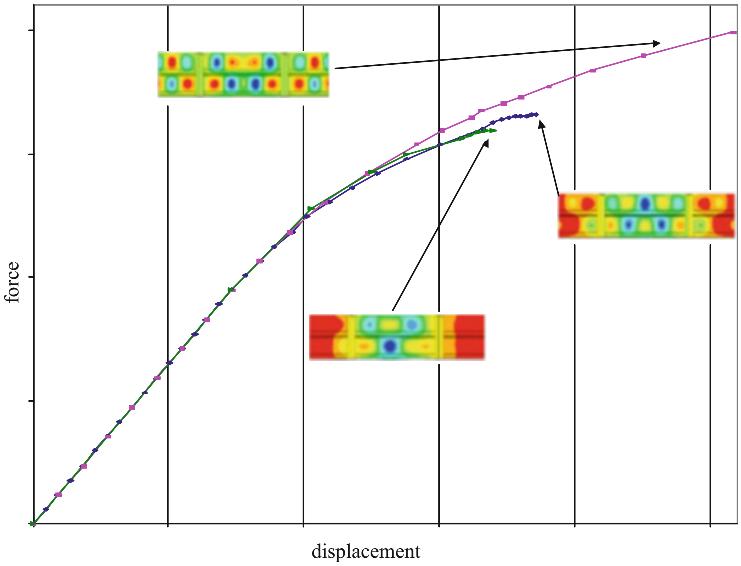

Fig. 3.17 Different load–displacement curves of a system depending on imperfections and the load incrementation

An example is shown in Fig. 3.18 for a similar system but a different load case. The second mode at load factor 1 becomes the first one if the load is slightly increased to 1.03 because the previously first mode is suppressed. The meaning of the vanishing mode is explained in Sect. 3.3.

The disappearance of a certain mode can be identified numerically (important for automation) regarding the fact that two different modes are M-orthogonal, i.e. the product of the mass matrix **M** times the transposed of the one eigenvector $\boldsymbol{\varphi}$ from the left and the other eigenvector from the right is zero. This holds for eigenvectors of the same matrices, here at the same equilibrium point. Since the eigenvectors under consideration are from different load levels and thus from different but similar tangential matrices this condition should be formulated as

$$\frac{\boldsymbol{\varphi}_{i-1}^T \mathbf{M} \boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} << 1 \tag{3.7}$$

where the index i counts the load levels where the eigenvalue analysis has taken place.

The problem of the suppressed modes and eigenvalues can be overcome by a solver for non-symmetric matrices or for natural frequencies of damped systems. In these cases complex eigenvalues are expected. If these solvers are applied to a

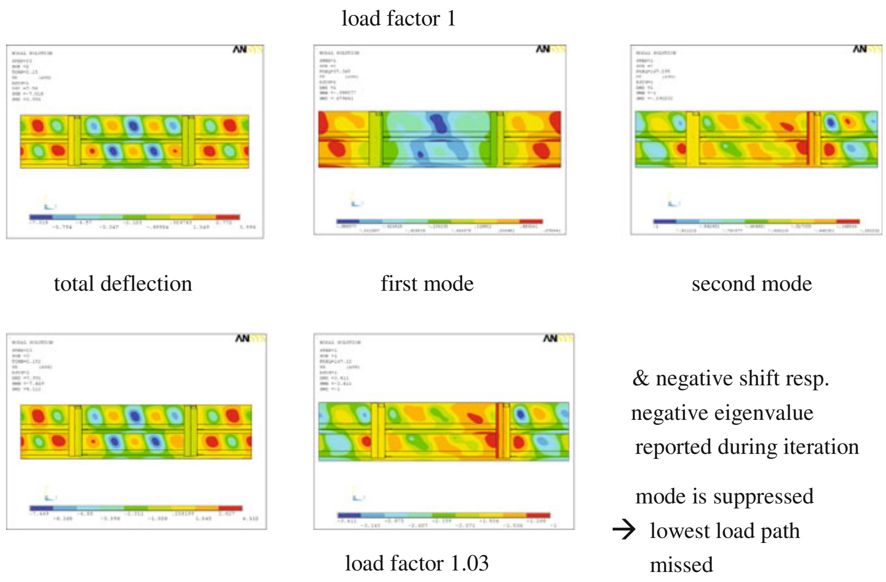

Fig. 3.18 Vanishing mode after reaching an unstable paths [21]

system not being damped and with symmetric matrices one natural frequency becomes purely imaginary when exceeding a critical load (Table 3.1).

The eigenvector is the probable buckling mode. In the example from Fig. 3.19 it is shaped below and above the critical load in the same way as depicted.

3.2.4 *Direct Identification of Critical Points by an Extended System*

At a critical load level λ_{crit} two conditions must be fulfilled [24]:

1. Equilibrium must be achieved:

   $$\mathbf{d}(\hat{\mathbf{u}}, \lambda) = \mathbf{f}^{int}(\hat{\mathbf{u}}) - \lambda \mathbf{f}_0^{ext}(\hat{\mathbf{u}}) = \mathbf{0}$$

2. The eigenvalue problems from Sect. 3.2.2 must deliver a critical eigenvalue (0 or 1 depending on the formulation). Thus:

   $$\mathbf{K}_T \boldsymbol{\varphi} = \mathbf{0}$$

   This results in two unknown vectors with n components each plus the unknown load factor λ, i.e. $2n + 1$ unknowns in $2n$ equations.

3. The missing equation can be found in conjunction with the eigenvector:

   a. The length of the eigenvector must be scaled, e.g. to 1:

Table 3.1 Natural frequencies below (upper table) and above a critical load (lower)

Mode	Real part [Hz]	Imaginary part [Hz]
1	32.149	0
2	224.28	0
3	297.9	0
4	545.01	0
5	804.38	0
6	957.56	0
Mode	Real part [Hz]	Imaginary part [Hz]
1	0	13.514
2	209.46	0
3	303.14	0
4	528.72	0
5	786.32	0
6	928.83	0

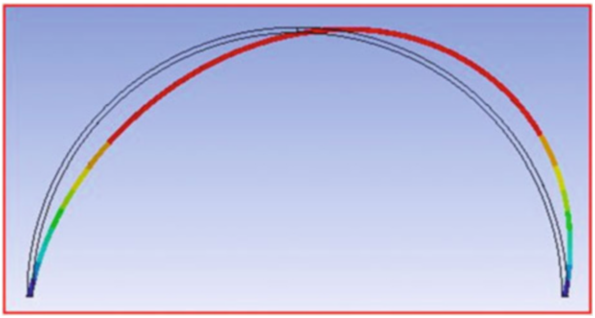

Fig. 3.19 First mode below and above the critical load

$$\boldsymbol{\varphi}^T\boldsymbol{\varphi} - 1 = 0$$

b. Only bifurcation points where the eigenvector is perpendicular to the external load vector should be found:

$$\boldsymbol{\varphi}^T\mathbf{f}_0^{ext} = 0$$

c. Only snap-through points where this is not the case should be found. Since the eigenvector can be scaled arbitrarily one can formulate:

$$\boldsymbol{\varphi}^T\mathbf{f}_0^{ext} - 1 = 0$$

These equations must be solved simultaneously by a Newton scheme. The tangential matrix contains the derivatives with respect to the unknowns, when using condition 3a:

$$\begin{bmatrix} \dfrac{\partial \mathbf{d}}{\partial \mathbf{u}} & \dfrac{\partial \mathbf{d}}{\partial \boldsymbol{\varphi}} & \dfrac{\partial \mathbf{d}}{\partial \lambda} \\ \dfrac{\partial \mathbf{K}_T\boldsymbol{\varphi}}{\partial \mathbf{u}} & \dfrac{\partial \mathbf{K}_T\boldsymbol{\varphi}}{\partial \boldsymbol{\varphi}} & \dfrac{\partial \mathbf{K}_T\boldsymbol{\varphi}}{\partial \lambda} \\ \dfrac{(\boldsymbol{\varphi}^T\boldsymbol{\varphi}-1)}{\partial \mathbf{u}} & \dfrac{(\boldsymbol{\varphi}^T\boldsymbol{\varphi}-1)}{\partial \boldsymbol{\varphi}} & \dfrac{(\boldsymbol{\varphi}^T\boldsymbol{\varphi}-1)}{\partial \lambda} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\boldsymbol{\varphi} \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{f}_0^{ext} - \mathbf{f}^{int} \\ -\mathbf{K}_T\boldsymbol{\varphi} \\ 1-\boldsymbol{\varphi}^T\boldsymbol{\varphi} \end{bmatrix} \tag{3.8}$$

The derivative in the second row and third column exists if the load vector depends on the displacements. Then

$$\mathbf{K}_p = -\frac{\partial \mathbf{f}^{ext}}{\partial \mathbf{u}} = -\frac{\partial\left(\lambda\mathbf{f}_0^{ext}\right)}{\partial \mathbf{u}} = -\lambda\frac{\partial\left(\mathbf{f}_0^{ext}\right)}{\partial \mathbf{u}} \tag{3.9}$$

is the only part of the tangential stiffness matrix $\mathbf{K}_T$ depending on the load factor λ. The derivative reads:

$$\frac{\partial \mathbf{K}_T\boldsymbol{\varphi}}{\partial \lambda} = \frac{\partial \mathbf{K}_p\boldsymbol{\varphi}}{\partial \lambda} = \frac{\partial \mathbf{K}_p}{\partial \lambda}\boldsymbol{\varphi} = -\frac{\partial\left(\mathbf{f}_0^{ext}\right)}{\partial \mathbf{u}}\boldsymbol{\varphi} = -\frac{1}{\lambda}\frac{\partial\left(\mathbf{f}^{ext}\right)}{\partial \mathbf{u}}\boldsymbol{\varphi} = -\frac{1}{\lambda}\mathbf{K}_p\boldsymbol{\varphi} \tag{3.10}$$

Thus the linear system of equations in Newton's method reads:

$$\begin{bmatrix} \mathbf{K}_T & \mathbf{0} & -\mathbf{f}_0^{ext} \\ \dfrac{\partial \mathbf{K}_T\boldsymbol{\varphi}}{\partial \mathbf{u}} & \mathbf{K}_T & -\dfrac{1}{\lambda}\mathbf{K}_p\boldsymbol{\varphi} \\ \mathbf{0} & 2\boldsymbol{\varphi}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\boldsymbol{\varphi} \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{f}_0^{ext} - \mathbf{f}^{int} \\ -\mathbf{K}_T\boldsymbol{\varphi} \\ 1-\boldsymbol{\varphi}^T\boldsymbol{\varphi} \end{bmatrix} \tag{3.11}$$

What looks like a rather complicated and huge system of equations can be reduced to mainly solving systems with $\mathbf{K}_T$ and different right hand sides as can be shown after presenting the idea of the arc-length method in Sect. 4.4.

$\mathbf{K}_T\boldsymbol{\varphi}$ and $\mathbf{K}_p\boldsymbol{\varphi}$ as well as their derivatives can be calculated on element level and then assembled to a global vector. For the term in the first column and second row this will be shown below after the derivation of the algorithm. For a multi-element-type program a numerical procedure might be appropriate to form the derivatives.

3.3 Meaning of the Eigenvector

In Fig. 3.20 the displacement states from two different load levels in the vicinity of the maximum load are shown. They are hard to distinguish but when forming the difference (left hand in Fig. 3.21) a certain pattern can be recognised. It is on the one hand similar to the vanishing mode from Fig. 3.18, on the other hand similar to the total displacement in the failure state (right hand in Fig. 3.21). That means this difference and the suppressed mode, the eigenvector to the zero eigenvalue at the critical point, show how the system must deform to reach the lowest load path.

The most important mode can change during the load history indicating further potential bifurcations. For the example in Fig. 3.20 [21] the deformation pattern is the result of an earlier buckling process with at first stable post-critical behaviour.

The eigenvector to the critical eigenvalue 0 resp. 1 shows how the system can deform without change in the load. In case of a snap-through problem the mode is similar to the deformation state reached, in case of a bifurcation problem it is completely different (compare Fig. 3.20 with Fig. 3.18). Furthermore, in the latter case the eigenvector is perpendicular to the load vector, i.e.

$$\boldsymbol{\varphi}^T \mathbf{f}^{ext} = 0 \tag{3.12}$$

For the snap-through problem this product yields a value being significantly apart from zero. Thus, a criterion to distinguish between snap-through and bifurcation is obtained as long as one has got a comparative value. This can be the product of the displacement vector and the load vector while the displacement vector should be normalised in the same way as the eigenvector:

$$\frac{\boldsymbol{\varphi}^T \mathbf{f}^{ext}}{\hat{\mathbf{u}}^T \mathbf{f}^{ext}} = \begin{cases} << 1 & \Rightarrow \quad bifurcation \;\; problem \\ else & \Rightarrow \quad snap-through \;\; problem \end{cases} \tag{3.13}$$

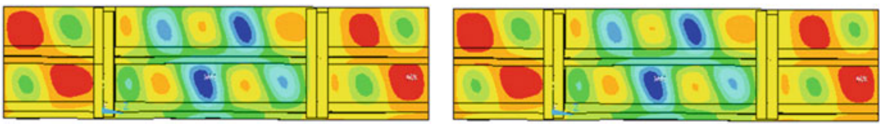

Fig. 3.20 Radial displacement for two subsequent load levels close to the instability point

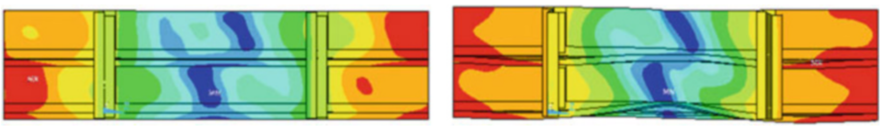

Fig. 3.21 Difference between two subsequent load levels (*left*), last solution (*right*)

3.4 Imperfections

In the case of a bifurcation problem a perfect system can remain on the primary unstable equilibrium path for loads much higher than the critical one. Often the buckling process must be initiated by an imperfection to obtain a physically meaningful solution. This can be done by appropriate perturbation loads or geometric imperfections.

3.4.1 Imperfection by Forces

Imperfections should initiate the buckling mode to the lowest load path. If this is not known the imperfection by forces should be chosen so that they do not overpredict the mode. Thus a small number of forces is preferred. This statement especially holds under the point of view that buckling patterns in experiments, even those being spread over the total system, usually are initiated by a local buckle. For example in Fig. 3.22 between five and seven half-waves are expected. Applying five single, equally spaced loads would be dangerous. Two forces, non-symmetrically placed are an initiation but leave the system enough degrees of freedom to find the correct buckling pattern.

Such an imperfection can only specially be chosen and requires an estimate of the buckling or failure mode.

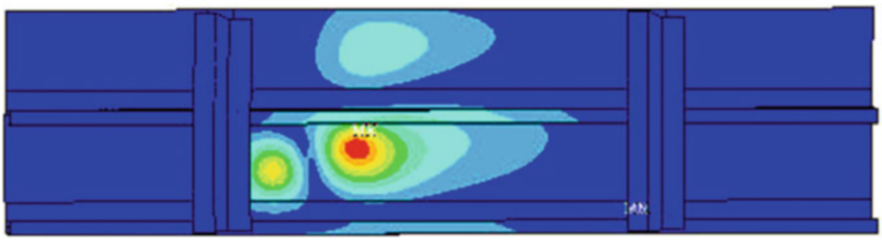

Fig. 3.22 Imperfection (radial displacements shown) resulting from two single loads

3.4.2 Imperfection by Geometric Prescriptions

For geometric imperfections, i.e. stress-free predeformations, that means changes of the nodal coordinates, one can try to find a suitable function to initiate the buckling but this is case-dependent. In particular it cannot be assured that the buckling mode to the lowest critical load is matched.

Randomly distributed changes in nodal coordinates turned out to be even less appropriate [22]. They can even lead to higher buckling loads than the ideal ones. Furthermore, it can happen that randomly the largest coordinate jumps occur from one node to its neighbour possibly leading to element warping which can deteriorate the accuracy. In order to avoid that such geometric distortions must be smoothed again.

Furthermore, in both cases conflicts occur if the system contains parts not being connected by common nodes but by contact. It is very likely that arbitrary changes of the contact surface geometry lead to interferences or gaps.

3.4.3 Imperfection by a Linear Buckling Analysis

A more general approach to generate geometric imperfections is the use of a mode from a buckling analysis, i.e. to change the nodal coordinates by the scaled eigenvector:

$$\mathbf{x}_0 \leftarrow \mathbf{x}_0 + c\boldsymbol{\varphi} \tag{3.14}$$

The assumptions of the linear buckling analysis, namely totally linear behaviour up to the incidence of buckling, must be fulfilled to make this procedure successful. Then the first buckling modes can constitute good imperfections. In case of multiple or—more often in practice—clustered eigenvalues at least all modes related to these values must be taken into account. Then each linear combination of these modes can be the proper buckling pattern.

If the system remains stable after the first bifurcation and later fails by a new bifurcation or a switch from one buckling pattern to the other, the linear buckling analysis of the system in its initial configuration often is not sufficient. Such a behaviour is typical for stiffened plates and shells. For example an initial buckling occurs in the zones bordered by the stiffeners, later followed by collapse of the stiffening construction. In the case of the panel shown in Figs. 3.22 and 3.23 [21] the seventh or eighth mode is similar to the pattern occurring later in the non-linear analysis, but how to know this beforehand?

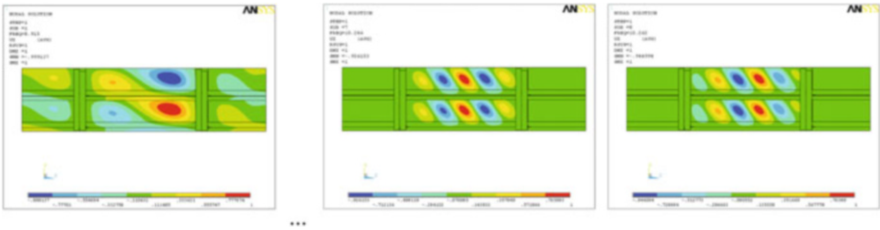

Fig. 3.23 First, seventh and eighth linear buckling mode

3.4.4 *Eigenvalue- and Mode-Tracking*

A more reliable but more complicated procedure than the linear buckling analysis is to use the first mode from an eigenvalue extraction which is performed repeatedly after resp. parallel to a non-linear analysis based on the actual tangential matrix (see Sect. 3.2.2). The question is how execute this buckling analysis and when as well as how to apply the buckling pattern as imperfection to the deformed system.

The purpose of the eigenvalue analysis is to identify bifurcation points resp. the remaining on an unstable path and to determine imperfections which have the system branch to the lowest load path. The following algorithms are appropriate:

Algorithm 1 Eigenvalue analysis *after* a non-linear analysis

- At certain load levels save the tangential matrix $\mathbf{K}_T$ or a restart file to recreate $\mathbf{K}_T$.
- Perform a series of eigenvalue analyses.
  - Keep the first mode when its eigenvalue becomes negative (eigenvalue problem 1 from Sect. 3.2.2 or modal analysis) resp. <1 (EVP 2 and 3) or
  - identify mode switching or suppression when

    $$\frac{\boldsymbol{\varphi}_{i-1}^T \mathbf{M} \boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} << 1$$

    where
    $\mathbf{M} = \mathbf{I}$ for EVP 1, $\mathbf{M} = \mathbf{K}_\sigma$ for EVP 2, $\mathbf{M} = \mathbf{K}_n + \mathbf{K}_\sigma$ for EVP 3 and $\mathbf{M}$ equals the mass matrix in a modal analysis.
- Repeat the non-linear analysis with a geometric imperfection obtained from the eigenvector.

In principle a repetition is only necessary if a bifurcation point is missed or no convergence could be achieved because the system tends to bifurcate but alternates between different intermediate disequilibrium states during the iteration.

This procedure cannot be repeated to often because each time a further geometric imperfection is added or replaces the preceding one. In the latter case it becomes doubtful at a certain time whether the bifurcation point for which the imperfection is suitable will be reached again.

More appropriate for multiple bifurcations is

Algorithm 2 Eigenvalue analysis *parallel* a non-linear analysis

- At certain load levels interrupt the non-linear analysis and solve an eigenvalue problem.
- Continue the non-linear analysis.
  As reaction on the eigenvalue there are two choices:
  - Stop the analysis if an unstable path is detected and restart from the beginning with a geometric imperfection based on the related eigenvector. Then, however, the procedure differs from algorithm 1 in the order of the steps only. It avoids too many increments on the unstable path but can show the same lack for multiple buckling.
    or
  - Perturb the Newton-Raphson iteration by the actual eigenvector when continuing the non-linear analysis.

The latter way is explained in more detail: The initial vector $\mathbf{u}^0{}_{i+1}$ for a Newton-Raphson scheme at a new load level $i+1$ usually is the last converged solution $\mathbf{u}^\infty{}_i$. Now the scaled eigenvector $\boldsymbol{\varphi}$ is added at the beginning of the iteration:

$$\mathbf{u}_{i+1}^0 = \mathbf{u}_i^\infty + c\boldsymbol{\varphi} \tag{3.15}$$

As long as the system is on a stable path the same solution $\mathbf{u}^\infty{}_{i+1}$ as from an undisturbed initial vector will be found. In the vicinity of a critical point, however, an appropriate mode will be found to initiate a bifurcation without having deformed the initial geometry. Further decisions are not necessary. If the system has reached a snap-through point, an eigenvalue analysis is not necessary but the perturbation will cause no problem because the eigenvector $\boldsymbol{\varphi}$ will be similar to the displacement increment.

Since only a perturbation of the initial displacement vector takes place, the accuracy of the eigenvalue extraction need not to be high. Furthermore, close to critical point it is likely that the eigenvalues are sufficiently separated. Thus the simplest method would be the inverse von-Mises iteration where the system of equations determined by $\mathbf{K}_T$ must be solved with a series of right hand sides which does not cause so much computational effort in a Gaussian type algorithm because the triangularisation has already happened.

Imperfections from an eigenvalue analysis are considered to be the worst case and thus the most appropriate ones if the real predeformation is not known.

3.5 Imperfection Sensitivity

If the safety of a system must be quantified it is necessary to study the sensitivity to the size of imperfections. It mainly depends on the post-critical behaviour: the steeper the (negative) slope in the post-critical branch, the more sensitive the

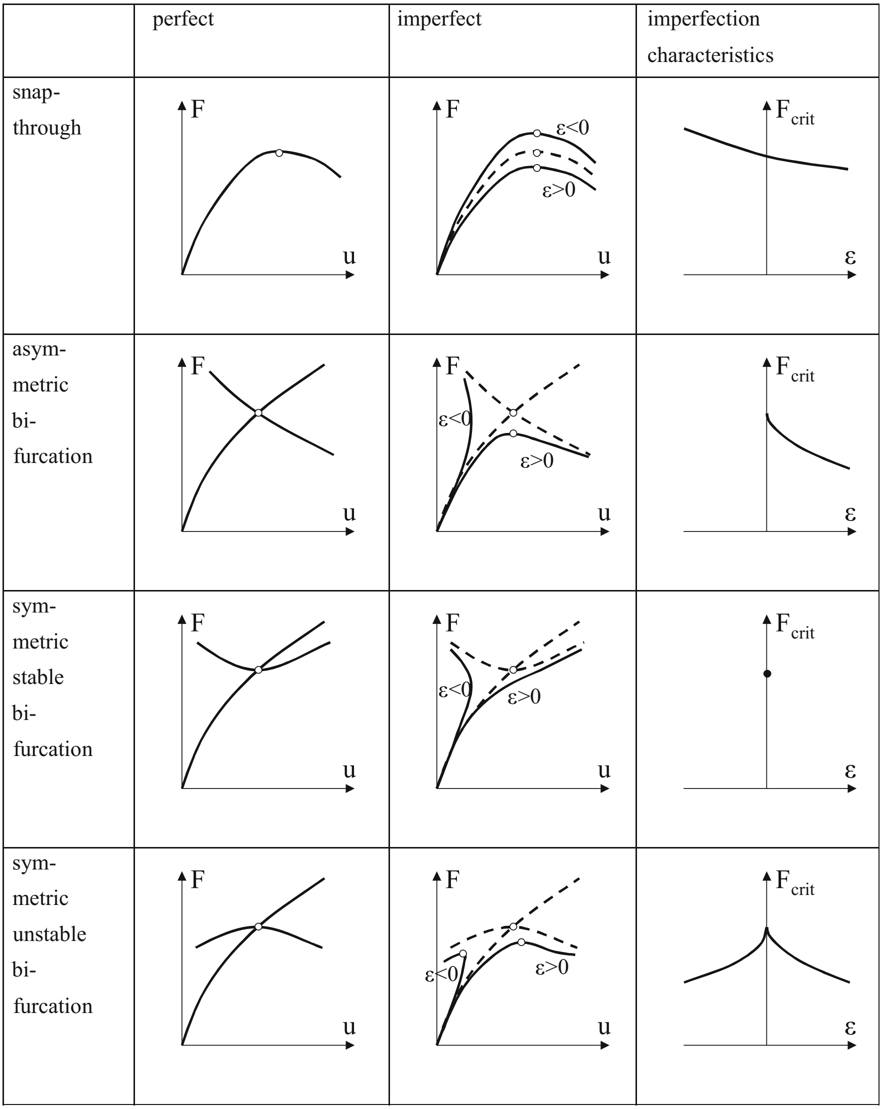

Fig. 3.24 Classification of the imperfection sensitivity

reaction to perturbations. The imperfection sensitivity can be classified by sensitivity diagrams (Fig. 3.24).

With an appropriate imperfection a snap-through problem remains a snap-through problem but the ultimate load is affected. A bifurcation problem, however, changes to either a snap-through problem, if the post-buckling branch is unstable, or to a non-linear stress problem without a strictly defined critical load if the post-critical path of the perfect system is stable. The direction of an imperfection is of importance, not only but especially in case of an asymmetric bifurcation.

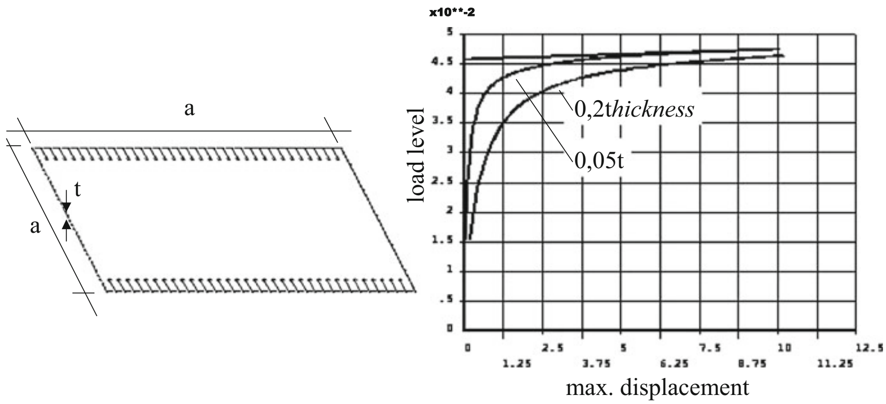

Fig. 3.25 Buckling plate and its load–displacement diagrams for different imperfection sizes

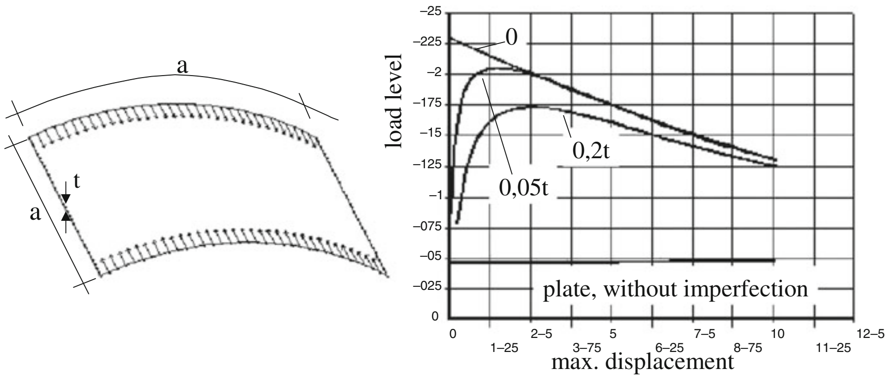

Fig. 3.26 Buckling cylindrical shell and its load–displacement diagrams for different imperfection sizes

In Figs. 3.25 and 3.26 the influence of the imperfection on the load carrying behaviour of two systems is shown, the one being a flat plate, the other a sector of a cylindrical shell made from the same material (length, width, thickness) as the plate. The shell shows an unstable post-critical behaviour. Therefore, the bifurcation problem changes to a snap-through problem by the imperfection showing a significant reduction of the maximum load. In case of the plate a transition happens to a still strongly non-linear but stable behaviour. A maximum load is no longer defined but in practice the deflection must be limited, at least under the loads in use. Furthermore, a comparison of the two systems shows that the shell reaches a significantly higher load level than the plate but the price is an unstable post-critical behaviour.

3.5.1 Size of the Imperfection

Which size must have the maximum value of the imperfection? This question can generally be answered in the following way only: Large enough to initiate buckling. More imperfection is mostly on the safe side but too much can be an economic problem. The best is to know the probable or maximum allowed imprecision due to prescribed tolerances. They must be multiplied by a safety factor. Technical standards often define such values.

For beam-like structures 1/500 to 1/250 of the buckling length is a usual choice. The buckling length is defined as the distance between two inflection points of the deflection curve of the buckling mode.

In case of surface structures a buckling diameter, also measured between inflection points could be a useful reference. In case of plates with stable post-critical behaviour 1/250 of that as an imperfection would lead to the fact that a buckling behaviour is hardly to identify any longer. In this case such an imperfection is probably considerably too large. A common choice for the maximum is 1/10 of the thickness. This often initiates the bifurcation but the buckling effect remains visible. This value has no theoretical background but is a rule of thumb. If in doubt one has to try a couple of imperfection sizes and to study their influence on the limit load.

3.6 Classification of Instability Analyses

This is kind of a summary of the procedures outlined above.

3.6.1 Linear Buckling Analysis (LBA)

- It requires a linear static analysis with a reference load to get a reference stress state.
- Assumption is linear behaviour until buckling.
- The method is suitable for perfect systems.
- It delivers a load multiplier and thus the ideal critical load.

This analysis is appropriate to *estimate* the critical load and thus to adjust settings for non-linear analyses. Furthermore, the modes can be used as imperfection to initiate *first* buckling provided the assumption above holds.

3.6.2 Geometrically Non-linear Analysis (GNA)

- It is necessary if the pre-critical behaviour is non-linear.
- It is always necessary if snap-through problems are explored.

- It uses the perfect system.
- At least large rotations must be activated.
- It describes the behaviour up to final buckling of very slender structures only.

This method can be sufficient in case of snap-through problems. It is a good base for eigenvalue tracking procedures to detect bifurcation points or unstable branches. It can also be used as long as the real bifurcations occur without imperfections.

3.6.3 Geometrically and Materially Non-linear Analysis (GMNA)

- It has the same assumptions and limitations as the geometrically non-linear analysis (GNA).
- Except that non-linear material behaviour is considered and influences the ultimate load.
- Material non-linearity is necessary for moderately slender structures.

The method should also be accompanied by eigenvalue tracking and/or the tracking of the minimum pivot element in the Gaussian algorithm.

3.6.4 Geometrically or Geometrically and Materially Non-linear Imperfect Analysis (GNIA or GMNIA)

- It includes the effect of imperfections
- It has the same distinction between considering material non-linearities or not as GNA and GNMA (usefulness depends on slenderness).
- The imperfections must be appropriate to guide the system on the lowest load path. Most reliable for their determination are eigenvalue analyses. Choices to apply imperfections are
  - imperfections by forces
  - geometric imperfections, especially by eigenvectors
  - perturbation of initial displacement states in iterative procedures
  - separately excited oscillations if the (quasi-)static analyses is stabilised by inertia effects.