L

Laplace Equation

Jeffrey R. Keaton
Amec Foster Wheeler, Los Angeles, CA, USA

Definition

The Laplace equation is a second-order partial differential equation of a mathematical function attributed to Pierre Simon Laplace based on a paper written in 1784 (Ball, 1908). In Cartesian coordinates x, y, and z, the second derivative of the function $f(x,y,z)$ is expressed with the use of the Laplacian operator $\nabla^2 f$ of the Laplace equation in the form

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

where $f(x,y,z)$ is a harmonic function for which the three independent variables x, y, and z are solutions (Jeffrey, 2002). The Laplace equation is important in potential fields including gravitational attraction, electrical current, and groundwater head. In steady-state groundwater flow following Darcy's law in an isotropic aquifer with no additions or losses of water, the Laplace equation is $\nabla^2 h = 0$. Darcy's law relates unit discharge, q in each direction per unit of aquifer width, to the product of hydraulic conductivity (K) and the change in head (h) in each direction ($q_x = -K(\partial h/\partial x)$, $q_y = -K(\partial h/\partial y)$, $q_z = -K(\partial h/\partial z)$). Steady-state groundwater flow has no transient flow (Domenico and Schwartz, 1998). Isotropic aquifer conditions require that hydraulic conductivity is equal in all three Cartesian directions ($K_x = K_y = K_z = K$).

Additions or losses of water (sources or sinks) in groundwater flow could be injections or withdrawals from a well completed in the aquifer. If steady-state groundwater flow is subject to an increase or decrease (W) across an aquifer reach, then the Laplace equation ($\nabla^2 h = 0$) becomes the Poisson equation (Domenico and Schwartz, 1998), which is $\nabla^2 h = W/K$.

The Laplace equation can be solved numerically using the finite difference method after the problem is discretized in the x-y plane. For the groundwater flow problem, the quantities at each node in the finite difference mesh would be head and flow. Head would be a scalar ($h_{i,j}$), whereas flow would be a vector separated into the x and y components ($q_{x(i,j)}$, $q_{y(i,j)}$). The second derivative is obtained by calculating the first derivative of the value and then repeating the procedure to the result.

An attempt to use the Laplace equation with age as a surrogate for potential was proposed by Hirano (1993) using bedding planes in sedimentary rock units as iso-potential surfaces with cross-cutting geologic structures being represented as boundary conditions.

References

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