Taylor Optimal Kernel for Derivative Etimation

Henri-Alex Esbelin and Remy Malgouyres

Clermont Universités, CNRS UMR 6158, LIMOS, Clermont-Ferrand, France
alex.esbelin@univ-bpclermont.fr, remy.malgouyres@udamail.fr

Abstract. In many geometry processing applications, the estimation of differential geometric quantities such as curvature or normal vector field is an essential step. In this paper, we investigate new estimators for the first and second order derivatives of a real continuous function f based on convolution of the values of noisy digitalizations of f. More precisely, we provide both proofs of multigrid convergence of the estimators (with a maximal error $O\left(h^{1-\frac{k}{2n}}\right)$ in the unnoisy case, where $k = 1$ for first order and $k = 2$ for second order derivatives and n is a parameter to be choosed *ad libitum*). Then, we use this derivative estimators to provide estimators for normal vectors and curvatures of a planar curve, and give some experimental evidence of the practical usefullness of all these estimators. Notice that these estimators have a better complexity than the ones of the same type previously introduced (cf. [4] and [8]).

Keywords: derivative estimation, curvature estimation, discrete derivation, convolution.

1 Introduction

In the framework of shape analysis, a common problem is to estimate derivatives of functions, or normals and curvatures of curves and surfaces, when only some (possibly noisy) sampling of the function or curve is available. This problem has been investigated through finite difference methods, scale-space methods, and discrete geometry, etc. ... For detailed informations about the state of the art, the reader is refered to [1], [5] and [7].

This paper focuses on estimating the derivatives on the boundary of digital planar shapes. Suppose that the digital data is distributed around the true sample of the Euclidean shape according to some noise. The curvature estimation is provided to be as close as possible to the curvature of the underlying Euclidean shape before digitization. More precisely, provided some formal models of the noise, the quality of the estimation should be improved as the digitization step gets finer and finer. This property is called the multigrid convergence(see [3], [2], [6] and [10]).

Our objective is to design estimators of successive derivatives for digital data which are provably multigrid convergent, accurate in practice, computable in an exact manner, robust to perturbations.

The first section provides definitions for first and second order discrete derivatives of digital curves. These discrete derivatives are proved to provide multigrid

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convergent estimators for the corresponding continuous derivatives, even in a noisy case.

The second section explains how to use these discrete differential notions to estimate continuous normal field and curvature.

The third section gives some experimental evidence to the multigrid convergence of all these estimators.

2 Discrete Derivatives for Discrete Functions

Throughout this paper, we call *discrete function* a function from $\mathbb{Z}$ to $\mathbb{Z}$ or $\mathbb{Z}^2$. . Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continous function. We say that the discrete function $\Gamma : \mathbb{Z} \mapsto \mathbb{Z}$ is a discretization of f on the interval $[a; b]$ for the discretization step $h > 0$ with error ϵ_h if, for any integer i such that $a \leq ih \leq b$, we have $h\Gamma(i) = f(ih) + \epsilon_h(i)$. We consider here the uniform noise case: $0 \leq \|\epsilon_h\|_\infty \leq Kh^\alpha$ with $0 < \alpha \leq 1$ and K a positive constant. Note that the rounded case and the floor case are particular cases of uniform noise with $\alpha = 1$.

2.1 First 0rder Derivatives

Definition 1. *Let m be a positive integer. The Taylor-optimal mask of size m for discrete first order derivation (TO_m mask for short) is the finite sequence of rational numbers $\mathbf{a_m}$ defined by $a_{m,0} = 0$ and for $-m \leq i < 0$ or $0 < i \leq m$ by*

$$a_{m,i} = \frac{(-1)^i}{i} \frac{\binom{m}{i}}{\binom{m+i}{i}} = \frac{(-1)^i}{i} \frac{\binom{2n}{n+i}}{\binom{2m}{m}}. \tag{1}$$

Definition 2. *Let $\Gamma : \mathbb{Z} \mapsto \mathbb{Z}$ be a discrete function. Let $p \in \mathbb{N}^*$ and $i_0 \in \mathbb{Z}$. The m-Taylor-optimal first order discrete derivative of Γ at point i_0 with step p is*

$$\left(\Delta_{m,p}\Gamma\right)(i_0) = \frac{1}{p} \sum_{i=-m}^{i=m} a_{m,i}\Gamma\left(i_0 - ip\right). \tag{2}$$

In order to show that the discrete derivative of a discretized function provides an estimate for the continuous derivative of the real function, we would like to evaluate the difference between $\left(\Delta_{m,p}\Gamma\right)(i_0)$ and $f'(i_0h)$.

Lemma 1. $\sum_{i=-m}^{i=m} i a_{m,i} = -1$ *and for $k = 0$ or $2 \leq k \leq 2m$,* $\sum_{i=-m}^{i=m} i^k a_{m,i} = 0$.

Proof. Let us first notice that the lemma is trivial for the even values of k because of the equality $a_{m,-i} = -a_{m,i}$.

Let us now consider the odd case. Let f be a polynomial and define the classical finite difference operator δ_+ by $\delta_+(f)(x) = f(x+1) - f(x)$. Iterating this operator leads to the well known equality:

$$\delta_+^{2m}(f)(x) = \sum_{i=0}^{2m} \binom{2m}{i} (-1)^{2m-i} f(x+i). \tag{3}$$

Notice that applying the operator δ_+ to a polynomial of degree d leads to a polynomial of degree $d-1$. Applying equality 3 to the polynomials defined by $(x-n)^{k-1}$ for $1 \leq k \leq 2m$, we get

$$0 = \sum_{i=0}^{2m} \binom{2m}{i} (-1)^{2m-i} (i+x-m)^{k-1} . \tag{4}$$

Let now $x = 0$ in equality 4, then for all $1 \leq k \leq 2m$ we have

$$0 = \sum_{i=0}^{2m} \binom{2m}{i} (-1)^{2m-i} (i-m)^{k-1} = (-1)^m \sum_{i=-m}^{m} \binom{2m}{m+i} (-1)^i \, i^{k-1}.$$

Hence, for all odd k such that $1 \leq k \leq 2m$, we have

$$\sum_{i=-m}^{i=m} i^k a_{m,i} = \frac{1}{\binom{2m}{m}} \left(\sum_{i=-m, i\neq 0}^{i=m} i^{k-1}(-1)^i \binom{2m}{m+i} \right) = \begin{cases} 0 & \text{if } k > 1 \\ -1 & \text{if } k = 1 \end{cases} .$$

Theorem 1. *Let $k \in \mathbb{N}$ with $k \geq 2$ and let f be C^k on $\mathbb{R}$ and let $k_0 = Min\{k, 2m+1\}$. Let $p = \lfloor h^{-1+\frac{\alpha}{k_0}} \rfloor$. Then*

$$|(\Delta_{m,p}\Gamma)(i_0) - f'(i_0 h)| = O\left(h^{\alpha - \frac{\alpha}{k_0}}\right). \tag{5}$$

Proof. The difference between the discrete derivative of Γ and the continuous derivative of f may be obviously seen as the sum of $EM(f, \Gamma, m, p, i_0)$ called method's error and $ED(f, \Gamma, m, p, i_0)$ called discretization's error respectively defined as

$$EM(f, \Gamma, m, p, i_0) = \left(\frac{1}{ph} \sum_{i=-m}^{i=m} a_{m,i} f\left((i_0 - ip)h\right) \right) - f'(i_0 h)$$

and

$$ED(f, \Gamma, m, p, i_0) = \frac{1}{ph} \sum_{i=-m}^{i=m} a_{m,i} \epsilon(i_0 - ip).$$

We intend to bound both errors. The choice of p will appear to equalize the speeds of growth of both bounds. From Taylor formula and lemma 1, we first majorize the method's error. Let us denote $\sum_{i=-m}^{i=m} a_{m,i} f\left((i_0 - ip)h\right)$ by S. We have $S = \sum_{i=-m}^{i=m} a_{m,i} \left(\sum_{j=0}^{j=k_0-1} \frac{f^{(j)}(i_0 h)}{j!} (-iph)^j + \frac{f^{(k_0)}(x_{i,j})}{k_0!} (-iph)^{k_0} \right)$ for some $x_{i,j}$ between $i_0 h$ and $(i_0 - ip)h$. Hence

$$S = \sum_{j=0}^{j=k_0-1} \frac{f^{(j)}(i_0 h)}{j!} (-ph)^j \left(\sum_{i=-m}^{i=m} a_{m,i} i^j \right) + \frac{(-ph)^{k_0}}{k_0!} \left(\sum_{i=-m}^{i=m} a_{m,i} i^{k_0} f^{(k_0)}(x_{i,j}) \right)$$

and from lemma 1, we have $S = f'(i_0h) + \frac{(-ph)^k}{k!}\left(\sum_{i=-m}^{i=m} a_{m,i} i^{k_0} f^{(k_0)}(x_{i,j})\right)$, which leads to the following majoration:

$$|EM(f,\Gamma,\mathbf{a},p,i_0)| \leq \frac{\|f^{(k_0)}\|_\infty}{k_0!}\left(\sum_{i=-m}^{i=m} |i^{k_0} a_{m,i}|\right) h^{k_0-1} p^{k_0-1}$$

$$|EM(f,\Gamma,\mathbf{a},p,i_0)| \leq \frac{\|f^{(k_0)}\|_\infty}{k_0!}\left(\sum_{i=-m}^{i=m} |i^{k_0} a_{m,i}|\right) h^{\alpha-\frac{\alpha}{k_0}}$$

Now we staightforwardly majorize the discretization's error: $|ED(f,\Gamma,m,p,i_0)| \leq \frac{\|\epsilon\|_\infty}{ph}\sum_{i=-m}^{i=m}|a_i| \leq K\left(\sum_{i=-m}^{i=m}|a_i|\right) h^{\alpha-\frac{\alpha}{k_0}} \frac{1}{1-h^{1-\frac{\alpha}{k_0}}}$ and the proof is complete.

In practice, $4 \leq m \leq 10$. It is easy to check that, for such mask sizes, we have $2 < \sum_{i=-m}^{i=m} |a_{m,i}| < 2.93$ and for all $0 \leq k \leq 2m+1$, we have $\frac{5}{2} < \frac{1}{k!}\sum_{i=-m}^{i=m} i^k |a_{m,i}| \leq 5$. Hence, for such mask sizes, we have for h great enough $|(\Delta_{m,p}\Gamma)(i_0) - f'(i_0h)| \leq (5\|f^{(k_0)}\|_\infty + 3K)h^{\alpha-\frac{\alpha}{k_0}}$

2.2 Second Order Derivatives

Definition 3. *Let m be a positive integer. The Taylor-optimal mask of size m for discrete second order derivation (TO_m^2 mask for short) is the finite sequence of rational numbers $\mathbf{b_m}$ defined for $i \neq 0$ by*

$$b_{m,i} = \frac{(-1)^{i+1}}{i^2}\frac{\binom{m}{i}}{\binom{m+i}{m}} = \frac{(-1)^{i+1}}{i^2}\frac{\binom{2m}{m+i}}{\binom{2m}{m}} \tag{6}$$

$b_{m,0}$ is such that $\sum_{i=-m}^{i=m} b_{m,i} = 0$

Definition 4. *Let $\Gamma : \mathbb{Z} \mapsto \mathbb{Z}$ be a discrete function. Let $p \in \mathbb{N}^*$ and $i_0 \in \mathbb{Z}$. The m-Taylor-optimal second order discrete derivative of Γ at point i_0 with step p is*

$$\left(\Delta_{m,p}^2\Gamma\right)(i_0) = \frac{2}{p^2h}\sum_{i=-m}^{i=m} b_{m,i}\Gamma(i_0 - ip) \tag{7}$$

Lemma 2. $\sum_{i=-m}^{i=m} i^2 b_{m,i} = 1$ *and for $0 \leq k \leq 1$ or $3 \leq k \leq 2m$, we have* $\sum_{i=-m}^{i=m} i^k b_{m,i} = 0$.

Proof. This lemma comes easily from lemma 1.

Theorem 2. *Let $k \in \mathbb{N}$ with $k \geq 3$ and let f be C^k on $\mathbb{R}$ and let $k_0 = Min\{k, 2m+1\}$. Let $p = \lfloor h^{-1+\frac{\alpha}{k_0}} \rfloor$. Then*

$$\left| \left(\Delta^2_{m,p}\Gamma\right)(i_0) - f"(i_0 h)\right| = O\left(h^{\alpha - \frac{2\alpha}{k_0}}\right) \quad (8)$$

Proof. The proof is analoguous to the one for theorem 1.

3 Normal Vectors and Curvature Estimation

In order to provide estimators for the normal vectors and the curvatures of a parametrized curve $g = (g_1, g_2) : (a, b) \mapsto \mathbb{R} \times \mathbb{R}$, we shall use the classical definitions and properties: for each real t_0 such that $a < t_0 < b$, the normal vector is $Ng(t_0) = (g_2'(t_0), -g_1'(t_0))$ and the curvature may be computed using $Ng(t_0) = \frac{g_1'(t_0)g_2''(t_0) - g_2'(t_0)g_1''(t_0)}{(g_1'^2(t_0) + g_2'^2(t_0))^{3/2}}$.

3.1 Normal Vectors Estimation

Definition 5. *Let $\Gamma = (\gamma_1, \gamma_2) : \mathbb{Z} \mapsto \mathbb{Z}^2$ be a discrete function. Let $p \in \mathbb{N}^*$ and $i_0 \in \mathbb{Z}$. The m-Taylor-optimal discrete normal vector of Γ at point i_0 with step p is*

$$\left(N_{m,p}\Gamma\right)(i_0) = \left(\left(\Delta_{m,p}\gamma_2\right)(i_0), -\left(\Delta_{m,p}\gamma_1\right)(i_0)\right) \quad (9)$$

We assume now that a planar simple closed C^1parameterized curve C (i.e., the parameterization is periodic and injective on a period) is given together with a family of parameterized discrete curves $(\Sigma_h)_{h \in H}$ with Σ_h contained in a tube with radius $H(h)$ around C. We estimate the continous normal vector at a point of C by a discrete normal vector at a not too far point of Σ_h. The following theorem gives a bound to the error of this estimation, and in particular shows that this error uniformly converges to 0 with h.

Theorem 3. *Let $g = (g_1, g_2)$ be a C^k parameterization of a simple closed curve C with $k \geq 2$. Let $Ng = (g_2', -g_1')$ be a normal vector field of g. Suppose that for all i we have $\|g(ih) - h\Sigma_h(i)\|_\infty \leq Kh^\alpha$. Let $p = \lfloor h^{-1+\frac{\alpha}{k_0}} \rfloor$ and $k_0 = Min\{k, 2m+1\}$. Then*

$$\left\| \left(N_{m,p}\Gamma\right)(i_0) - Ng\right\| = O\left(h^{\alpha - \frac{\alpha}{k_0}}\right) \quad (10)$$

Proof. The proof is straightforward from theorem 1.

3.2 Curvature Estimation

Definition 6. *Let $\Gamma = (\gamma_1, \gamma_2) : \mathbb{Z} \mapsto \mathbb{Z}^2$ be a discrete function. Let $p \in \mathbb{N}^*$ and $i_0 \in \mathbb{Z}$. The m-Taylor-optimal discrete curvature of Γ at point i_0 with step p is*

$$\left(C_{m,p}\Gamma\right)(i_0) = \frac{\left(\Delta_{m,p}\gamma_2\right)(i_0)\left(\Delta^{(2)}_{m,p}\gamma_1\right)(i_0) - \left(\Delta_{m,p}\gamma_1\right)(i_0)\left(\Delta^{(2)}_{m,p}\gamma_1\right)(i_0)}{\left(\left(\left(\Delta_{m,p}\gamma_1\right)(i_0)\right)^2 + \left(\left(\Delta_{m,p}\gamma_2\right)(i_0)\right)^2\right)^{\frac{3}{2}}} \quad (11)$$

Under the same assumption than in the previous subsection, we estimate the continous curvature at a point of C by a discrete curvature at a not too far point of Σ_h.

4 Experimental Evaluation

In this section, we present an experimental evaluation of our various differential estimators. We need to compare the estimated differential quantities values with expected Euclidean ones on graphs of functions or on parametric curves on which such information is known. The considered shapes are simple continous shapes such as the sine function graph, discs or ellipses. These continuous objects have been digitized, with an eventually additional uniform noise. In the 2D shapes cases, we have got a 8-connexe parametrization of the eventually noisy boundary (without ouliers). Then we compare the values of the discrete differential quantity with the corresponding continuous one computed for close points. Considering the empirical multigrid convergence, we always compute the worst case error for a familly of points on the curves for various resolution steps. In the noisy cases, we consider five random curves and compute the average of the worst case errors for these five curves.

4.1 First Order Derivation

First Order Derivative of the Sine function. Figure 1 shows the estimated values of the first order discrete derivatives of noisy digitizations of the sine function graph on the interval $[2; 2.25]$. The discretization step is $h = \frac{1}{1000}$. We use the estimator $\Delta_{7,200}$. The noise is uniform on a set of values $\{-n, ..., +n\}$, with $n = 0, 1, 2, 5$.

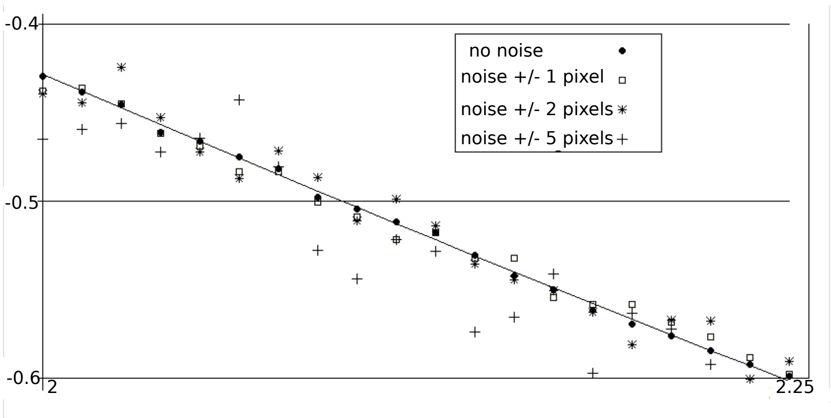


Fig. 1. Estimations of the first order derivatives of the sine function using $\Delta_{7,200}$ for digitizations with $h = \frac{1}{1000}$ as a discretization step and with various levels of noise

Empirical Multigrid Convergence. Here we compute estimations for the first order derivative of the sine function for various resolution step, using Δ_{7,h^n} for various values of n. For each resolution step, the computation is achieve for one hundred points with abcissae $(x_{h,k} = 2 + \frac{k}{h})_{0 \leq k < 100}$. Then we evaluate the maximal errors for these points:

$$Max\left\{\left|\left(\Delta_{7,h^n}\Gamma\right)(x_{h,k}) - cos(x_{h,k})\right|; 0 \leq k < 100\right\}$$

The graph of the function defined by $2h^{0.9}$ which is less than the best theoretical bounding function ($8h^{0.9}$, see theorem 1) is drawn on the figure 2.

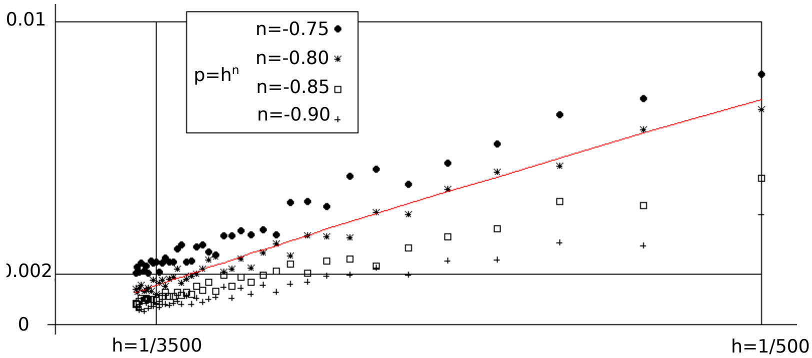


Fig. 2. Maximal error for approximations of the first order derivatives of the sine function using Δ_{7,h^n} at one hundred points for various n

4.2 Second Order Derivation

Second Order Derivative of the Sine Function. Figure 3 shows the computed values of the second order discrete derivatives of rounded digitalizations of the sine function using $\Delta^2_{7,h^{-0.9}}$ for the same values of h than the one considered in [9].

Empirical Multigrid Convergence. Figure 4 shows the estimations of the second order derivative of the sine function for various resolutions, using Δ_{7,h^n} for various n. For each resolution step, the computation is achieve for two hundred points with abcissae $(x_{h,k} = 2 + \frac{k}{h})_{0 \leq k < 200}$. Then we evaluate the maximal absolute errors for these points:

$$Max\left\{\left|\left(\Delta^2_{7,h^n}\Gamma\right)(x_{h,k}) + sin(x_{h,k})\right|; 0 \leq k < 200\right\}$$

4.3 Curvatures

Here we compare the proposed 2D curvature estimators with the continuous one.

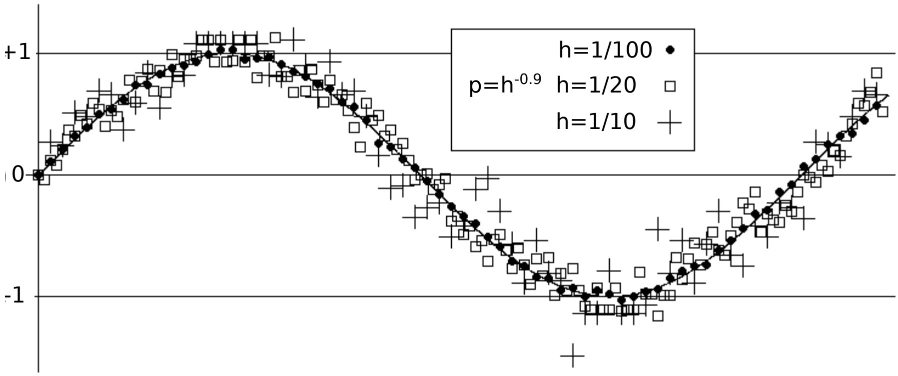


Fig. 3. Estimations of the second order derivative of the sine function using $\Delta_{7,h^{-0.9}}$ at different resolutions

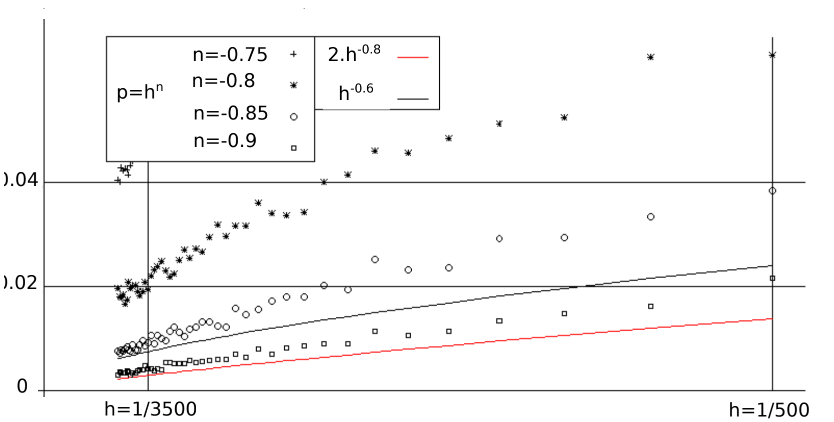
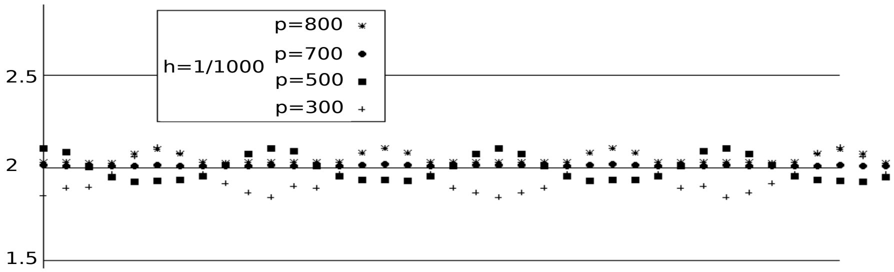


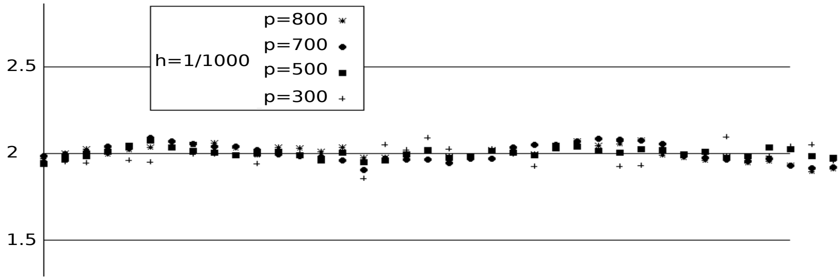
Fig. 4. Maximal error for approximations of the second order derivatives of the sine function using Δ^2_{7,h^n} for various n at two hundred points

Curvature of a Noisy Circle. Figure 5 shows the computed values of the discrete curvature of noisy digitizations of a circle of radius $\frac{1}{2}$. These values have been computed for forty points around the discrete curve. The considered digitization step is $h = \frac{1}{1000}$. We have introduced various uniform noises. Each graph presents results for different values of the computation steps p. The parametrized discrete curves have been obtained by drawing a noisy disc and an ellipse, then extracting the boundary (hence eliminating the outliers).

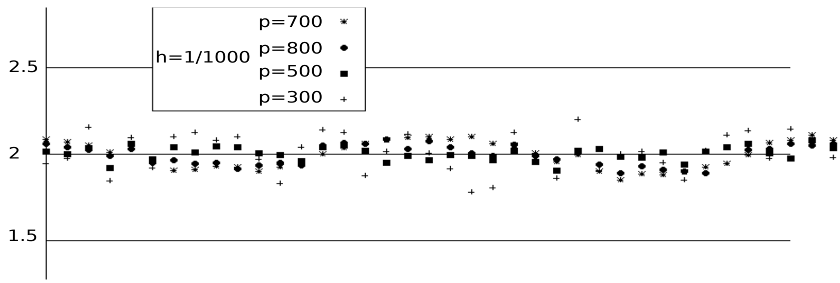
Empirical Multigrid Convergence Here we compute the curvature of digitizations of a circle of radius 1 and of an ellipse of equation $4x^2 + y^2 = 1$ for various digitization steps, using the $C_{7,p}$ mask for various computations steps. The



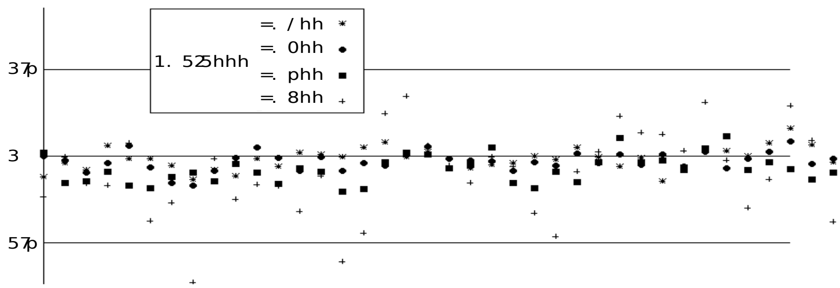
(a) curvature of an unoisy circle of radius $\frac{1}{2}$



(b) curvature of a noisy circle of radius $\frac{1}{2}$, with noise ± 1 pixel

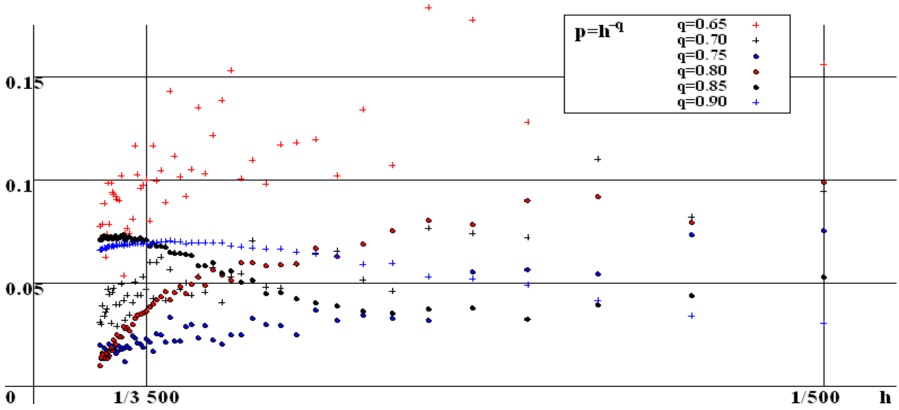


(c) curvature of a noisy circle of radius $\frac{1}{2}$, with noise ± 2 pixel

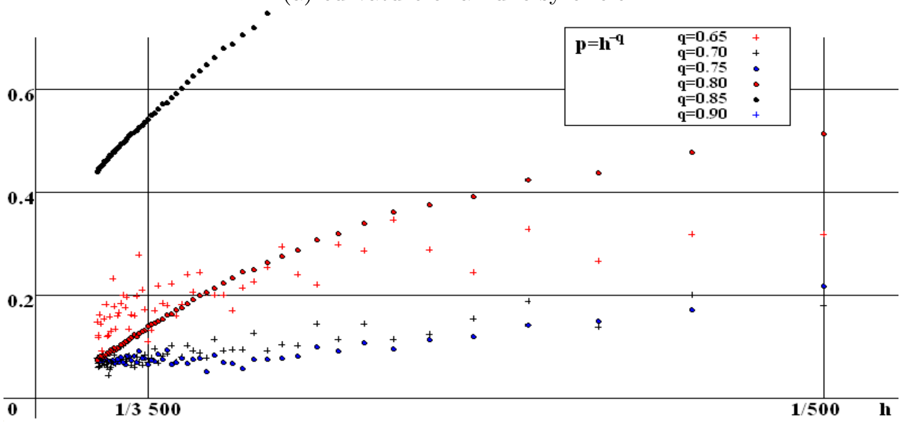


(d) curvature of a noisy circle of radius $\frac{1}{2}$, with noise ± 4 pixel

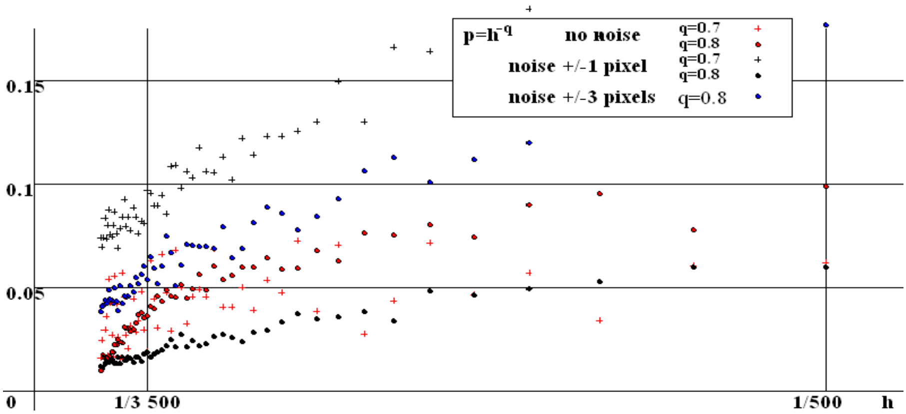
Fig. 5. Estimations of the curvature of a circle of radius $\frac{1}{2}$ for digitization step $h = \frac{1}{1000}$ and various steps p and noises



(a) curvature of an unoisy circle



(b) curvature of an unoisy ellipse



(c) curvature of a noisy circle

Fig. 6. Maximal relative error for curvature approximation of (a) the unnoisy circle of radius 1, (b) the unnoisy ellipse of equation $4x^2 + y^2 = 1$ (c) a noisy circle of radius 1, using the $C_{7,p}$ estimators for various computation steps p

computation is achieved for twenty points $(p_{h,k})_{0\le k<20}$ around the discrete curves in the same quadrant. Then we evaluate the maximal error for these points:

$$Max\left\{\left|\frac{(C_{7,p}\Gamma)\,(p_{h,k}) - c(\bar{p}_{h,k})}{c(\bar{p}_{h,k})}\right|; 0 \le k < 20\right\}$$

(see Figure 6). Here $c(\bar{p}_{h,k})$ is the curvature of the continuous shape at a point $\bar{p}_{h,k}$ close to the point $p_{h,k}$ (namely the point of the continuous shape having the same abcissa).

5 Conclusion

We have presented a new way for estimating differential quantities using convolution. The main idea is to use sparse nodes. Using this technic allows a better complexity than the other known convolution-based methods. The use of rational numbers as coefficients of the convolution mask is not very heavy, because they all have a common constant denominator. Moreover, this method provides a theoretical multigrid convergence and simulations show a good estimation in practice. However, we have to compare carefully our method with the alternative ones in a further work. A bottleneck of the Taylor optimal kernels is that the step of discretization needs to be known to determine the value of the parameter p. This is generally not the case. We thank the anonymous referees for valuable remarks.

References

[1] Coeurjolly, D., Miguet, S., Tougne, L.: Discrete Curvature Based on Osculating Circle Estimation. In: Arcelli, C., Cordella, L.P., Sanniti di Baja, G. (eds.) IWVF 2001. LNCS, vol. 2059, pp. 303–312. Springer, Heidelberg (2001)
[2] Coeurjolly, D., Lachaud, J.-O., Roussillon, T.: Multigrid Convergence of Discrete Geometric Estimators. In: Brimkov, V., Barneva, R. (eds.) Digital Geometry Algorithms, Theoretical Foundations and Applications of Computational Imaging. LNCVB, vol. 2, pp. 395–424. Springer (2012)
[3] Coeurjolly, D., Lachaud, J.-O., Levallois, J.: Integral Based Curvature Estimators in Digital Geometry. In: Gonzalez-Diaz, R., Jimenez, M.-J., Medrano, B. (eds.) DGCI 2013. LNCS, vol. 7749, pp. 215–227. Springer, Heidelberg (2013)
[4] Esbelin, H.-A., Malgouyres, R., Cartade, C.: Convergence of Binomial-Based Derivative Estimation for 2 Noisy Discretized curves. Theoretical Computer Science 412, 4805–4813 (2011)
[5] Kerautret, B., Lachaud, J.-O., Naegel, B.: Comparison of Discrete Curvature Estimators and Application to Corner Detection. In: Bebis, G., et al. (eds.) ISVC 2008, Part I. LNCS, vol. 5358, pp. 710–719. Springer, Heidelberg (2008)
[6] Klette, R., Rosenfeld, A.: Digital Geometry: Geometric Methods for Digital Picture Analysis. Series in Computer Graphics and Geometric Modeling. Morgan Kaufmann (2004)
[7] Lachaud, J.-O., Vialard, A., de Vieilleville, F.: Analysis and Comparative Evaluation of Discrete Tangent Estimators. In: Andrès, É., Damiand, G., Lienhardt, P. (eds.) DGCI 2005. LNCS, vol. 3429, pp. 240–251. Springer, Heidelberg (2005)

[8] Malgouyres, R., Brunet, F., Fourey, S.: Binomial Convolutions and Derivatives Estimation from Noisy Discretizations. In: Coeurjolly, D., Sivignon, I., Tougne, L., Dupont, F. (eds.) DGCI 2008. LNCS, vol. 4992, pp. 370–379. Springer, Heidelberg (2008)

[9] Provot, L., Gérard, Y.: Estimation of the Derivatives of a Digital Function with a Convergent Bounded Error. In: Debled-Rennesson, I., Domenjoud, E., Kerautret, B., Even, P. (eds.) DGCI 2011. LNCS, vol. 6607, pp. 284–295. Springer, Heidelberg (2011)

[10] Roussillon, T., Lachaud, J.-O.: Accurate Curvature Estimation along Digital Contours with Maximal Digital Circular Arcs. In: Aggarwal, J.K., Barneva, R.P., Brimkov, V.E., Koroutchev, K.N., Korutcheva, E.R. (eds.) IWCIA 2011. LNCS, vol. 6636, pp. 43–55. Springer, Heidelberg (2011)