Non-additive Bounded Sets of Uniqueness in $\mathbb{Z}^n$

Sara Brunetti[1], Paolo Dulio[2,⋆], and Carla Peri[3]

[1] Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Via Roma, 56, 53100 Siena, Italy
sara.brunetti@unisi.it

[2] Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy
paolo.dulio@polimi.it

[3] Università Cattolica S. C., Via Emilia Parmense, 84, 29122 Piacenza, Italy
carla.peri@unicatt.it

Abstract. A main problem in discrete tomography consists in looking for theoretical models which ensure uniqueness of reconstruction. To this, lattice sets of points, contained in a multidimensional grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$ (where for $p \in \mathbb{N}$, $[p] = \{0, 1, ..., p-1\}$), are investigated by means of X-rays in a given set S of lattice directions. Without introducing any noise effect, one aims in finding the minimal cardinality of S which guarantees solution to the uniqueness problem.

In a previous work the matter has been completely settled in dimension two, and later extended to higher dimension. It turns out that $d+1$ represents the minimal number of directions one needs in $\mathbb{Z}^n$ ($n \geq d \geq 3$), under the requirement that such directions span a d-dimensional subspace of $\mathbb{Z}^n$. Also, those sets of $d+1$ directions have been explicitly characterized.

However, in view of applications, it might be quite difficult to decide whether the uniqueness problem has a solution, when X-rays are taken in a set of more than two lattice directions. In order to get computational simpler approaches, some prior knowledge is usually required on the object to be reconstructed. A powerful information is provided by additivity, since additive sets are reconstructible in polynomial time by using linear programming.

In this paper we compute the proportion of non-additive sets of uniqueness with respect to additive sets in a given grid $\mathcal{A} \subset \mathbb{Z}^n$, in the important case when d coordinate directions are employed.

Keywords: Additive set, bad configuration, discrete tomography, non-additive set, uniqueness problem, X-ray.

1 Introduction

One of the main problems of discrete tomography is to determine finite subsets of the integer lattice $\mathbb{Z}^n$ by means of their X-rays taken in a finite number of

⋆ Corresponding author.

E. Barcucci et al. (Eds.): DGCI 2014, LNCS 8668, pp. 226–237, 2014.

lattice directions. Given a lattice direction u, the X-ray of a finite lattice set E in the direction u counts the number of points in E on each line parallel to u. The points in E can model the atoms in a crystal, and new techniques in high resolution transmission electron microscopy allow the X-rays of a crystal to be measured so that the main goal of discrete tomography is to use these X-rays to deduce the local atomic structure from the collected counting data, with a view to applications in the material sciences. The high energies required to produce the discrete X-rays of a crystal mean that only a small number of X-rays can be taken before the crystal is damaged. Therefore, discrete tomography focuses on the reconstruction of binary images from a small number of X-rays.

In general, it is hopeless to obtain uniqueness results unless the class of lattice sets is restricted. In fact, for any fixed set S of more than two lattice directions, to decide whether a lattice set is uniquely determined by its X-rays along S is NP-complete [9]. Thus, one has to use a priori information, such as convexity or connectedness, about the sets that have to be reconstructed (see for instance [8], where convex lattice sets are considered). An important class, which provides a computational simpler approach to the uniqueness and reconstruction problems, is that of additive sets introduced by P.C. Fishburn and L.A. Shepp in [7] (see the next section for all terminology). Additive sets with respect to a finite set of lattice directions are uniquely determined by their X-rays in the given directions, and they are also reconstructible in polynomial time by use of linear programming. The notions of additivity and uniqueness are equivalent when two directions are employed, whereas, for three or more directions, additivity is more demanding than uniqueness, as there are non-additive sets which are unique [6]. More recently, additivity have been reviewed and settled by a more general treatment in [10]. Thanks to this new approach, the authors showed that there are non-additive lattice sets in $\mathbb{Z}^3$ which are uniquely determined by their X-rays in the three standard coordinate directions by exhibiting a counter-example (see [10, Remark 2 and Figure 2]). This answers in the negative a question raised by Kuba at a conference on discrete tomography in Dagsthul (1997), that every subset E of $\mathbb{Z}^3$ might be uniquely determined by its X-rays in the three standard unit directions of $\mathbb{Z}^3$ if and only if E is additive.

In previous works we restricted our attention to bounded sets, i.e. lattice sets contained in a given grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$. In particular, in [2] we addressed the problem in dimension two and we proved that for a given set S of four lattice directions, there exists a rectangular grid $\mathcal{A}$ such that all the subsets of $\mathcal{A}$ are uniquely determined by their X-rays in the directions in S. In [4] we extended the previous uniqueness results to higher dimensions, by showing that $d+1$ represents the minimal number of directions one needs in $\mathbb{Z}^n$ $(n \geq d \geq 3)$, under the requirement that such directions span a d-dimensional subspace of $\mathbb{Z}^n$. Also, those sets of $d+1$ directions have been explicitly characterized.

We also recall that Fishburn et al. [7] noticed that an explicit construction of non-additive sets of uniqueness has proved rather difficult even though it might be true that non-additive uniqueness is the rule rather than exception. In particular they suggest that for some set of X-ray directions of cardinality larger than two, the proportion of lattice sets E of uniqueness that are not also additive approaches 1 as E gets large. They leave it as an open question in the discussion section. In [3] we presented a procedure for constructing non-additive sets in $\mathbb{Z}^2$ and we showed that when S contains the coordinate directions this proportion does not depend on the size of the lattice sets into consideration.

In the present paper we focus on non-additive sets in $\mathbb{Z}^n$ and estimate the proportion of non-additive sets of uniqueness with respect to additive sets in a given grid $\mathcal{A}$, when the set S contains d coordinate directions (see Theorem 1). It turns out that such proportion tends to zero as $\mathcal{A}$ gets large so that the probability to have an additive set is high. From the viewpoint of the applications, this suggest the use of linear programming for good quality solutions as the reconstruction problem for additive sets is polynomial.

2 Definitions and Preliminaries

The standard orthonormal basis for $\mathbb{Z}^n$ will be $\{e_1, \ldots, e_n\}$, and the coordinates with respect to this orthonormal basis $x_1, \ldots, x_n$. A vector $u = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, where $a_1 \geq 0$, is said to be a *lattice direction*, if $\gcd(a_1, ..., a_n) = 1$. We refer to a finite subset E of $\mathbb{Z}^n$ as a *lattice set*, and we denote its cardinality by $|E|$. For a finite set $S = \{u_1, u_2, ..., u_m\}$ of directions in $\mathbb{Z}^n$, the *dimension* of S, denoted by $\dim S$, is the dimension of the vector space generated by the vectors $u_1, u_2, ..., u_m$. Moreover, for each $I \subseteq S$, we denote $u(I) = \sum_{u \in I} u$, with $u(\emptyset) = 0 \in \mathbb{Z}^n$. Any two lattice sets E and F are *tomographically equivalent* if they have the same X-rays along the directions in S. Conversely, a lattice set E is said to be S-*unique* if there is no lattice set F different from but tomographically equivalent to E.

An S-*weakly bad configuration* is a pair of lattice sets (Z, W) each consisting of k lattice points not necessarily distinct (counted with multiplicity), $z_1, ..., z_k \in Z$ and $w_1, ..., w_k \in W$ such that for each direction $u \in S$, and for each $z_r \in Z$, the line through z_r in direction u contains a point $w_r \in W$. If all the points in each set Z, W are distinct (multiplicity 1), then (Z, W) is called an S-*bad configuration*. If for some $k \geq 2$ an S-(weakly) bad configuration (Z, W) exists such that $Z \subseteq E$, $W \subseteq \mathbb{Z}^n \setminus E$, we then say that a lattice set E has an S-(weakly) bad configuration. This notion plays a crucial role in investigating uniqueness problems, since a lattice set E is S-unique if and only if E has no S-bad configurations [7].

For $p \in \mathbb{N}$, denote $\{0, 1, ..., p-1\}$ by $[p]$. Let $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$ be a fixed lattice grid in $\mathbb{Z}^n$. We shall restrict our considerations to lattice sets contained in a given lattice grid $\mathcal{A}$, referred to as *bounded sets*. We say that a set S is a *valid* set of directions for $\mathcal{A}$, if for all $i \in \{1, \ldots, n\}$, the sum h_i of the absolute values of the i-th coordinates of the directions in S satisfies the condition $h_i < m_i$. Notice that this definition excludes trivial cases when S contains a direction with so large (or so small) slope, with respect to $\mathcal{A}$, such that each line with this slope meets $\mathcal{A}$ in no more than a single point. If each subset $E \subset \mathcal{A}$ is S-unique in $\mathcal{A}$, we then say that S is a *set of uniqueness* for $\mathcal{A}$. For our purpose, we define additivity in terms of solutions of linear programs.

The reconstruction problem can be formulated as an integer linear program (ILP). Since the NP-hardness of the reconstruction problem for more than two directions reflects in the integrality constraint, relaxation of ILP are considered (see, for instance [1], [11],[14],[15]). In this setting, a lattice set is S-*additive* if it is the unique solution of the relaxed linear program (LP). Moreover, a set is S-additive if and only if it has no S-weakly bad configurations. Additivity is also fundamental for treating uniqueness problems, due to the following facts (see [5, Theorem 2]):

1. Every S-*additive* set is S-*unique*.
2. There exist S-*unique* sets which are not S-*additive*.

A set which is not S-additive will be simply said *non-additive*, when confusion is not possible.

In [2] we characterized all the minimal sets S of planar directions which are sets of uniqueness for $\mathcal{A}$, and in [4] we studied the problem in higher dimension. In particular, we stated the following necessary condition on minimal sets S of lattice directions to be sets of uniqueness for $\mathcal{A}$.

Proposition A ([4, Proposition 8]). *Let $S \subset \mathbb{Z}^n$ be a set of distinct lattice directions such that $|S| = d+1$ and $\dim S = d \geq 3$ ($n \geq d \geq 3$). Suppose that S is a valid set of uniqueness for a finite grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n] \subset \mathbb{Z}^n$. Then S is of the form*

$$S = \{u_1, ..., u_d, w = u(I) - u(J)\}, \tag{1}$$

where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$.

Examples of sets S of the form (1) which are contained in $\mathbb{Z}^3$ and $\mathbb{Z}^4$ are presented in Subsections 3.1 and 3.2, respectively.

Among the sets S of the form (1) we then specified which ones are sets of uniqueness for $\mathcal{A}$, by employing an algebraic approach introduced by Hajdu and Tijdeman in [12]. To illustrate the result we need some further definitions (see also [4]).

For a vector $u = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we simply write x^u in place of the monomial $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$. Consider now any lattice vector $u \in \mathbb{Z}^n$, where $u \neq 0$. Let $u_- \in \mathbb{Z}^n$ be the vector whose entries equal the corresponding entries of u if negative, and are 0 otherwise. Analogously, let $u_+ \in \mathbb{Z}^n$ be the vector whose entries equal the corresponding entries of u if positive, and are 0 otherwise.

For any finite set S of lattice directions in $\mathbb{Z}^n$, we define the polynomial

$$F_S(x_1, \ldots, x_n) = \prod_{u \in S} \left(x^{u_+} - x^{-u_-} \right). \tag{2}$$

For example, for $S = \{e_1, e_2, e_3, e_1 + e_2 - e_3\} \subset \mathbb{Z}^3$ we get

$$F_S(x_1, x_2, x_3) = (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_1 x_2 - x_3) = -x_1 x_2 x_3^2 + x_2 x_3^2 + x_1 x_3^2 - x_3^2 + \\ + x_1^2 x_2^2 x_3 - x_1 x_2^2 x_3 - x_1^2 x_2 x_3 + 2 x_1 x_2 x_3 - x_2 x_3 - x_1 x_3 + x_3 - x_1^2 x_2^2 + x_1 x_2^2 + x_1^2 x_2 - x_1 x_2.$$

Given a function $f : \mathcal{A} \to \mathbb{Z}$, its *generating function* is the polynomial defined by

$$G_f(x_1, \ldots, x_n) = \sum_{(a_1,\ldots,a_n)\in\mathcal{A}} f(a_1, \ldots, a_n) x_1^{a_1} \ldots x_n^{a_n}.$$

Conversely, we say that the function f is *generated* by a polynomial $P(x_1, \ldots, x_n)$ if $P(x_1, \ldots, x_n) = G_f(x_1, \ldots, x_n)$. Notice that the function f generated by the polynomial $F_S(x_1, \ldots, x_n)$ vanishes outside $\mathcal{A}$ if and only if the set S is valid for $\mathcal{A}$.

Furthermore, to a monomial $kx_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ we associate the lattice point $z = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, together with its *weight* k. We say that a point $(a_1, \ldots, a_n) \in \mathcal{A}$ is a *multiple positive point* for f (or G_f) if $f(a_1, \ldots, a_n) > 1$. Analogously, $(a_1, \ldots, a_n) \in \mathcal{A}$ is said to be a *multiple negative point* for f if $f(a_1, \ldots, a_n) < -1$. Such points are simply referred to as *multiple points* when the signs are not relevant. For a polynomial $P(x_1, \ldots, x_n)$ we denote by P^+ (resp. P^-) the set of lattice points corresponding to the monomials of $P(x_1, \ldots, x_n)$ having positive (resp. negative) sign, referred to as positive (resp. negative) lattice points. We also write $P = P^+ \cup P^-$.

The *line sum* of a function $f : \mathcal{A} \to \mathbb{Z}$ along the line $x = x_0 + tu$, passing through the point $x_0 \in \mathbb{Z}^n$ and with direction u, is the sum $\sum_{x=x_0+tu, x\in\mathcal{A}} f(x)$. Further, we denote $||f|| = \max_{x\in\mathcal{A}}\{|f(x)|\}$. We can easily check that the function f generated by $F_S(x_1, \ldots, x_n)$ has zero line sums along the lines taken in the directions belonging to S.

Hajdu and Tijdeman proved that if $g : \mathcal{A} \to \mathbb{Z}$ has zero line sums along the lines taken in the directions of S, then $F_S(x_1, \ldots, x_n)$ divides $G_g(x_1, \ldots, x_n)$ over $\mathbb{Z}$ (see [12, Lemma 3.1] and [13]). We recall that two functions $f, g : \mathcal{A} \subset \mathbb{Z}^n \to \{0, 1\}$ are *tomographically equivalent* with respect to a given finite set S of lattice directions if they have equal line sums along the lines corresponding to the directions in S. Note that two non trivial functions $f, g : \mathcal{A} \to \{0, 1\}$ which are tomographically equivalent can be interpreted as characteristic functions of two lattice sets which are tomographically equivalent. Further, the difference $h = f - g$ of f and g has zero line sums. Hence there is a one-to-one correspondence between S-bad configurations contained in $\mathcal{A}$ and non-trivial functions $h : \mathcal{A} \to \mathbb{Z}$ having zero line sums along the lines corresponding to the directions in S and $||h|| \leq 1$.

Let us consider a set $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$, where I, J are disjoint subsets of $\{u_1, ..., u_d\}$. We define

$$D = \{\pm v : v = u(X) - u(I) \neq 0, X \subseteq I \cup J \cup \{w\}\}. \tag{3}$$

In [4] we proved the following result.

Theorem B ([4, Theorem 12]). *Let $S \subset \mathbb{Z}^n$ be a set of distinct lattice directions such that $S = \{u_r = (a_{r1}, \ldots, a_{rn}) : r = 1, \ldots, d + 1\}$ ($n \geq d \geq 3$), where $u_1, ..., u_d$ are linearly independent, $u_{d+1} = u(I) - u(J)$, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| (\mathrm{mod} 2)$. Suppose S is valid for the grid*

$\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$. *Denote* $\sum_{r=1}^{d+1} |a_{ri}| = h_i$*, for each* $i \in \{1, \ldots, n\}$*. Suppose that* $g : \mathcal{A} \to \mathbb{Z}$ *has zero line sums along the lines in the directions in* S*, and* $||g|| \leq 1$*. Then* g *is identically zero if and only if for each* $v = (v_1, \ldots, v_n) \in D$*, there exists* $s \in \{1, \ldots, n\}$ *such that* $|v_s| \geq m_s - h_s$*.*

From the geometrical point of view, a set S of lattice directions is a set of uniqueness for a grid $\mathcal{A}$ if and only if S and $\mathcal{A}$ are chosen according to assumptions in Theorem B, and the resulting set D is such that its members satisfy the conditions of the theorem.

3 Non-additive Bounded Set of Uniqueness

In this section we study non-additive sets of uniqueness contained in a given grid $\mathcal{A}$, in the important case when S contains the coordinate directions. In [3] we showed that when $S \subset \mathbb{Z}^2$ contains the coordinate directions the proportion of lattice sets E of uniqueness that are not also additive does not depend on the size of the lattice sets into consideration and is given by

$$\frac{2}{2^{|F_S|} - 2},$$

where $|F_S|$ denotes the cardinality of the set of points corresponding to the polynomial $F_S(x_1, x_2)$.

In the present paper we aim to extend this estimate to higher dimension. Before presenting the general result, we wish to consider two preliminary cases, concerning $\mathbb{Z}^3$ and $\mathbb{Z}^4$ respectively, which motivate the general result presented below.

3.1 Non-additive Sets in $\mathbb{Z}^3$

Let us consider the case $n = d = 3$. Let $S = \{e_1, e_2, e_3, w = u(I) - u(J)\} \subset \mathbb{Z}^3$, where I, J are disjoint subsets of $\{e_1, e_2, e_3\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Since w is a direction distinct from e_1, e_2, e_3, we have $e_1 \in I$ and, up to exchanging the role of e_2 and e_3, we have the following choices for w.

1. $w = e_1 + e_2 + e_3 = (1, 1, 1)$, where $I = \{e_1, e_2, e_3\}, J = \emptyset$.
2. $w = e_1 + e_2 - e_3 = (1, 1, -1)$, where $I = \{e_1, e_2, \}, J = \{e_3\}$.
3. $w = e_1 - e_2 - e_3 = (1, -1, -1)$, where $I = \{e_1\}, J = \{e_2, e_3\}$.

In order to apply Theorem B, we now evaluate the set D defined by (3), in all these cases.

1. If $I = \{e_1, e_2, e_3\}, J = \emptyset$, then by choosing $X = \{w, e_i\}$, for $i = 1, 2, 3$, we get $v = e_i \in D$.
2. If $I = \{e_1, e_2, \}, J = \{e_3\}$, then for $X = \{e_1, e_2, e_3\}$ we get $v = e_3 \in D$. For $X = \{e_1\}$ we get $v = -e_2 \in D$, and for $X = \{e_2\}$ we get $v = -e_1 \in D$.

3. If $I = \{e_1\}, J = \{e_2, e_3\}$, then for $X = \{e_1, e_i\}$ we get $v = e_i \in D$, where $i = 2, 3$. For $X = \{w, e_1, e_2, e_3\}$ we get $v = e_1 \in D$.

Since in all the previous cases we have $h_i = 2$, for $i = 1, 2, 3$, then the set S is a set of uniqueness for the grid $\mathcal{A} = [m_1] \times [m_2] \times [m_3]$ if and only if $m_i - h_i \leq 1$ for each $i = 1, 2, 3$, that is $m_i = 3$. This implies that $\mathcal{A}$ contains a unique S-weakly bad configuration given by F_S. The non-additive sets of uniqueness in F_S are precisely F_S^- and F_S^+. All the other subsets of F_S, are additive. Therefore, the proportion of bounded non-additive sets of uniqueness w.r.t. those additive is given by

$$\frac{2 \cdot 2^{|\mathcal{A} \setminus F_S|}}{2^{|\mathcal{A}|} - 2 \cdot 2^{|\mathcal{A} \setminus F_S|}} = \frac{2}{2^{|F_S|} - 2}. \tag{4}$$

3.2 Non-additive Sets in $\mathbb{Z}^4$

Let us now consider the case $n = 4$. We first note that the condition $|I| \equiv |\{w\} \cup J| \pmod 2$ in Theorem B implies that $|I \cup J|$ must be odd. Therefore, we have $|I \cup J| = 3$ and we can distinguish the following cases:

1. $n = 4 > d = |I \cup J| = 3$;
2. $n = 4 = d > |I \cup J| = 3$.

1. Suppose $n = 4 > d = |I \cup J| = 3$. Up to permutations of the standard orthonormal vectors, we can assume $S = \{e_1, e_2, e_3, w = u(I) - u(J)\} \subset \mathbb{Z}^4$, where I, J are disjoint subsets of $\{e_1, e_2, e_3\}$ such that $I \cup J = \{e_1, e_2, e_3\}$, and $|I| \equiv |\{w\} \cup J| \pmod 2$. By identifying $\mathbb{Z}^3$ with the subspace $H = \{(z_1, z_2, z_3, 0) : z_1, z_2, z_3 \in \mathbb{Z}\}$ in $\mathbb{Z}^4$, we can repeat the same considerations as in the previous subsection. Therefore, we have $S, D \subset H$ and the set S is a set of uniqueness for the grid $\mathcal{A} = [m_1] \times [m_2] \times [m_3] \times [m_4]$ if $m_i = 3$ for $i = 1, 2, 3$, and $m_4 \geq 1$. We shall simply write $m = m_4$. In this case the grid $\mathcal{A}$ can be arbitrary large in one direction and we shall compute the proportion of bounded non-additive sets of uniqueness w.r.t. those additive in $\mathcal{A}$ as a function of m. For the sake of simplicity, we assume $w = e_1 + e_2 + e_3 = (1, 1, 1, 0)$, since all the other cases are analogous. We have
$$F_S(x_1, x_2, x_3) = (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_1 x_2 x_3 - 1),$$
and all the S-weakly bad configurations contained in $\mathcal{A}$ correspond to polynomials of the form $F_S(x_1, x_2, x_3)P(x_4)$, where $P(x_4)$ is a polynomial in x_4 with degree less than or equal to $m - 1$. Thus $P(x_4) = a_{m-1}x_4^{m-1} + \cdots + a_0$, where the coefficients $a_{m-1}, \cdots, a_0$ are not all zero.
Let us consider two polynomials
$$Q_1(x_1, x_2, x_3, x_4) = F_S(x_1, x_2, x_3)P_1(x_4) = F_S(x_1, x_2, x_3)\left(a_{m-1}x_4^{m-1} + \cdots + a_0\right),$$
$$Q_2(x_1, x_2, x_3, x_4) = F_S(x_1, x_2, x_3)P_2(x_4) = F_S(x_1, x_2, x_3)\left(b_{m-1}x_4^{m-1} + \cdots + b_0\right).$$

The corresponding sets of points Q_1, Q_2 are equal if and only if $a_i = 0$ implies $b_i = 0$, for all $i = 0, \cdots, m-1$. Thus the number of S-weakly bad configurations contained in $\mathcal{A}$ equals the number of polynomials $P(x_4) = a_{m-1}x_4^{m-1} + \cdots + a_0$ whose coefficients belong to the set $\{0, 1\}$, except the null polynomial.

Let us denote by $\mathcal{F}$ the set of all points in $\mathcal{A}$ which belong to some S-weakly bad configuration. Notice that each S-weakly bad configuration contains two non-additive sets consisting of the set of positive (resp. negative) points. By multiplying the corresponding polynomial by -1, these two non-additive sets exchange each other. Therefore, the number of non-additive sets which are contained in $\mathcal{F}$ equals the number of polynomials $P(x_4) = a_{m-1}x_4^{m-1} + \cdots + a_0$ whose coefficients belong to the set $\{-1, 0, 1\}$, except the null polynomial. Thus we have $3^m - 1$ non-additive sets in $\mathcal{F}$. Any other non-additive set in $\mathcal{A}$ is obtained by adding some points of $\mathcal{A} \setminus \mathcal{F}$ to a non-additive set in $\mathcal{F}$. Therefore, the number of non-additive sets contained in $\mathcal{A}$ is given by $2^{|\mathcal{A} \setminus \mathcal{F}|}(3^m - 1)$.

The proportion of bounded non-additive sets of uniqueness in $\mathcal{A}$ with respect to those additive is given by

$$\frac{2^{|\mathcal{A} \setminus \mathcal{F}|}(3^m - 1)}{2^{|\mathcal{A}|} - 2^{|\mathcal{A} \setminus \mathcal{F}|}(3^m - 1)} = \frac{3^m - 1}{2^{|F_S|m} - 3^m + 1}. \tag{5}$$

For $m = 1$ we get

$$\frac{2}{2^{|F_S|} - 2},$$

as in (4). Moreover, as $\mathcal{A}$ gets large, we get

$$\lim_{m \to \infty} \frac{3^m - 1}{2^{|F_S|m} - 3^m + 1} = 0,$$

so that the set of non-additive sets is negligible.

2. Let us now suppose $n = 4 = d > |I \cup J| = 3$. Up to permutations of the standard orthonormal vectors, we can assume $S = \{e_1, e_2, e_3, e_4, w = u(I) - u(J)\} \subset \mathbb{Z}^4$, where I, J are disjoint subsets of $\{e_1, e_2, e_3\}$ such that $I \cup J = \{e_1, e_2, e_3\}$, and $|I| \equiv |\{w\} \cup J| \pmod 2$. By identifying $\mathbb{Z}^3$ with the subspace $H = \{(z_1, z_2, z_3, 0) : z_1, z_2, z_3 \in \mathbb{Z}\}$ in $\mathbb{Z}^4$, we can repeat the same considerations as in the previous subsection. In particular, we have $h_i = 2$, for $i = 1, 2, 3$, $h_4 = 1$, and the set S is a set of uniqueness for the grid $\mathcal{A} = [m_1] \times [m_2] \times [m_3] \times [m_4]$ if and only if $m_i = 3$ for each $i = 1, 2, 3$, and $m_4 = m \geq 2$. Again we assume $w = e_1 + e_2 + e_3 = (1, 1, 1, 0)$, so that we have

$$F_S(x_1, x_2, x_3, x_4) = (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_1x_2x_3 - 1)(x_4 - 1) = F_T(x_1, x_2, x_3)(x_4 - 1),$$

where $T = \{e_1, e_2, e_3, w\}$. All the S-weakly bad configurations contained in $\mathcal{A}$ correspond to polynomials of the form $F_S(x_1, x_2, x_3, x_4)P(x_4)$, where $P(x_4)$ is a polynomial in x_4 with degree less than or equal to $m - 2$. Thus

$P(x_4) = a_{m-2}x_4^{m-2} + \cdots + a_0$, where the coefficients $a_{m-2}, \cdots, a_0$ are not all zero. Denote by $\mathcal{F}$ the set of all points in $\mathcal{A}$ which belong to some S-weakly bad configuration. Then, by the same arguments as in the previous case we have that $\mathcal{F}$ contains $3^{m-1} - 1$ non-additive sets. Thus, we obtain the following estimate for the proportion of bounded non-additive sets of uniqueness in $\mathcal{A}$ with respect to those additive.

$$\frac{2^{|\mathcal{A}\setminus\mathcal{F}|}\left(3^{m-1}-1\right)}{2^{|\mathcal{A}|} - 2^{|\mathcal{A}\setminus\mathcal{F}|}\left(3^{m-1}-1\right)} = \frac{3^{m-1}-1}{2^{|\mathcal{F}|} - \left(3^{m-1}-1\right)} = \frac{3^{m-1}-1}{2^{|F_T|m} - 3^{m-1} + 1}, \quad (6)$$

as $|\mathcal{F}| = m|F_T|$.
For $m = 2$ we get

$$\frac{2}{2^{|F_S|} - 2},$$

as $|F_S| = 2|F_T|$.
Again, as $\mathcal{A}$ gets large, we get

$$\lim_{m\to\infty} \frac{3^{m-1}-1}{2^{15m} - 3^{m-1} + 1} = 0,$$

so that the set of non-additive sets is negligible.

3.3 Non-additive Sets in $\mathbb{Z}^n$

We now consider the general case. In the following, for $p, q \in \mathbb{N}$ with $1 \leq p < q$, we denote $(p, q] = \{z \in \mathbb{N} : p < z \leq q\}$. Further, to unify different cases, when $p = q$ we still adopt the notation $(p, q]$ with the convention that $\prod_{j\in(p,q]} z_j = 1$, for every $z_j \in \mathbb{Z}$.

Theorem 1. *Let $S = \{e_1, \ldots, e_d, w = u(I) - u(J)\}$ be a set of $d + 1$ distinct directions in $\mathbb{Z}^n$, where $n \geq d \geq 3$, $I \cup J = \{e_1, \ldots, e_k\}$ $(3 \leq k \leq d)$, and $|I| \not\equiv |J| \pmod 2$. Let $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n] \subset \mathbb{Z}^n$, where $m_i = 3$ for $i = 1, \ldots, k$, $m_i \geq 2$ for $i = k + 1, \ldots, d$, and $m_i \geq 1$ for $i = d + 1, \ldots, n$. Then the set S is a set of uniqueness for $\mathcal{A}$ and the proportion of non-additive sets of uniqueness in $\mathcal{A}$ with respect to those additive is given by*

$$\frac{3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1}{2^{(2^{k+1}-1)\prod_{i\in(k,n]} m_i} - \left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1\right)}. \quad (7)$$

Proof. By Theorem B, in order to prove that S is a set of uniqueness for $\mathcal{A}$, we have to show that for each $v = (v_1, \ldots, v_n) \in D$ there exists $i \in \{1, \ldots, n\}$ such that $|v_i| \geq m_i - h_i$.
We have $w = \sum_{i=1}^{k} \delta_i e_i$, where $\delta_i = 1$ if $e_i \in I$, and $\delta_i = -1$ if $e_i \in J$, so that $h_i = 2$ for $i = 1, \ldots, k$, $h_i = 1$ for $i = k+1, \ldots, d$, and $h_i = 0$ for $i = d+1, \ldots, n$. If $v = (v_1, \ldots, v_n) \in D$, then $v_j = 0$ for $k + 1 \leq j \leq n$, and $v_i \neq 0$ for some $i_o \in \{1, \ldots, k\}$, so that $|v_{i_o}| \geq m_{i_o} - h_{i_o} = 1$. This proves that S is a set of uniqueness for $\mathcal{A}$.

We have

$$F_S(x_1,\ldots,x_d) = \left(x^{w_+} - x^{-w_-}\right)\prod_{i=1}^{d}(x_i-1).$$

Let us denote $F_T(x_1,\ldots,x_k) = (x^{w_+} - x^{-w_-})\prod_{i=1}^{k}(x_i-1)$. Then

$$F_S(x_1,\ldots,x_d) = F_T(x_1,\ldots,x_k)\prod_{i\in(k,d]}(x_i-1).$$

All the S-weakly bad configurations contained in $\mathcal{A}$ correspond to polynomials of the form $F_S(x_1,\ldots,x_d)P(x_{k+1},\ldots,x_n)$, where the degree $deg_iP(x_{k+1},\ldots,x_n)$ of $P(x_{k+1},\ldots,x_n)$ with respect to x_i, where $i=k+1,\ldots,n$, satisfies the conditions

$$\begin{array}{ll} deg_iP(x_{k+1},\ldots,x_n) < m_i - 1 & \text{for } \; i = k+1,\ldots,d, \\ deg_iP(x_{k+1},\ldots,x_n) < m_i & \text{for } \; i = d+1,\ldots,n. \end{array} \tag{8}$$

Thus we have

$$P(x_{k+1},\ldots,x_n) = \sum a_{r_{k+1},\ldots,r_n} x_{k+1}^{r_{k+1}}\ldots x_n^{r_n}, \tag{9}$$

where $r_{k+1}\in[m_{k+1}-1],\ldots,r_d\in[m_d-1], r_{d+1}\in[m_{d+1}],\ldots,r_n\in[m_n]$. Each S-weakly bad configuration contained in $\mathcal{A}$ corresponds to a polynomial of the form

$$P(x_{k+1},\ldots,x_n)F_T(x_1,\ldots,x_k)\prod_{i\in(k,d]}(x_i-1),$$

where $P(x_{k+1},\ldots,x_n)$ is given by (9).

Let us denote by $\mathcal{F}$ the set of points in $\mathcal{A}$ which belong to some S-weakly bad configuration. Notice that each S-weakly bad configuration contains two non-additive sets consisting of the set of positive (resp. negative) points. By multiplying the corresponding polynomial by -1, these two non-additive sets exchange each other. Therefore, the number of non-additive sets which are contained in $\mathcal{F}$ equals the number of polynomials $P(x_{k+1},\ldots,x_n)$ given by (9), whose coefficients belong to the set $\{-1,0,1\}$, except the null polynomial. Such a number is given by

$$3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]}m_j} - 1.$$

Any other non-additive set in $\mathcal{A}$ is obtained by adding some points of $\mathcal{A}\setminus\mathcal{F}$ to a non-additive set in $\mathcal{F}$. Thus, the proportion of non-additive sets of uniqueness in $\mathcal{A}$ with respect to those additive results

$$\begin{aligned} &\frac{2^{|\mathcal{A}\setminus\mathcal{F}|}\left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]}m_j} - 1\right)}{2^{|\mathcal{A}|} - 2^{|\mathcal{A}\setminus\mathcal{F}|}\left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]}m_j} - 1\right)} \\ &= \frac{3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]}m_j} - 1}{2^{|\mathcal{F}|} - \left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]}m_j} - 1\right)}. \end{aligned} \tag{10}$$

Since $|\mathcal{F}| = |F_T| \prod_{i\in(k,n]} m_i$ and $|F_T| = 2^{k+1} - 1$, we get

$$\frac{3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1}{2^{(2^{k+1}-1)\prod_{i\in(k,n]} m_i} - \left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1\right)},$$

as required. □

When $n = d = k = 3$, as in Subsection 3.1, we have

$$\prod_{i\in(k,d]} (m_i - 1) = \prod_{j\in(d,n]} m_j = \prod_{i\in(k,n]} m_i = 1, \quad 2^{k+1} - 1 = 2^4 - 1 = |F_S|,$$

so that formula (7) gives (4).

When $4 = n = d > k = 3$, as in Subsection 3.2 case 2, we have

$$\prod_{i\in(k,d]} (m_i-1) = m_4-1 = m-1, \quad \prod_{j\in(d,n]} m_j = 1, \quad \prod_{i\in(k,n]} m_i = m_4 = m, \quad |F_T| = 2^{k+1}-1 = 15,$$

so that formula (7) gives (6).

Moreover, if $3 \leq k < n$, then for $(m_{k+1}, \ldots, m_n) \to (\infty, \ldots, \infty)$ we have

$$\frac{3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1}{2^{(2^{k+1}-1)\prod_{i\in(k,n]} m_i} - \left(3^{\prod_{i\in(k,d]}(m_i-1)\prod_{j\in(d,n]} m_j} - 1\right)} \to 0.$$

4 Conclusions

We have determined explicitly the proportion of bounded non-additive sets of uniqueness with respect to those additive. The resulting ratio has been computed as a function of the dimensions of the confining grid. This allows us to prove that, in the limit case when the grid gets large, the above proportion tends to zero, meaning that the probability that a random selected set is additive increases. Further improvements could be explored by considering more general sets S of directions. In this case the tomographic grid, obtained as intersections of lines parallel to the X-ray directions corresponding to nonzero X-ray, is not necessarily contained in the confining rectangular grid $\mathcal{A}$, and the computation of the proportion of bounded non-additive sets of uniqueness with respect to those additive, seems to be a more challenging problem to be investigated.

References

1. Aharoni, R., Herman, G.T., Kuba, A.: Binary vectors partially determined by linear equation systems. Discr. Math. 171, 1–16 (1997)
2. Brunetti, S., Dulio, P., Peri, C.: Discrete Tomography determination of bounded lattice sets from four X-rays. Discrete Applied Mathematics 161(15), 2281–2292 (2013), doi:10.1016/j.dam.2012.09.010

3. Brunetti, S., Dulio, P., Peri, C.: On the Non-additive Sets of Uniqueness in a Finite Grid. In: Gonzalez-Diaz, R., Jimenez, M.-J., Medrano, B. (eds.) DGCI 2013. LNCS, vol. 7749, pp. 288–299. Springer, Heidelberg (2013)
4. Brunetti, S., Dulio, P., Peri, C.: Discrete Tomography determination of bounded sets in $\mathbb{Z}^n$. Discrete Applied Mathematics, doi:10.1016/j.dam.2014.01.016
5. Fishburn, P.C., Lagarias, J.C., Reeds, J.A., Shepp, L.A.: Sets uniquely determined by projections on axes II. Discrete case. Discrete Math. 91, 149–159 (1991), doi:10.1016/0012-365X(91)90106-C
6. Fishburn, P.C., Schwander, P., Shepp, L., Vanderbei, R.: The discrete Radon transform and its approximate inversion via linear programming. Discrete Applied Math. 75, 39–61 (1997), doi:10.1016/S0166-218X(96)00083-2
7. Fishburn, P.C., Shepp, L.A.: Sets of uniqueness and additivity in integer lattices. In: Herman, G.T., Kuba, A. (eds.) Discrete Tomography: Foundations, Algorithms and Application, pp. 35–58. Birkhäuser, Boston (1999)
8. Gardner, R.J., Gritzmann, P.: Discrete tomography: Determination of finite sets by X-rays. Trans. Amer. Math. Soc. 349, 2271–2295 (1997), doi:10.1090/S0002-9947-97-01741-8
9. Gardner, R.J., Gritzmann, P., Prangenberg, D.: On the computational complexity of reconstructing lattice sets from their X-rays. Discrete Math. 202, 45–71 (1999), doi:10.1016/S0012-365X(98)00347-1
10. Gritzmann, P., Langfeld, B., Wiegelmann, M.: Uniquness in Discrete Tomography: three remarks and a corollary. SIAM J. Discrete Math. 25, 1589–1599 (2011), doi:10.1137/100803262
11. Gritzmann, P., Prangenberg, D., de Vries, S., Wiegelmann, M.: Success and failure of certain reconstruction and uniqueness algorithms in discrete tomography. Intern. J. of Imaging System and Techn. 9, 101–109 (1998), doi:10.1002/(SICI)1098-1098(1998)9:2/3<101::AID-IMA6>3.0.CO;2-F
12. Hajdu, L., Tijdeman, R.: Algebraic aspects of discrete tomography. J. Reine Angew. Math. 534, 119–128 (2001), doi:10.1515/crll.2001.037
13. Hajdu, L., Tijdeman, R.: Algebraic Discrete Tomography. In: Herman, G.T., Kuba, A. (eds.) Advances in Discrete Tomography and Its Applications, pp. 55–81. Birkhäuser, Boston (2007), doi:10.1007/978-0-8176-4543-4_4
14. Weber, S., Schnoerr, C., Hornegger, J.: A linear programming relaxation for binary tomography with smoothness priors. Electr. Notes Discr. Math. 12, 243–254 (2003)
15. Weber, S., Schüle, T., Hornegger, J., Schnörr, C.: Binary tomography by iterating linear programs from noisy projections. In: Klette, R., Žunić, J. (eds.) IWCIA 2004. LNCS, vol. 3322, pp. 38–51. Springer, Heidelberg (2004)