Planar and Plane Slope Number of Partial 2-Trees

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Abstract. We prove tight bounds (up to a small multiplicative or additive constant) for the plane and the planar slope numbers of partial 2-trees of bounded degree. As a byproduct of our techniques, we answer a long standing question by Garg and Tamassia about the angular resolution of the planar straight-line drawings of series-parallel graphs of bounded degree.

1 Introduction

A *drawing* of a graph G in $\mathbb{R}^2$ maps each vertex of G to a point and each edge of G to a Jordan arc such that an edge does not contain a vertex other than its endpoints, no edge crosses itself, edges do not meet tangentially, and edges sharing a common end-vertex do not cross each other. A *planar graph* is a graph that admits a *planar drawing*, i.e. a drawing such that no two edges intersect except at their common end-points. A *plane graph* is a planar graph together with a combinatorial embedding, i.e. a prescribed set of faces including a prescribed outer face. A *plane drawing* of a plane graph G is a planar drawing that realizes the combinatorial embedding of G.

The *slope number* of a straight-line drawing Γ of a planar graph G is the number of distinct slopes of the edges of Γ. Every plane (planar) graph admits a plane (planar) straight-line drawing [1], i.e. a drawing where the edges are mapped to straight line segments. The *planar slope number* of G is the smallest slope number over all planar straight-line drawings of G. If G is a plane graph, the *plane slope number* of G is the smallest slope number over all plane straight-line drawings of G.

The problem of computing drawings of planar graphs with maximum degree four, using only horizontal and vertical slopes, has long been studied in graph drawing through the research on orthogonal and rectilinear graph drawing (see, e.g., [1]). In a seminal paper, Dujmović et al. [2] extend this study to non-orthogonal slopes, and give tight upper and lower bounds (expressed as functions of the number n of vertices) on the plane slope numbers of several graph families including plane 3-trees and plane 3-connected graphs. They also ask whether the plane slope number of a plane graph of maximum degree Δ can be bounded by a function $f(\Delta)$. Keszegh et al. [7] answer the question affirmatively proving that, for a suitable constant c, the plane slope number of a plane

graph of bounded degree Δ is at most $O(c^\Delta)$. In the same paper, Keszegh et al. establish a $3\Delta - 6$ lower bound for the plane slope number of the plane graphs of maximum degree at most Δ, which motivates additional research on reducing the gap between upper and lower bound. The question is studied by Jelínek et al. [5] who prove that the plane slope number of plane partial 3-trees is $O(\Delta^5)$. Also Kant, Dujmović et al., Mondal et al. independently show that the plane slope number of cubic 3-connected plane graphs is six [2,6,9], whereas the slope number (i.e., when the drawings may contain edge crossings) of cubic graphs is four [10].

In this paper we prove tight bounds (up to a small multiplicative or additive constant) for the plane and the planar slope numbers of planar 2-trees of bounded degree. Our results extend previous papers concerning the planar and plane slope numbers of proper subfamilies of the partial 2-trees. Namely, Jelínek et al. [5] prove that the planar slope number of series-parallel graphs with maximum degree three is at most three. Knauer et al. [8] show that the plane slope number of outerplane graphs with maximum degree $\Delta \geq 4$ is at most $\Delta - 1$ and that $\Delta - 1$ slopes are sometimes necessary. As a byproduct of our techniques, we answer a long standing open problem by Garg and Tamassia [3], who ask whether $\Omega(\frac{1}{\Delta^2})$ is a tight lower bound on the *angular resolution* of series-parallel graphs of degree Δ (i.e. they ask whether these graphs admit planar straight-line drawings where minimum angle between any two consecutive edges is $\Omega(\frac{1}{\Delta^2})$). More precisely, our results can be listed as follows.

- We prove that the planar slope number of a partial 2-tree of maximum degree Δ is at most 2Δ and there exist partial 2-trees whose planar slope number is at least Δ if Δ is odd and at least $\Delta + 1$ if Δ is even (Section 3).
- We prove that the plane slope number of a plane partial 2-tree of maximum degree Δ is at most 3Δ and there exist plane 2-trees whose plane slope number is at least $3\Delta - 3$ if Δ is even and at least $3\Delta - 4$ if Δ is odd (Section 4).
- We show that a partial 2-tree G of maximum degree Δ admits a planar straight-line drawing with angular resolution $\frac{\pi}{2\Delta}$. If G is a plane graph, a plane straight-line drawing of G exists whose angular resolution is $\frac{\pi}{3\Delta}$ (Section 5). The previously best known bound was $\frac{1}{48\pi\Delta^2}$, established by varying the input embedding [3].

2 Decomposition Trees and Universal Slope Sets

In this section we recall some known concepts. Throughout the paper "drawing" means "planar straight-line drawing"; "plane drawing" means "plane straight-line drawing".

SPQ Trees and Block-cut Vertex Trees. Let G be a 2-connected graph. A *separation pair* is a pair of vertices whose removal disconnects G. A *split pair* of G is either a separation pair or a pair of adjacent vertices. A *split component* of a split pair $\{u, v\}$ is either an edge (u, v) or a maximal subgraph G_{uv} of G such that $\{u, v\}$ is not a split pair of G_{uv}. We call vertices u and v the *poles* of G_{uv}. Note that a split component of G need no t be 2-connected.

A *2-connected series-parallel graph* is recursively defined as follows. A simple cycle with three edges is a 2-connected series-parallel graph. The graph obtained by

replacing an edge of a 2-connected series-parallel graph with a path is a 2-connected series-parallel graph. The graph obtained by adding an edge between the vertices of a non-adjacent separation pair $\{u, v\}$ of a 2-connected series-parallel graph is a 2-connected series-parallel graph. Let G be a 2-connected series-parallel graph. An SPQ-tree T of G is a rooted tree describing a recursive decomposition of G into its split components. The nodes of T are of three types: S, P, or Q. Each node μ has an associated graph called the *skeleton of* μ and denoted by $skeleton(\mu)$. Starting from a split pair $\{s, t\}$ of G, T recursively describes the split components of G as follows. The root of T is a P-node corresponding to G; its skeleton is defined as in the "Parallel case" below.

- *Base case:* The split component H is an edge. Then H corresponds to a Q-node of T whose skeleton is this edge. The Q-nodes are the leaves of T.
- *Series case:* The split component H is a 1-connected graph with split components $H_1, \ldots H_k$ $(k \geq 2)$ and cut vertices $c_i = H_i \cap H_{i+1}$. Then H corresponds to an S-node μ of T. The graph $skeleton(\mu)$ is a chain $e_1, \ldots, e_k$ of edges such that $e_i = (c_{i-1}, c_i)$, where $c_0 = s$ and $c_k = t$. The children of μ are the roots of the SPQ-trees of $H_1, \ldots, H_k$.
- *Parallel case:* Otherwise, the split component H is 2-connected and its split components are $H_1, \ldots, H_k (k \geq 2)$. Then H has $skeleton(\mu)$ consisting of a set of parallel edges $e_1, \ldots, e_k$ between s and t, one for each H_i. The children of μ are the roots of the SPQ-trees of $H_1, \ldots, H_k$.

Figure 1(a) and (b) show a 2-connected series-parallel graph and its SPQ-tree, which is uniquely determined by the choice of the initial split pair. Note that no P-node (S-node) has a P-node (S-node) as a child. Let T be an SPQ-tree of a 2-connected series-parallel graph G and let μ be a node of T. The *pertinent graph* of μ is the subgraph of G whose SPQ-tree is the subtree of T rooted at μ, as shown in Figure 1(c). The *virtual edge* of μ is an edge in the skeleton of the parent of μ that represents the pertinent graph of μ. Hence for every internal (i.e., non-Q) node μ in T, each edge in $skeleton(\mu)$ is a virtual edge of some child of μ.

If μ is P-node, then we associate with μ another graph $frame(\mu)$, called the *frame of* μ, which is formed by replacing each edge e in $skeleton(\mu)$ with the skeleton of the child node whose virtual edge is e, as shown in Figure 1(d). Every vertex in a frame corresponds to a unique vertex of G. Given a vertex v of G, the *first frame of* v is the frame that is closest to the root of T and contains v. For any split pair $\{u, v\}$ in a 2-connected series-parallel graph G with n vertices, an SPQ-tree having $\{u, v\}$ as reference pair can be computed in $O(n)$ time [4].

A graph G is a *partial 2-tree* (or, has tree-width at most 2) if and only if each 2-connected component of G is either series-parallel or consists of a single edge. Let G be a 1-connected graph. The *block-cut vertex tree of* G, denoted by BC-tree, is a graph with vertex set $B \cup C$ such that B consists of one vertex for each *block* (maximal 2-connected subgraph) of G and C consists of one vertex for each cut vertex of G. There is an edge from $b \in B$ to $c \in C$ in the BC-tree if and only if the vertex of G represented by c belongs to the block represented by b.

Universal Slope Sets and Free Wedges. Let G be a graph with vertex set V. For a vertex $v \in V$, we denote the degree of v by $\delta(v)$. Hence the *maximum degree of G is*
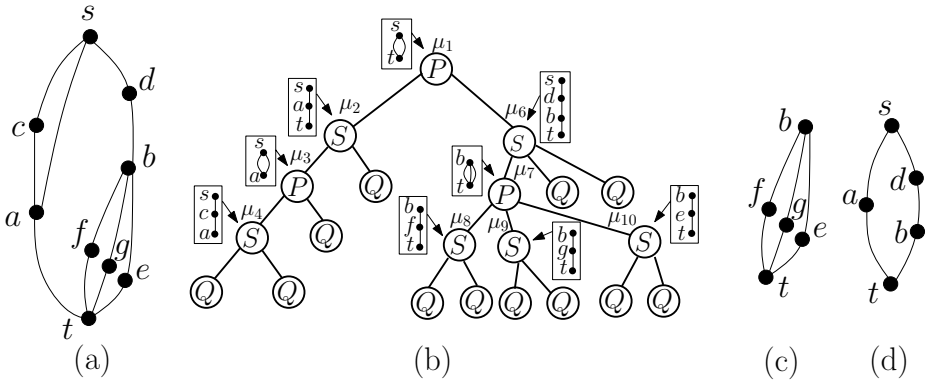
Fig. 1. (a) A 2-connected series-parallel graph G. (b) A SPQ-tree of G, where the internal nodes are labeled with $\mu_1, \ldots, \mu_{10}$. The skeleton of $\mu_i, 1 \leq i \leq 10$, is drawn in the box associated with μ_i. (c) The pertinent graph of μ_7. (d) The frame graph of μ_1.

$\Delta = \max_{v \in V} \delta(v)$. The *excess of v* is $\epsilon(v) = \Delta - \delta(v)$. If H is a subgraph of G, $\delta_H(v)$ and $\epsilon_H(v)$ are the restrictions of $\delta(v)$ and $\epsilon(v)$ to H.

A *set S of slopes is universal* for a family of planar graphs $\mathcal{G}$ if every graph $G \in \mathcal{G}$ admits a planar straight-line drawing such that the slope of every edge in the drawing belongs to S. We consider universal slope sets defined as follows.

Definition 1. *Given a positive integer $k \geq 2$, let $\alpha = \frac{\pi}{2k}$. Define S_k to be the set of slopes $i \cdot \alpha$, for $0 \leq i \leq 2k - 1$.*

We prove the upper bounds on plane and planar slope numbers showing that there is a value of k depending on Δ such that S_k is universal for the partial 2-trees. In our constructions, we guarantee that some wedge shaped regions of the plane can be used for recursive drawing. In particular, for any $r > 0$, point $p \in \mathbb{R}^2$, and angle ϕ, a ϕ-*wedge at p of radius r* is a sector of angular measure ϕ in the disk of radius r centered at p. For convenience, we will often omit reference to r, since any suitably small value of r suffices for our purposes. Let v be a vertex in some planar straight-line drawing Γ. A wedge with its apex at v in Γ is a *free wedge at v* if the wedge intersects the drawing Γ only at v. The angular measure of our free wedges will depend on the degree and excess of the corresponding vertices.

3 Slope Number of Partial 2-trees

In this section we present upper and lower bounds on the planar slope number of partial 2-trees of maximum degree Δ. We start by studying 2-connected series-parallel graphs (Section 3.1) and then we extend the study to all partial 2-trees (Section 3.2).

3.1 2-connected Series-parallel Graphs

In this section we show that S_Δ is a universal slope set for the family of 2-connected series-parallel graphs with maximum degree Δ.

Lemma 1. *Let $\mathcal{G}$ be the family of 2-connected series-parallel graphs having maximum degree at most Δ. Then S_Δ is universal for $\mathcal{G}$.*

Proof. The argument is based on a construction that recursively computes a drawing of G; the proof is by induction on the number of P-nodes in an SPQ-tree of G. Let T be an SPQ-tree of G having split pair $\{s, t\}$ associated with its root and let $m \in S_\Delta$. Since G is 2-connected, T must have at least one P-node, e.g., the root of T. We show that G admits a drawing Γ using only slopes from S_Δ that satisfies the following properties.

(1) Graph G is drawn within a triangle $\triangle abc$ having $\angle bac = (\delta(s) - 0.5)\alpha$ and $\angle abc = (\delta(t) - 0.5)\alpha$ (i.e. every edge is either in the interior or on the boundary of $\triangle abc$).
(2) Vertices s and t are located at a and b, respectively.
(3) Segment $\overline{ab}$ has slope m.
(4) The edges incident to s are drawn using consecutive slopes of S_Δ, as are the edges incident to t.
(5) At each vertex $v \notin \{s, t\}$ in the drawing of G, there is a free $\epsilon(v)\alpha$-wedge at v contained in $\triangle abc$.

Let a and b be two distinct points on a line of slope $m \in S_\Delta$, and let c be the point of intersection of the lines through a and b having slopes $m + (\delta(s) - 0.5)\alpha$ and $m - (\delta(t) - 0.5)\alpha$, respectively.

Base Case: Assume that T has a single P-node, which must be the root of T since G is 2-connected. The frame of the P-node consists of a set of paths of length at least 1 (and at most one path of length 1). We draw s and t on a and b, respectively. We draw one of these paths between s and t along the segment ab with slope m (if there is a path of length one, then we draw that path along ab; otherwise, any of the paths can be used). The remaining paths are drawn inside $\triangle abc$ using slopes $m + i\alpha, i = 1, \ldots, \delta(s) - 1$ at a and slopes $m - i\alpha, i = 1, \ldots, \delta(t) - 1$ at b for the edges incident to s and t, respectively; we use slope m for all other edges. See Figure 2 (a) for an example.

Let Γ be the computed drawing; by construction, Γ is crossing-free and it only uses slopes from S_Δ. Also, the paths from s to t lie within $\triangle abc$ and for each vertex $v \notin \{s, t\}$ there is an empty wedge of angle at least $\epsilon(v)\alpha$ with its apex at v that is completely contained within $\triangle abc$; this wedge is in the "c side" of the path containing v, i.e., in the half-plane defined by the line through a, b and containing c. Hence, Γ satisfies all invariant Properties (1)-(5).

Induction Step: Suppose now that any 2-connected series-parallel graph having at most j P-nodes in some SPQ-tree admits a drawing that only uses slopes from S_Δ and that satisfies Properties (1)-(5). Let G be a 2-connected series-parallel graph having $j + 1$ P-nodes in some SPQ-tree T. As above, the root of T is a P-node and its frame consists of a set of paths $\Pi_1, \ldots, \Pi_k$ of length at least 1. We will draw them in a fashion similar to the base case but with one important difference: we do not use consecutive slopes for the edges of the paths incident to s and t, but we leave room for the (recursive) drawings of the pertinent graphs associated with each virtual edge incident to s or t.

To do this, for each $i = 1, \ldots, k$, let e_i be the virtual edge incident to s in Π_i and let μ_{e_i} be the node of T corresponding to the virtual edge e_i (note that μ_{e_i} is either a P- or Q-node of T). Further, let $\delta_{e_i}(s)$ be the degree of s in the pertinent graph of μ_{e_i}. Then e_1 is drawn using slope m, and for each $i > 1$, e_i is drawn using
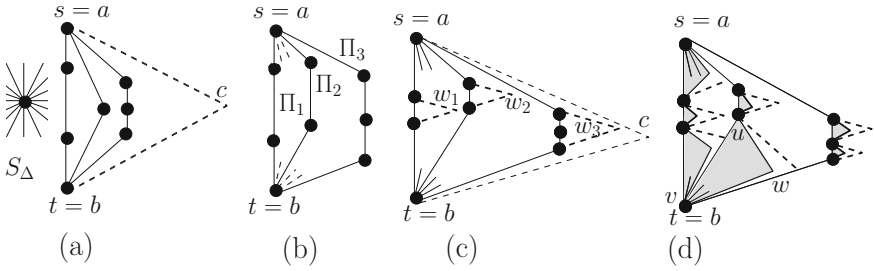
Fig. 2. (a) An example drawing for Base case, where $\delta(s) = \delta(t) = 3$ and $\Delta = 4$. (b) Illustration for Inductive step, where $\delta_{e_1}(s) = 3$, $\delta_{e_2}(s) = 1$, $\delta_{e_1}(t) = 2$, and $\delta_{e_2}(t) = 3$. (c) Illustration for $u_i v_i w_i$ in dashed line, and (d) recursive construction.

slope $slope(e_{i-1}) + \delta_{e_{i-1}}(s)\alpha$. The edges of Π_i incident to t are positioned similarly, beginning with slope m but decreasing the slopes as we move from one path to the next. See Figure 2 (b) for an example.

For every subpath $\Pi_i \setminus \{s, t\}$ with at least two vertices ($1 \leq i \leq k$), we draw the subpath using a sufficiently small line segment as follows. Let u_i, v_i be the endvertices of that subpath $\Pi_i \setminus \{s, t\}$. Let w_i be the point of intersection of the half-lines from u_i and v_i having slopes $slope(u_i v_i) + (\Delta - 0.5)\alpha$ and $slope(u_i v_i) - (\Delta - 0.5)\alpha$, respectively. We draw the paths such that $\triangle u_i v_i w_i$ lies in the region bounded by Π_i and Π_{i+1}. If $i + 1 < k$, then $u_i v_i w_i$ lies within $\triangle abc$. An example is illustrated in Figure 2(c) in dashed line.

Now that the frame of the root of T has been drawn, let $e = uv$ be a (drawn) virtual edge of a path Π_i of the frame and let μ_e be its corresponding P- or Q-node in T. Let w be the point of intersection of the lines from u and v having slopes $slope(e) + (\delta_e(u) - 0.5)\alpha$ and $slope(e) - (\delta_e(v) - 0.5)\alpha$, respectively. We recursively draw the pertinent graph of μ_e within $\triangle uvw$. If $\{u, v\} \cap \{s, t\} = \phi$, then $\triangle uvw$ is contained in $\triangle u_i v_i w_i$, which by construction does not intersect any part of the already drawn edges. If e is incident to s or t (i.e. either $u = s$ or $v = t$), then $\triangle uvw$ does not intersect the edge e' of Π_{i+1} incident to s or t, because by construction, the slope of e' is $slope(e) + \delta_e(s)\alpha$. Finally, observe that $\triangle uvw$ does not intersect any other triangle that contains the drawing of a pertinent graph associated with a node of T which is not in the subtree rooted at μ_e. See Figure 2(d) for an illustration.

The observations above, together with the fact that the drawing of the frame graph of the root is crossing-free and that it satisfies Properties (1)-(4), imply that G admits a drawing that only uses slopes from S_Δ satisfying Properties (1)-(4). To see that Property (5) is also satisfied, note that each path Π_i is drawn as a convex (or linear, for $i = 1$) chain. Thus at each vertex $v \notin \{s, t\}$ has an angle of at least π between its two consecutive (virtual) edges in its first frame. The drawing of G uses two consecutive sets of slopes at v, since the pertinent graph for the two nodes corresponding to each of those two virtual edges is drawn using consecutive slopes. This leaves a free wedge of angular measure $\pi - \delta(v)\alpha = 2\Delta\alpha - \delta(v)\alpha = \Delta\alpha + \epsilon(v)\alpha > \epsilon(v)\alpha$ at v, establishing Property (5). $\qquad\square$

3.2   Partial 2-trees

In this section we extend the result of Lemma 1 to partial 2-trees of maximum degree Δ by proving that S_Δ is universal for these graphs. We shall focus on connected partial 2-trees, since every connected components can be drawn independently of the others.

Lemma 2. *Let $\mathcal{G}$ be the family of partial 2-trees having maximum degree at most Δ. Then S_Δ is universal for $\mathcal{G}$.*

Proof. We assume that G is connected and has at least one edge. Let T be the block-cutvertex tree of G (see Figures 3(a)–(b)). We build the desired drawing of G by drawing subgraphs of G corresponding to subtrees of T, starting with the leaves of T. The drawing of G produced will have the following properties:

(a) For some split pair $\{s, t\}$ of G, G is drawn inside a $\delta(s)\alpha$-wedge with apex s.
(b) s and t are located on a line of slope $m \in S_\Delta$.
(c) The edges incident to s are drawn using consecutive slopes of S_Δ, as are the edges incident to t.
(d) The wedge of Property (a) is bounded by rays from s in directions $m - 0.5\alpha$ and $m + (\delta(s) - 0.5)\alpha$.

If G is 2-connected, then Lemma 1 establishes the existence of a drawing with the desired properties: Property (b) follows obviously from Property (1) of Lemma 1, Property (c) from Property (3) of Lemma 1, and properties (a) and (d) from properties $(1), (2)$ and (4) of Lemma 1.

Otherwise, let s be a cut vertex of G and choose s as the root of T. The parent of each block vertex B of T is a cut vertex s_B of G. For each such block B, choose a vertex t_B in B such that $\{s_B, t_B\}$ is a split pair for B. Each leaf of T is a block vertex B and by Lemma 1, it can be drawn with split pair $\{s_B, t_B\}$ inside a wedge with apex s_B and angular measure $\delta_B(s_B)\alpha$ so that properties (a)–(d) hold.

Assume now that x is a vertex of T for which all subgraphs of T have been drawn respecting properties (a)–(d). Then x represents either a block of G or a cut vertex of G. If x represents a cut vertex v of G, then let $T_1, \ldots, T_k$ represent the subtrees of x. Let G_v be the subgraph of G corresponding to the subtree of T with root v. For each of the trees T_i, v is at the apex of the $\delta_{T_i}(v)\alpha$-wedge in which the subgraph G_i of G corresponding to T_i has been drawn. These wedges can all be rotated about v so that they use consecutive slopes in S_Δ, as shown in Figure 3(c). Thus the subgraph of G corresponding to the union of $T_1, \ldots, T_k$ has been drawn in a wedge of angle $\delta_{G_1}(v)\alpha + \ldots + \delta_{G_k}(v)\alpha = \delta_{G_v}(v)\alpha$ with apex v such that properties (a)–(d) are satisfied. Note that by Property (a), the drawing of G_v can be made small enough to lie completely inside the free wedge and so does not intersect any other portion of the drawing.

Suppose now that x represents a block B of G. The children of B in T represent the cut vertices of G that belong to B; let v be one of the child cut vertices of B. Draw B in the manner described by Lemma 1. We consider two cases: $v \neq t_B$ and $v = t_B$ (note that s_B is the parent cut vertex of B, which is handled by the previous case).
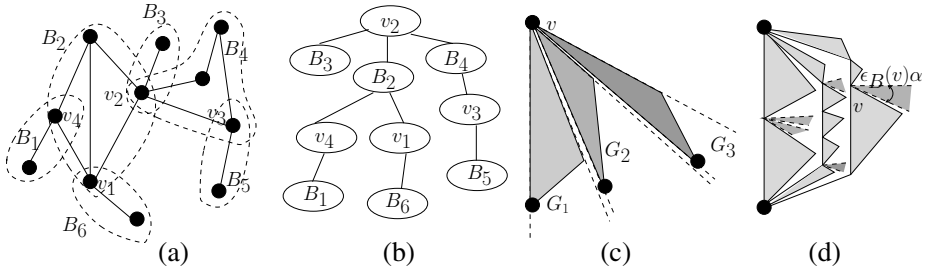
Fig. 3. (a) A 1-connected graph, and (b) corresponding block-cut vertex tree. (c)–(d) Illustration for the proof of Lemma 2.

Assume first that $v \neq t_B$. Then by Lemma 1, there is a free $\epsilon_B(v)\alpha$-wedge with apex v. Now $\epsilon_B(v)\alpha = (\Delta - \delta_B(v))\alpha > (\delta_G(v) - \delta_B(v))\alpha = \delta_{G_v}(v)\alpha$ and so G_v can be drawn completely inside the free $\epsilon_B(v)\alpha$-wedge with apex v as shown in Figure 3(d).

If $v = t_B$, then the drawing of G_v already produced can be rotated so that all of the edges in B and of G_v adjacent to t_B use consecutive (clockwise from the line containing s_B and t_B) directions in S_Δ as shown in Figure 3(d). It is an easy observation that the addition of the subgraphs G_v for each cut vertex of B into the free wedge preserves properties (a)–(d) of the drawing of B (and of the drawing of G). □

Lemma 3. *For any $\Delta > 3$, there exists a 2-connected series-parallel graph G of maximum degree Δ whose planar slope number is at least $\Delta + 1$ if Δ is even and at least Δ if Δ is odd.*

Proof. Consider the graph G obtained from $K_{2,\Delta-1}$ by adding the edge (u, v) connecting the two vertices u and v of degree $\Delta - 1$. Thus G has maximum degree Δ. Now consider any drawing of G. At least half of the remaining $\Delta - 1$ vertices are on one side of the line determined by the segment representing uv in the drawing. Each of these $\lfloor \Delta/2 \rfloor$ vertices forms a triangle with uv and these triangles are nested. Thus no two of the $2\lfloor \Delta/2 \rfloor + 1$ edges in this portion of the drawing of G have the same slope. □

The following theorem is an immediate consequence of Lemmas 2, and 3.

Theorem 1. *Let G be a partial 2-tree having maximum degree Δ and let $psl(G)$ denote the planar slope number of G. Then $psl(G) \leq 2\Delta$. Also, for every even $\Delta > 3$ there exists a partial 2-tree G such that $psl(G) \geq \Delta + 1$ and for every odd $\Delta \geq 3$ there exists a partial 2-tree G such that $psl(G) \geq \Delta$.*

4   Plane Slope Number of Partial 2-trees

In this section we show that the plane slope number of 2-connected series-parallel graphs, i.e., when the output drawings respect the input embeddings, is at least $3\Delta - 4$ and at most 3Δ. In fact, we show that $S_{1.5\Delta}$ is universal for the family of 2-connected series-parallel graphs with fixed embeddings.

We introduce some additional notation. For an embedded planar 2-connected series-parallel graph G with poles s and t, we call the edge (s,t) (if exists) the *central edge* of G. Since G is 2-connected, the root of its SPQ-tree is a P-node μ. Observe that each of the edges in $skeleton(\mu)$ corresponds to either the edge (s,t) or a *2-connected series-parallel subgraph of G*. If the edge (s,t) exists, then we categorize each of those subgraphs as a *left or right series-parallel subgraph* of G depending on whether it lies to the left or right of the edge (s,t), while walking from s to t. By $G_1^-, \ldots, G_l^-$, (respectively, $G_1^+, \ldots, G_r^+$) we denote the left (respectively, right) series-parallel subgraphs of G. If (s,t) does not exist, then we assume that all the series-parallel subgraphs are right series-parallel, i.e., $G_1^+, \ldots, G_r^+$. Furthermore, we assume that the subgraphs are ordered as follows: $G_l^-, \ldots, G_1^-, (s,t), G_1^+, \ldots, G_r^+$, reflecting their left to right ordering in the embedding.

Lemma 4. *Let $\mathcal{G}$ be the family of plane 2-connected series-parallel graphs of maximum degree at most Δ. Then $S_{1.5\Delta}$ is universal for $\mathcal{G}$.*

Sketch of Proof: Similar to Lemma 1 we employ an induction on the number of P-nodes in an SPQ-tree T of G. Since the proof follows a similar argument, for reasons of space we sketch here the main idea of the proof. Let $\{s,t\}$ be the split pair associated with the root of T, and let $\overline{ab}$ be a straight line segment with slope m, where $m \in S_{1.5\Delta}$. We show that G admits a drawing Γ using only slopes from $S_{1.5\Delta}$ such that the following properties hold.

(1) Graph G is drawn within a convex quadrilateral $\square adbc$ having $\angle dac = \delta(s)\alpha$ and $\angle dbc = \delta(t)\alpha$ (i.e. every edge is in the interior of $\square adbc$).
(2) Vertices s and t are located at a and b, respectively.
(3) Segment $\overline{ab}$ has slope m.
(4) The edges incident to s are drawn using consecutive slopes of $S_{1.5\Delta}$, as are the edges incident to t.
(5) At each vertex $v \notin \{s,t\}$ in the drawing of G, there are two free $\epsilon(v)\alpha$-wedges at v contained in $\square adbc$, one in the region between the subgraph G_i^+ (or G_i^-) containing v and the previous series-parallel subgraph of G in the ordering, and one in the region between G_i^+ (or G_i^-) and the next series-parallel subgraph of G in the ordering.

Base Case: Assume that T has a single P-node, which must be the root of T since G is 2-connected. We draw s and t on a and b, respectively. The frame of the P-node consists of a set of paths of length at least one (and at most one path of length one). If the central edge exists, then we draw that edge along $\overline{ab}$ (otherwise, we draw the leftmost path along $\overline{ab}$). We then draw the paths $\Pi_i^+, i = 1, \ldots, r$ corresponding to each G_i^+ between a and b using consecutive slopes at s and t, as in the proof of Lemma 1. All the remaining paths $\Pi_i^-, i = 1, \ldots, l$ corresponding to G_i^- are drawn symmetrically to the left of segment $\overline{ab}$, as shown in Figure 4(a). While drawing the paths, we maintain the input embedding. To construct the quadrilateral $\square adbc$, let d be the intersection of the line through a having slope 0.5α plus the slope of the edge of Π_r^+ incident to s with the line through b having slope -0.5α plus the slope of the edge of Π_r^+ incident to t. Similarly, let c be the intersection of the line through a having slope -0.5α plus the
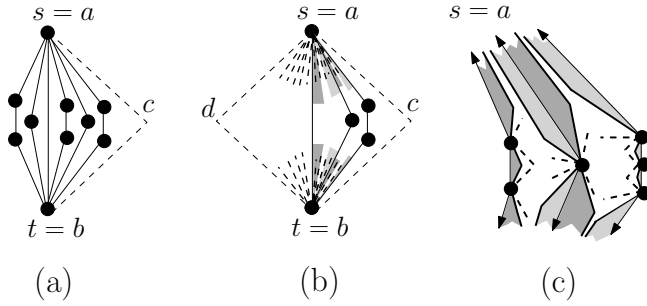
Fig. 4. (a) Base case. (b) Drawing frame, where the light-gray (respectively, dark-gray) regions correspond to the right (respectively, left) series-parallel subgraphs of the corresponding pertinent graph. (c) Recursive construction.

slope of the edge of Π_l^- incident to s with the line through b having slope 0.5α plus the slope of the edge of Π_l^- incident to t. This convex quadrilateral has angle $\delta(s)\alpha$ at a and $\delta(t)\alpha$ at b. See Figure 4 (a) for an example.

It is now straightforward to observe that the resulting drawing is planar and satisfies properties (1)-(4). As for Property (5), consider $v \notin \{s, t\}$ in one of the frame paths and its neighbors u and w on that path. Each of the two angles $\angle uvw$ is either non-acute or at least $\pi - (\delta(u) - 1)\alpha - (\delta(w) - 1)\alpha \geq (\Delta + 2)\alpha \geq \epsilon(v)\alpha$, and so the required empty wedges exist at v.

Induction Step: In a way similar to the proof of Lemma 1, we first draw the frame of the root of the SPQ-tree and then define disjoint convex quadrilaterals for each of the virtual edges. Finally, we recursively compute the drawings of the pertinent graphs inside the corresponding quadrilaterals. The idea is illustrated in Figures 4(b)-(c).   □

By Property (5) of Lemma 4, for every vertex $v \notin \{s, t\}$ in G, there is a free wedge on each side of the drawing of the pertinent subgraph G_i^+ (G_i^-) of G containing v. Thus the arguments used in the proof of Lemma 2 can be directly applied in the fixed embedding case to establish the following lemma.

Lemma 5. *Let $\mathcal{G}$ be the family of plane partial 2-trees of maximum degree at most Δ. Then $S_{1.5\Delta}$ is universal for $\mathcal{G}$.*

We observe that the $\Delta - 1$ lower bound proved by Knauer et al. [8] for the outerplane slope number implies a lower bound for plane slope number of plane partial 2-trees, because outerplane graphs are plane partial 2-trees. The next lemma shows a better lower bound for partial 2-trees.

Lemma 6. *For every $\Delta \geq 2$, there exists a plane 2-connected series-parallel graph G of maximum degree Δ whose plane slope number is at least $3\Delta - 3$ if Δ is even and at least $3\Delta - 4$ if Δ is odd.*

Proof. Suppose first that $\Delta \geq 2$ is an even number and consider a plane partial 2-tree G defined as follows. The external face of G is a 3-cycle with vertices a, b, c. In its interior there are $\Delta/2 - 1$ paths of length two connecting each pair from $\{a, b, c\}$.

The external face of every plane drawing Γ of G is a triangle $\triangle abc$ that contains the paths of length two in its interior. From elementary geometry, no two edges of Γ can have common slope, and hence the graph, which has $3\Delta - 3$ edges, has plane slope number at least $3\Delta - 3$. Suppose now that Δ is odd and let $\Delta' = \Delta + 1$. Construct a graph G' of maximum degree Δ' as described above. Now remove one of those paths of length 2 connecting a and b and one connecting c and b from G'. This new graph, G, has maximum degree $\Delta = \Delta' - 1$ and it requires a different slope for each edge. Since we deleted four edges from G', G has $(3\Delta' - 3) - 4 = 3\Delta - 4$ edges. Thus $3\Delta - 4$ slopes are required, and the result is established. $\square$

Lemmas 4 and 6 imply the following.

Theorem 2. *Let G be a plane partial 2-tree having maximum degree Δ and let $epsl(G)$ denote the plane slope number of G. Then $epsl(G) \leq 3\Delta$. Also, for every even $\Delta > 3$ there exists a plane partial 2-tree G such that $epsl(G) \geq 3\Delta - 3$ and for every odd $\Delta \geq 3$ there exists a plane partial 2-trees G such that $epsl(G) \geq 3\Delta - 4$.*

5   Angular Resolution

The *angular resolution* of a planar straight-line drawing is the minimum angle between any two edges incident to a common vertex. The angular resolution of a planar graph G is the maximum angular resolution over all possible drawings of G.

Malitz and Papakostas [11] show that the angular resolution of a planar graph of maximum degree Δ is $\Omega(\frac{1}{7^\Delta})$. Garg and Tamassia [3] show that there exist planar 3-trees of maximum degree Δ that require angular resolution $O(\sqrt{\frac{\log \Delta}{\Delta^3}})$. They also show that for a subfamily of the partial 2-trees of bounded degree, namely the series-parallel graphs of maximum degree at most Δ, the angular resolution is at least $\frac{1}{48\pi\Delta^2}$. Their drawing technique does not apply to plane graphs since it may vary a given combinatorial embedding. Garg and Tamassia leave as open the problem about whether $\Omega(\frac{1}{\Delta^2})$ is a tight lower bound for the angular resolution of the series-parallel graphs.

An implication of the drawing techniques of the previous sections of this paper is that the angular resolution of partial 2-trees (and thus also of series-parallel graphs) is in fact $\Omega(\frac{1}{\Delta})$. Namely, the constructions of Lemmas 2 and 4 either use the universal set S_Δ or the universal set $S_{1.5\Delta}$ which consist of equally spaced slopes; therefore, the minimum angle between any two edges sharing a common end-vertex is either $\frac{\pi}{2\Delta}$ in the variable embedding setting or it is $\frac{\pi}{3\Delta}$ in the fixed embedding setting. Therefore:

Theorem 3. *A partial 2-tree of maximum degree Δ admits a planar straight-line drawing with angular resolution $\frac{\pi}{2\Delta}$. A plane partial 2-tree of maximum degree Δ admits a planar straight-line drawing with angular resolution $\frac{\pi}{2\Delta}$.*

6   Open Problems

An interesting research direction is to study the trade-off between the slope number and the area requirement of planar graphs. Similar studies have been carried out for the

angular resolution and the area requirements of planar graphs having maximum degree at most Δ (see, e.g., [3]).

Another fascinating open problem is to close the gap between upper and lower bounds on the planar/plane slope number of planar/plane graphs of bounded degree (see [7]). This would be interesting even restricted to partial 3-trees (see [5]).

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