

A Duality Transform for Constructing Small Grid Embeddings of 3D Polytopes^{*}

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Abstract. We study the problem of how to obtain an integer realization of a 3d polytope when an integer realization of its dual polytope is given. We focus on grid embeddings with small coordinates and develop novel techniques based on Colin de Verdière matrices and the Maxwell–Cremona lifting method.

As our main result we show that every truncated 3d polytope with n vertices can be realized on a grid of size polynomial in n . Moreover, for a class \mathcal{C} of simplicial 3d polytopes with bounded vertex degree, at least one vertex of degree 3, and polynomial size grid embedding, the dual polytopes of \mathcal{C} can be realized on a polynomial size grid as well.

1 Introduction

By Steinitz’s theorem the graphs of convex 3d polytopes¹ are exactly the planar 3-connected graphs [16]. Several methods are known for realizing a planar 3-connected graph G as a polytope with graph G on the grid [4,7,11,12,13,15]. It is challenging to find algorithms that produce polytopes with small integer coordinates. Having a realization with small grid size is a desirable feature, since then the polytope can be stored and processed efficiently. Moreover, grid embeddings imply good vertex and edge resolution. Hence, they produce “readable” drawings.

In 2d, it is well known that planar 3-connected graphs with n vertices can be drawn on a $O(n) \times O(n)$ grid without crossings [5], and a drawing with convex faces can be realized on a $O(n^{3/2} \times n^{3/2})$ grid [2]. For the realization as a polytope the best algorithm guarantees an integer embedding with coordinates at most $O(147.7^n)$ [3,11]. The current best lower bound is $\Omega(n^{3/2})$ [1]. Closing this large gap is probably one of the most interesting open problems in lower dimensional polytope theory. Recently, progress has been made for a special class of 3d polytopes, the so-called *stacked polytopes*. A *stacking operation* replaces a triangular face of a polytope with a tetrahedron, while maintaining the convexity of the embedding. A polytope that can be constructed from a tetrahedron and a

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¹ In our terminology polytopes are always considered *convex*.

sequence of stacking operation is called stacked polytope. The graphs of stacked polytopes are planar 3-trees. Stacked polytopes can be embedded on a grid that is polynomial in n [6]. This is, however, the only nontrivial polytope class for which such an algorithm is known.

In this paper we introduce a duality transform that maintains a polynomial grid size. In other words, we provide a technique that takes a grid embedding of a polytope with graph G and generates a grid embedding of a polytope whose skeleton is G^* , the dual graph of G . We call a 3d polytope with graph G^* a dual polytope. If the original polytope has integer coordinates bounded by a polynomial in n , then the dual polytope obtained with our techniques has also integer coordinates bounded by a (different) polynomial in n . Our methods can only be applied to special polytopes. Namely, we require that the graph of the polytope is a triangulation (the polytope is simplicial), that it contains a K_4 , and that the maximum vertex degree is bounded.

For the class of stacked polytopes (although their maximum vertex degree is not bounded) we can also apply our approach to show that all graphs dual to planar 3-trees can be embedded as polytopes on a polynomial size grid. These polytopes are known as truncated polytopes. Truncated polytopes are simple polytope, that can be generated from a tetrahedron and a sequence of *truncations*. A truncation is the dual operation to stacking. This means that a degree-3 vertex of the polytope is cut off by adding a new bounding hyperplane that separates this vertex from the remaining vertices of the polytope. We show that all truncated polytopes can be realized with integer coordinates in $O(n^{44})$. The approach for this class is more direct, since stronger results for realizations of stacked polytopes on the grid are known [6].

Duality. There exist several natural approaches how to construct for a given polytope a dual. The most prominent construction is polarity. Let P be some polytope that contains the origin. Then $P^* = \{y \in \mathbb{R}^d: x^T y \leq 1 \text{ for all } x \in P\}$ is a realization of a polytope dual to P , called its *polar*. The vertices of P^* are intersection points of planes with integral normal vectors, and hence not necessarily integer points. In order to scale to integrality one has to multiply P^* with the product of all denominators of its vertex coordinates, which may cause an exponential increase of the grid size.

A second approach uses the classic Maxwell–Cremona correspondence technique (also known as lifting approach) [10], which is applied in many embedding algorithms for 3d polytope realization. The idea here is to first draw the graph of the polytope as a convex 2d embedding with an additional equilibrium condition. The equilibrium condition guarantees that the 2d drawing is a projection of a convex 3d polytope, furthermore the polytope can be reconstructed from its projection in a canonical way (called lifting) in linear time. There is a classical transformation that constructs for a 2d drawing in equilibrium a 2d drawing of its dual graph, also in equilibrium. This drawing is called the *reciprocal diagram*. The induced lifting realizes the dual polytope, but it does not provide small integer coordinates for two reasons. First, the weights that define the equilibrium of the reciprocal diagram are the reciprocals of the weights in the original graph.

Second, the lifting realizes the dual polytope in projective space with one point “over the horizon”. The second property can be “fixed” with a projective transformation. This, however, makes a large scaling factor for an integer embedding unavoidable. Also the reciprocal weights are difficult to handle without scaling by a large factor.

Structure and notation. As a novelty we work with Colin de Verdière matrices to construct *small* grid embeddings. In order to make these techniques (as introduced by Lovász) applicable we extend this framework slightly; see Sect. 2. In Sect. 3 we then present the main idea, combining the classical lifting approach with the methods of Sect. 2, which finds applications in the following sections, where the results on truncated polytopes and triangulations are presented.

Throughout the paper we denote by G the graph of the original polytope, and by G^* its dual graph. For any graph H we write $V(H)$ for its vertex set, $E(H)$ for its edge set and $N(v, H)$ for the set of neighbors of a vertex v in H . Since we consider 3-connected planar graphs, the facial structure of the graph is predetermined up to a global reflection [17]. The set of faces is therefore predetermined, and we name it $F(H)$. For convenience we denote an edge (v_i, v_j) as (i, j) . A face spanned by vertices $v_i, v_j,$ and v_k is denoted as $(v_i v_j v_k)$. A graph obtained from H by stacking a vertex v_1 on a face $(v_2 v_3 v_4)$, is denoted as $\text{Stack}(H; v_1; v_2 v_3 v_4)$. For convenience we use $|p|$ for the Euclidean norm of the vector p . We denote the maximum vertex degree of a graph G as Δ_G . Finally, we write $G[X]$ for the induced subgraph of a vertex set $X \subseteq V(G)$.

2 3d Representations with CDV Matrices

In this section we review some of the methods Lovász introduced in his paper on Steinitz representations [9]. In our constructions throughout the paper every face of any graph is realized such that all its vertices lie on a common plane. From this perspective drawings of graphs in \mathbb{R}^3 and the realizations of their corresponding polyhedra are the same objects.

Definition 1. We call a straight-line embedding $(u_1, \dots, u_n) \in (\mathbb{R}^3)^n$ of a planar 3-connected graph G in \mathbb{R}^3 a cone-convex embedding iff the cones over its faces, $C_f = \{\lambda x \mid x \in f, \lambda > 0\}$, $f \in F(G)$ are convex and have disjoint interiors.

In other words, an embedding is a cone-convex embedding if its projection to the sphere $S = \{|x| = 1\}$ is a convex drawing of G with edges drawn as geodesic arcs. We remark that the vertices of a cone-convex embedding are not supposed to form a convex polytope.

Definition 2. Let (u_1, \dots, u_n) be an embedding of a graph G into \mathbb{R}^d . We call a symmetric matrix $M = [M_{ij}]_{1 \leq i, j \leq n}$ a CDV matrix of the embedding if

1. $M_{ij} = 0$ for $i \neq j, (i, j) \notin E(G)$, and
2. $\sum_{1 \leq j \leq n} M_{ij} u_j = 0$ for $1 \leq i \leq n$.

We call a CDV Matrix positive if $M_{ij} > 0$ for all $(i, j) \in E(G)$.

We call the second condition in the above definition the *CDV equilibrium condition*.

The CDV equilibrium condition can also be expressed in a slightly different, more geometric form as

$$\sum_{j \in N(i, G)} M_{ij} u_j = -M_{ii} u_i \quad \text{for } 1 \leq i \leq n. \tag{1}$$

Hence, a positive CDV Matrix witnesses that every vertex of the embedding can be written as a convex combination of its neighbors using symmetric weights. The following lemma appears in [9], we include the proof since it illustrates how to construct a realization out of a CDV matrix.

Lemma 1 (Lemma 4, [9]). *Let (u_1, \dots, u_n) be a cone-convex embedding of a graph G with a positive CDV matrix $[M_{ij}]$. Then every face f in G can be assigned a vector ϕ_f s.t. for each adjacent face g and separating edge (i, j)*

$$\phi_f - \phi_g = M_{ij}(u_i \times u_j), \tag{2}$$

where f lies to the left and g lies to the right from $\overrightarrow{u_i u_j}$. The set of vectors $\{\phi_f\}$ is uniquely defined up to translations.

Proof. To construct the family of vectors $\{\phi_f\}$, we start by assigning an arbitrary value to ϕ_{f_0} (for an arbitrary face f_0); then we proceed iteratively. To prove the consistency of the construction, we show that the vectors $(\phi_f - \phi_g)$ sum to zero over every cycle in G^* . Since G as well as G^* is planar and 3-connected, it suffices to check this condition for all elementary cycles of G^* , which are the faces of G^* . Let $\tau(i)$ denote the set of counterclockwise oriented edges of the face in G^* dual to $v_i \in V(G)$. Then, combining 1 and 2 yields

$$\sum_{(f, g) \in \tau(i)} (\phi_f - \phi_g) = \sum_{j \in N(i, G)} M_{ij}(u_i \times u_j) = u_i \times \left(\sum_{j \in N(i, G)} M_{ij} u_j \right) = u_i \times (-M_{ii} u_i) = 0.$$

The vectors $\{\phi_f\}$ are unique up to the initial choice of ϕ_{f_0} . □

Note that there is a canonical way to derive a CDV matrix from a 3d polytope [9]. Every 3d embedding of a graph G as a polytope (u_i) possesses a positive CDV matrix defined by the vertices (ϕ_i) of its polar and equation (2). We refer to this matrix as the *canonical CDV matrix*.

The following theorem, which is a variation of Lemma 5 in [9], is the main tool in our construction.

Theorem 1 (based on Lovász [9]). *Let (u_1, \dots, u_n) be a cone-convex embedding of a graph G and M a positive CDV matrix for this embedding. Then for any set of vectors $\{\phi_f\}_{f \in F(G)}$ fulfilling (2), the convex hull $\text{Conv}(\{\phi_f\}_{f \in F(G)})$ is a convex polytope with graph G^* ; and the isomorphism between G^* and the skeleton of $\text{Conv}(\{\phi_f\}_{f \in F(G)})$ is given by $f \rightarrow \phi_f$.*

The proof of the theorem is included in the full version of the paper. It relies on a projection of the cone-convex embedding onto the sphere and an appropriate “scaling” of the CDV matrix.

3 Construction of Cone-Convex Embeddings

In this section we describe how to go from a convex 2d embedding with a positive equilibrium stress to a cone-convex 3d embedding with a positive CDV matrix.

Definition 3. We call a set of reals $\{\omega_{ij}\}_{(i,j) \in E(G)}$ an equilibrium stress for an embedding (u_1, \dots, u_n) of a graph G into \mathbb{R}^d if for each $i \in V(G)$

$$\sum_{j \in N(i,G)} \omega_{ij}(u_j - u_i) = 0.$$

We call an equilibrium stress of a 2d embedding with a distinguished boundary face f_0 positive if it is positive on every edge that does not belong to f_0 .

The concept of equilibrium stresses plays a central role in the classical Maxwell–Cremona lifting approach and it is also a crucial concept in our embedding algorithm. The equilibrium stress on a realization of a complete graph arises as a “building block” in later constructions. The complete graph K_n , embedded in \mathbb{R}^{n-2} , has a unique equilibrium stress up to multiplication with a scalar. This stress has an easy expression in terms of volumes related to the embedding. We use the *square bracket notation*²

$$[q_i q_j q_k q_l] := \det \begin{pmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

to obtain a formulation for the equilibrium stress on the K_5 embedding.

Lemma 2 (Rote, Santos, and Streinu [14]). Let (u_0, u_1, \dots, u_4) be an integer embedding of the complete graph K_5 onto \mathbb{R}^3 . Then the set of real numbers:

$$\omega_{ij} := [u_{i-2} u_{i-1} u_{i+1} u_{i+2}] [u_{j-2} u_{j-1} u_{j+1} u_{j+2}]$$

(indices in cyclic notation) defines an integer equilibrium stress on this embedding.

Theorem 2. Let (p_2, \dots, p_n) be a convex 2d drawing of a planar 3-connected graph G_\uparrow with positive equilibrium stress $\{\omega_{ij}\}$ and designated triangular face $f_0 = (p_2 p_3 p_4)$ embedded as the boundary face. Then we can define a cone convex embedding (q_i) of the graph $G = \text{Stack}(G_\uparrow; v_1; v_2 v_3 v_4)$ into \mathbb{R}^3 equipped with a positive CDV matrix $[M_{ij}]$, such that

$$M_{ij} = \omega_{ij} \quad \text{for each internal edge } (i, j) \text{ of the 2d drawing of } G_\uparrow$$

and each entry of M is bounded by $O(n \max_{ij} |\omega_{ij}| \cdot \max_i |p_i|^6)$.

² For 2d vectors $[p_i p_j p_k]$ is defined similarly.

Proof. We can assume that $(0, 0)^T$ lies inside the embedding of f_0 . Let (q_1, \dots, q_n) be the embedding of the graph G , defined as follows: The embedding of G_\uparrow is realized in the plane $\{z = 1\}$ and the stacked vertex is placed at $(0, 0, -1)^T$. The embedding is cone-convex since it describes a tetrahedron containing the origin with one face that is refined with a plane convex subdivision.

Following the structure of $G = \text{Stack}(G_\uparrow; v_1; v_2v_3v_4)$, we decompose G into two subgraphs: $G_\uparrow = G[\{v_2, \dots, v_n\}]$ and $G_\downarrow := G[\{v_1, v_2, v_3, v_4\}]$.

We first compute a CDV matrix $[M'_{ij}]_{2 \leq i, j \leq n}$ for the embedding (q_2, \dots, q_n) of G_\uparrow . The plane embedding (p_i) of G_\uparrow has the equilibrium stress $\{\omega_{ij}\}_{2 \leq i, j \leq n}$. Since $\{q_2, \dots, q_n\}$ is just a translation of $\{p_2, \dots, p_n\}$, clearly, $\{\omega_{ij}\}_{2 \leq i, j \leq n}$ is as well an equilibrium stress for the embedding (q_2, \dots, q_n) and we can assign:

$$M'_{ij} := \begin{cases} -\sum_{k \in N(i, G_\uparrow)} \omega_{ik} & i = j, \\ \omega_{ij} & (i, j) \in E(G_\uparrow), \\ 0 & \text{else.} \end{cases}$$

Now we check the CDV equilibrium condition: for every $2 \leq i \leq n$

$$\begin{aligned} \sum_{2 \leq j \leq n} M'_{ij} q_j &= \sum_{j \in N(i, G_\uparrow)} M'_{ij} q_j + M'_{ii} q_i = \sum_{j \in N(i, G_\uparrow)} M'_{ij} (q_j - q_i) + (M'_{ii} + \sum_{j \in N(i, G_\uparrow)} M'_{ij}) q_i \\ &= \sum_{j \in N(i, G_\uparrow)} \omega_{ij} (q_j - q_i) + (M'_{ii} + \sum_{j \in N(i, G_\uparrow)} M'_{ij}) q_i = 0. \end{aligned}$$

The last transition holds since both summands equal 0. Hence, $[M'_{ij}]$ is a valid CDV matrix for the embedding $(q_i)_{2 \leq i \leq n}$ of G_\uparrow .

As a second step we compute a CDV matrix $[M''_{ij}]_{1 \leq i, j \leq 4}$ for the embedding of the tetrahedron G_\downarrow . We apply Lemma 2 for the embedding of the K_5 formed by $\{q_0 = (0, 0, 0)^T, q_1, q_2, q_3, q_4\}$ and receive an equilibrium stress $\{\omega''_{ij}\}_{0 \leq i, j \leq 4}$. We can now derive a CDV matrix $[M''_{ij}]_{1 \leq i, j \leq 4}$ for the tetrahedron $\{q_1, q_2, q_3, q_4\}$ based on the equilibrium stresses $\{\omega''_{ij}\}_{0 \leq i, j \leq 4}$ as follows: We set

$$M''_{ij} := \begin{cases} -\sum_{0 \leq j \leq 4, j \neq i} \omega''_{ij}, & i = j, \\ \omega''_{ij}, & \text{otherwise,} \end{cases}$$

and see that the CDV equilibrium condition holds, by noting

$$\begin{aligned} \forall i \quad \sum_{1 \leq j \leq 4} M''_{ij} q_j &= \sum_{1 \leq j \leq 4, j \neq i} M''_{ij} q_j + M''_{ii} q_i = \sum_{1 \leq j \leq 4, j \neq i} \omega''_{ij} q_j + M''_{ii} q_i \\ &= \sum_{0 \leq j \leq 4, j \neq i} \omega''_{ij} (q_j - q_i) - \omega''_{i0} q_0 + \left(\sum_{0 \leq j \leq 4, j \neq i} \omega''_{ij} + M''_{ii} \right) q_i = 0. \end{aligned}$$

The last transition holds since $\sum_{0 \leq j \leq 4, j \neq i} \omega''_{ij} (q_j - q_i) = 0$ by the definition of $\{\omega_{ij}\}$, $q_0 = 0$, and $\sum_{0 \leq j \leq 4, j \neq i} \omega''_{ij} + M''_{ii} = 0$ due to the choice of M''_{ii} . One can easily check that as soon as the origin lies inside the tetrahedron $\{q_1, q_2, q_3, q_4\}$ all entries M''_{ij} have the same sign. We can assume that $[M''_{ij}]$ is positive, otherwise we reorder the vertices $\{v_2, v_3, v_4\}$.

In the final step we extend the two CDV matrices M' and M'' to G and combine them. Clearly, a CDV matrix padded with zeros remains a CDV matrix. Furthermore, any linear combination of CDV matrices is again a CDV matrix. Thus, we form a CDV matrix for the whole embedding (q_1, \dots, q_n) of G by setting:

$$M := M' + \lambda M'',$$

where λ is a positive integer chosen such that M is a positive CDV matrix. This can be done as follows.

Recall that $\{\omega_{ij}\}$ is a positive stress and $[M''_{ij}]$ is a positive CDV matrix. Hence, the only six entries in $[M_{ij}]$ that may be negative are: M_{23} , M_{34} and M_{42} (and their symmetric entries), for which $M_{ij} := M'_{ij} + \lambda M''_{ij}$ with $M'_{ij} < 0$ and $M''_{ij} > 0$. Thus, we choose λ such that M is positive at these entries. To satisfy this condition we pick

$$\lambda = \left\lceil \max_{(i,j) \in \{(2,3), (3,4), (4,2)\}} (|M'_{ij}| / |M''_{ij}|) \right\rceil + 1.$$

To bound M_{ij} we notice that the entries of M''_{ij} are strictly positive integers, so $\lambda = O(\max |M'_{ij}|)$, while $|M'_{ij}| = O(n \cdot \max |\omega_{ij}|)$ and $|M''_{ij}| = O(\max |\omega''_{ij}|) = O(\max |p_i|^6)$. The bound $|M_{ij}| = O(n \cdot \max_{ij} |\omega_{ij}| \cdot \max_i |p_i|^6)$ follows. \square

4 Realizations of Truncated Polytopes

In this section we sum up previous results in Theorem 3 and present an embedding algorithm for truncated 3d polytopes in Theorem 4. We will apply Theorem 3 also in the more general setup of Sect. 5.

Theorem 3. *Let $G = \text{Stack}(G_\uparrow; v_1; v_2 v_3 v_4)$ and (p_2, \dots, p_n) be an integer planar embedding of G_\uparrow with boundary face $(v_2 v_3 v_4)$ and with positive integer equilibrium stress $\{\omega_{ij}\}$. Then one can construct a grid embedding (ϕ_f) of a convex polytope with graph G^* such that*

$$|\phi_f| = O(n^2 \cdot \max |\omega_{ij}| \cdot \max |p_i|^8).$$

Proof. We first apply Theorem 2 to obtain a cone-convex embedding (q_1, \dots, q_n) of G with a positive CDV matrix $[M_{ij}]_{1 \leq i, j \leq n}$. We then apply Lemma 1 and obtain a family of vectors $\{\phi_f\}_{f \in F(G^*)}$ fulfilling

$$\phi_f - \phi_g = M_{ij}(q_i \times q_j), \quad \forall (f, g) \text{ dual to } (i, j) \text{ - edges of } G^* \text{ and } G.$$

Due to Theorem 1 the vectors $\{\phi_f\}$ form a realization of G^* as a polytope.

To finish the proof we estimate how large the coordinates of the embedding (ϕ_f) are. To do so, let us again follow the construction of (ϕ_f) as outlined in the proof of Lemma 1. We pick one face as $f_0 \in F(G)$, and assign $\phi_{f_0} = (0, 0, 0)^T$. Let us now evaluate ϕ_{f_k} for some face $f_k \in F(G)$. The following algebraic expression holds for all values $\{\phi_{f_i}\}$:

$$\phi_{f_k} = \phi_{f_0} + (\phi_{f_1} - \phi_{f_0}) + \dots + (\phi_{f_{k-1}} - \phi_{f_{k-2}}) + (\phi_{f_k} - \phi_{f_{k-1}}).$$

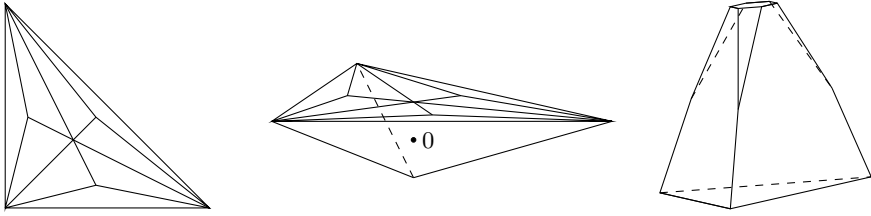


Fig. 1. A 2d embedding of G_\uparrow (left), the cone-convex embedding of G (center), and the resulting embedding of the dual (right).

Let us now consider the shortest path f_0, f_1, \dots, f_k in G^* connecting the faces f_0 and f_k . Clearly, k is less than $2n - 3$, and hence

$$\begin{aligned}
 |\phi_{f_k}| &\leq 2n \cdot \max_{(f_a, f_b) \in E(G^*)} |\phi_{f_a} - \phi_{f_b}| = 2n \cdot \max_{v_i, v_j \in V(G)} |M_{ij}(q_i \times q_j)| \\
 &= O(n \cdot (n \cdot \max |\omega_{ij}| \cdot \max |p_i|^6) \cdot \max |q_i|^2) = O(n^2 \cdot \max |\omega_{ij}| \cdot \max |p_i|^8).
 \end{aligned}$$

The bound for the entries of M is due to Theorem 2. □

Next we apply Theorem 3 to construct an integer polynomial size grid embedding for truncated polytopes. To construct small integer 2d embeddings with small integer equilibrium stresses we use a Lemma by Demaine and Schulz [6], which states that the graph of a stacked polytope with n vertices and any distinguished face f_0 can be embedded on a $10n^4 \times 10n^4$ grid with boundary face f_0 and with integral positive equilibrium stress $\{\omega_{ij}\}$ such that, for every edge (i, j) , we have $|\omega_{ij}| = O(n^{10})$.

Theorem 4. *Any truncated 3d polytope with n vertices can be realized with integer coordinates of size $O(n^{44})$.*

Proof. Let G^* be the graph of the truncated polytope and $G := (G^*)^*$ its dual. Clearly, G is the graph of a stacked polytope with $(n + 4)/2$ vertices. We denote the last stacking operation (for some sequence of stacking operations producing G) as the stacking of the vertex v_1 onto the face $(v_2v_3v_4)$ of the graph $G_\uparrow := G[V \setminus \{v_1\}]$. The graph G_\uparrow is again a stacked graph, and hence, by the Lemma of Demaine and Schulz, there exists an embedding $(p_i)_{2 \leq i \leq n}$ of G_\uparrow into \mathbb{Z}^2 with an equilibrium stress $\{\omega_{ij}\}$ satisfying the properties of Theorem 3. We apply the theorem and obtain a polytope embedding (ϕ_f) of G^* with bound $|\phi_f| = O(n^2 \cdot \max |\omega_{ij}| \cdot \max |p_i|^8) = O(n^{44})$. □

Figure 1 shows an example of our method. The computations for this example are included in the full paper.

5 A Dual Transform for Simplicial Polytopes

As we have seen a small grid embedding of a 3d polytope can be computed when a small integer (though, not necessarily convex) embedding of its dual polytope

with a small integral positive CDV matrix is known. However, if one wants to build a dual for an already embedded polytope, one usually does not possess such a matrix. The canonical CDV matrix associated with any embedding of a 3d polytope is not helpful, since its entries, when scaled to integers, might become exponentially large. We show in this section how one can tackle this problem for a special class of polytopes. In particular, we require that the original polytope is simplicial, it contains a vertex of degree 3, and its maximum vertex degree is bounded.

Before proceeding, let us review how the canonical stress associated with an orthogonal projection of a 3d polytope in the $\{z = 0\}$ plane can be described. The assignment of heights to the interior vertices of a 2d embedding resulting in a polyhedral surface is called a (polyhedral) *lifting*. By the Maxwell-Cremona correspondence the equilibrium stresses of a 2d embedding of a planar 3-connected graph and its liftings are in 1-1 correspondence. Moreover, the bijection between liftings and stresses can be defined as follows. Let (p_i) be a 2d drawing of a triangulation and let (q_i) be the 3d embedding induced by some lifting. We map this lifting to the equilibrium stress $\{\omega_{ij}\}$ by assigning to every edge (i, j) separating the faces $(v_i v_j v_k)$ (on the left) and $(v_i v_j v_l)$ (on the right)

$$\omega_{ij} := \frac{[q_i q_j q_k q_l]}{[p_i p_j p_k][p_l p_j p_i]}. \tag{3}$$

This mapping is a bijection between the space of liftings and the space of equilibrium stresses. The expression (3) is a slight reformulation of the form presented in Hopcroft and Kahn [8, Equation 11].

We continue by studying the spaces of equilibrium stresses for triangulations. A graph formed by a cycle v_1, \dots, v_n with an additional vertex v_0 , called *center*, that is adjacent to every other vertex, is called a *wheel*; we denote it as $W(v_0; v_1 \dots v_n)$. A wheel that is a subgraph of a triangulation G with $v_i \in V(G)$ as center is denoted by W_i . Every triangulation can be “covered” with a set of wheels $\{W_i\}_{v_i \in V(G)}$, such that every edge is covered four times.

Lemma 3. *Let (p_0, \dots, p_n) be an embedding of a wheel $W(v_0; v_1 \dots v_n)$ in \mathbb{R}^2 . Then the following expression defines an equilibrium stress:*

$$\omega_{ij} = \begin{cases} -1/[p_i p_{i+1} p_0] & j = i + 1, 1 \leq i \leq n, \\ [p_{i-1} p_i p_{i+1}] / ([p_{i-1} p_i p_0][p_i p_{i+1} p_0]) & j = 0, 1 \leq i \leq n. \end{cases}$$

The equilibrium stress for the embedding (p_i) is unique up to a renormalization.

Proof. This stress coincides with (3) from the lifting of W with $z_0 = 1$ and $z_i = 0$ for $1 \leq i \leq n$ and so is an equilibrium stress. The space of the stresses is 1-dimensional, since the space of the polyhedral liftings is 1-dimensional. \square

Definition 4. *1. For a wheel W embedded in the plane we refer to the equilibrium stress defined in Lemma 3 as its small atomic stress and denote it as $\{\omega_{ij}^a(W)\}$.*

2. We call the stress $\{\Omega_{ij}^a(W)\}$ that is obtained by the renormalization of $\{\omega_{ij}^a(W)\}$ by the factor $\prod_{1 \leq j \leq n} [p_j p_{j+1} p_0]$, the large atomic stress of W .

We point out that the large atomic stresses are products of $\deg(v_0) - 1$ triangle areas multiplied by 2, and so, $\{\Omega_{ij}^a(W)\}$ is a set of integers if W is realized with integer coordinates.

Theorem 5 (Wheel-decomposition). *Let G be a triangulation. Every equilibrium stress $\{\omega_{ij}\}$ of an embedding (p_1, \dots, p_n) of G can be expressed as a linear combination of the small atomic stresses on the wheels $\{W_i\}$:*

$$\omega = \sum_{i \leq n} \alpha_i \omega^a(W_i),$$

where the coefficients α_i are the heights (i.e., z -coordinates) of the corresponding vertices v_i in the Maxwell–Cremona lifting of (p_1, \dots, p_n) induced by $\{\omega_{ij}\}$.

Proof. Let (q_1, \dots, q_n) be the Maxwell–Cremona lifting of (p_i) by means of the stress $\{\omega_{ij}\}$. We rewrite this stress (given by Equation 3) using

$$[q_1 q_2 q_3 q_4] = \sum_{1 \leq i \leq 4} (-1)^{i+1} z_i [p_{i+1} p_{i+2} p_{i+3}],$$

(with cyclic notation for indices) and obtain

$$\omega_{ij} = z_i \frac{[p_j p_k p_l]}{[p_i p_j p_k][p_l p_j p_i]} + z_j \frac{[p_l p_k p_i]}{[p_i p_j p_k][p_l p_j p_i]} - z_k \frac{1}{[p_i p_j p_k]} - z_l \frac{1}{[p_l p_j p_i]},$$

which is exactly the decomposition of ω_{ij} into small atomic wheel stresses. \square

Theorem 6. *Let (q_1, \dots, q_n) be an embedding of a triangulation G into \mathbb{Z}^3 , whose projection (p_1, \dots, p_n) to the plane $\{z = 0\}$ is a noncrossing embedding of G with boundary face $(v_1 v_2 v_3)$. Then one can construct a positive integer equilibrium stress $\{\omega_{ij}\}$ for the embedding (p_1, \dots, p_n) such that*

$$|\omega_{ij}| < (\max_{i \leq n} |q_i|)^{2 \Delta_G + 5}.$$

Proof. We start with the equilibrium stress $\{\tilde{\omega}_{ij}\}$ as specified by (3) for the embedding (p_i) . Since all the coordinates are integers, all stresses are bounded by

$$\frac{1}{L^4} \leq \frac{1}{|[p_i p_j p_k][p_l p_j p_i]|} \leq |\tilde{\omega}_{ij}| \leq |[q_i q_j q_k q_l]| \leq L^3,$$

for $L = 2 \max_{i \leq n} |q_i|$.

We are left with making these stresses integral while preserving a polynomial bound. The stress $\{\tilde{\omega}_{ij}\}$ can be written as a linear combination of large atomic stresses of the wheels W_i by means of the Wheel-decomposition Theorem,

$$\tilde{\omega}_{ij} = \alpha_i \Omega_{ij}^a(W_i) + \alpha_j \Omega_{ij}^a(W_j) + \alpha_k \Omega_{ij}^a(W_k) + \alpha_l \Omega_{ij}^a(W_l).$$

Since all points p_i have integer coordinates, the large atomic stresses are integers as well. Moreover, each of them, as a product of $\deg(v_k) - 1$ triangle areas, is bounded by $|\Omega_{ij}^a(W_k)| \leq L^{2(\Delta_G - 1)}$.

To make the $\tilde{\omega}_{ij}$ s integral we round the coefficients α_i down. To guarantee that the rounding does not alter the signs of the stress, we scale the atomic stresses (before rounding) with the factor

$$C = 4 \max_{i,j,k} |\Omega_{ij}^a(W_k)| / \min_{i,j} |\tilde{\omega}_{ij}|$$

and define as the new stress:

$$\omega_{ij} := \lfloor C\alpha_i \rfloor \Omega_{ij}^a(W_i) + \lfloor C\alpha_j \rfloor \Omega_{ij}^a(W_j) + \lfloor C\alpha_k \rfloor \Omega_{ij}^a(W_k) + \lfloor C\alpha_l \rfloor \Omega_{ij}^a(W_l).$$

Clearly,

$$\begin{aligned} |\omega_{ij} - C\tilde{\omega}_{ij}| &= \left| \sum_{\tau=i,j,k,l} (\lfloor C\alpha_\tau \rfloor - C\alpha_\tau) \Omega_{ij}^a(W_\tau) \right| \\ &\leq \sum_{\tau=i,j,k,l} |\Omega_{ij}^a(W_\tau)| \leq 4 \max_{i,j,k} |\Omega_{ij}^a(W_k)| = C \min_{i,j} |\tilde{\omega}_{ij}| \leq C|\tilde{\omega}_{ij}| \end{aligned}$$

and so $\text{sign}(\omega_{ij}) = \text{sign}(C\tilde{\omega}_{ij}) = \text{sign}(\tilde{\omega}_{ij})$.

Therefore, the constructed equilibrium stress $\{\omega_{ij}\}$ is integral and positive. We conclude the proof with an upper bound on its size. Since $C < 4L^{2(\Delta_G - 1)}L^4$,

$$\begin{aligned} |\omega_{ij}| &\leq \left| \sum_{\tau=i,j,k,l} (C\alpha_\tau \pm 1) \Omega_{ij}^a(W_\tau) \right| \leq C|\tilde{\omega}_{ij}| + \sum_{\tau=i,j,k,l} |\Omega_{ij}^a(W_\tau)| \\ &\leq C \max |\tilde{\omega}_{ij}| + 4 \max |\Omega_{ij}^a(W_k)| \leq 4L^{2\Delta_G + 2} \cdot L^3 + 4L^{2\Delta_G - 2} = O(L^{2\Delta_G + 5}). \end{aligned}$$

□

Combining Theorem 6 and Theorem 3 leads to the following result:

Theorem 7. *Let (q_2, \dots, q_n) be an integer embedding of a simplicial 3d polytope with graph G_\uparrow , such that the orthogonal projection into the plane $\{z = 0\}$ gives a planar 2d embedding (p_2, \dots, p_n) with boundary face $(v_2v_3v_4)$. Then we can construct an embedding $(\phi_f)_{f \in F(G)}$ of a graph dual to $G = \text{Stack}(G_\uparrow; v_1; v_2v_3v_4)$ with integer coordinates bounded by*

$$|\phi_f| = O(n^2 \max |q_i|^{2\Delta_G + 13}).$$

We remark that the algorithms following the lifting approach generate embeddings that fulfill the conditions of the above theorem. Using a more technical analysis we can even show that the following stronger version of Theorem 7 holds. The proof of the theorem can be found in the full version of the paper.

Theorem 8. *Let G be a triangulation with at least one vertex of degree 3, and let (q_i) be an integer realization of G as a convex polytope. Then there is a realization $(\phi_f)_{f \in F(G)}$ of the dual graph G^* as a convex polytope with integer coordinates bounded by*

$$|\phi_f| < \max |q_i|^{O(\Delta_G)}.$$

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