Strongly-Connected Outerplanar Graphs with Proper Touching Triangle Representations

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Abstract. A *proper touching triangle representation* $\mathcal{R}$ of an n-vertex planar graph consists of a triangle divided into n non-overlapping triangles. A pair of triangles are considered to be adjacent if they share a partial side of positive length. Each triangle in $\mathcal{R}$ represents a vertex, while each pair of adjacent triangles represents an edge in the planar graph. We consider the problem of determining when a proper touching triangle representation exists for a *strongly-connected outerplanar graph*, which is biconnected and after the removal of all degree-2 vertices and outeredges, the resulting *connected* subgraph only has chord edges (w.r.t. the original graph). We show that such a graph has a proper representation if and only if the graph has at most two *internal faces* (*i.e.,* faces with no outeredges).

1 Introduction

Although the node-link model has been the traditional form of drawing a planar graph $G(V, E)$, many application areas demand alternate models of representing graphs, such as polygon *edge-contact representations*. Here vertices are represented by simple polygons and edges are represented by adjacent polygons that have at least a partial side in common. As pointed out by de Fraysseix *et al.* [2], one can easily find such a representation of any planar graph with non-convex polygons with complexity as high as $|V| - 1$, where much area is unused leading to many gaps and holes within the representation. Recently, convex hexagons have been shown to always be sufficient in producing hole-free representations [1], although, 6-sided polygons are sometimes necessary. The problem thus arises in determining which classes of planar graphs can be represented by polygons with fewer than six sides.

In this context, we focus on the case of minimal polygonal complexity, where all the representing polygons are triangles. Specifically, an n-vertex *touching triangle graph* (TTG) has a representation $\mathcal{R}$ where each vertex is represented by one of n non-overlapping triangles and each edge is represented by a pair of adjacent triangles in $\mathcal{R}$. Again, triangles are only considered to be adjacent if they share at least a partial side of positive length. Thus, pairs of triangles having only point contacts are not considered to be adjacent, and hence, do not represent edges.

The most natural looking edge-contact representations have triangular boundaries where their interiors contains no gaps or holes. If this is the case, then $\mathcal{R}$ is a *proper* TTG representation, and the graph is a proper TTG. Visually, $\mathcal{R}$ can be thought of as a triangle that has been subdivided into n non-overlapping triangles, where adjacent

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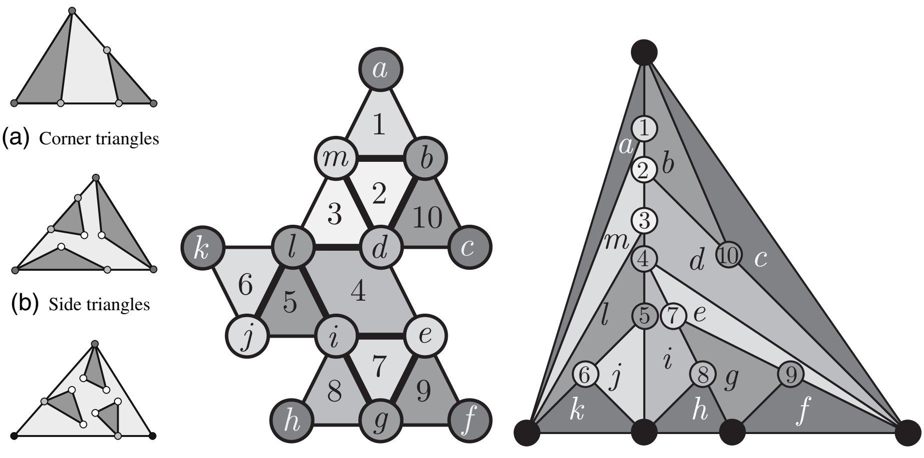


(a) Corner triangles

(b) Side triangles

(c) Internal triangles

(d) Graph G

(e) Representation $\mathcal{R}$

Fig. 1. (a–c) Three types of representing triangles; (d) a strongly-connected outerplanar graph G with two internal faces: 2 and 7; and (e) a proper TTG representation $\mathcal{R}$ of G

triangles do not necessarily share entire sides as they do in a triangulation. Not all planar graphs are TTGs [3], let alone proper. While it was also shown in [3] that all biconnected outerplanar graphs have hole-free TTG representations, their boundaries are not necessarily triangular, and hence, are not necessarily proper. This raises the question as to which outerplanar graphs have proper TTG representations.

A proper outerplanar TTG has several restrictions. Degree-1 vertices can only be represented by *corner triangles of* $\mathcal{R}$ with two edges along the boundary T of $\mathcal{R}$, while degree-2 vertices can also be represented by *side triangles of* $\mathcal{R}$ with one side along T. All other vertices are represented by *internal triangles of* $\mathcal{R}$; *cf.* Figs. 1(a)–1(c).

A biconnected outerplanar graph G is *strongly-connected* if after the removal of all degree-2 vertices and outeredges, the resulting *connected* subgraph only has chord edges (w.r.t. G) as in Fig. 1(d). Such graphs are not necessarily maximal. We characterize this graph class in terms of *internal faces* (*i.e.* faces with no outeredges) as follows:

(1) First, we construct proper TTG representations for strongly-connected outerplanar graphs, as in Fig. 1(e), using a *chord-to-endpoint assignment* that pairs each chord (except for one) with a distinct vertex that is also an endpoint of the chord.

(2) Second, we show that having at most two internal faces is sufficient when the graph is strongly-connected, since a chord-to-endpoint assignment exists in this case.

(3) Third, we finish our characterization by proving that having at most two internal faces is also necessary in order for G to have a proper TTG representation.

To the best of our knowledge, the only other results specifically for proper TTGs are in [4], where a fixed-parameter tractable decision algorithm for 3-connected planar max-degree-Δ graphs is described, and where it it shown that planar 3-connected cubic graphs are proper TTGs.

2 Proper Strongly-Connected Outerplanar TTGs

Let G be a strongly-connected outerplanar graph. A *strong reversed peeling order* σ is an ordering of the inner faces $F_1, \ldots, F_q$ of G such that the subgraph $G_i = F_1 \cup \cdots \cup F_i$ is also strongly-connected for each $i \in [1 .. q]$. We index the chords $\mathcal{C} = \{c_2, \ldots, c_q\}$ of G by σ such that c_i is the chord of face F_i that becomes an outeredge of G_{i-1} when F_i is "peeled" from G_i. Thus, c_i is common to $F_{i'}$ and F_i for some $i' < i$. Finally, a *chord-to-endpoint assignment* $\tau : \mathcal{C}' \to V'$ of a strong reversed peeling order σ assigns the chords $\mathcal{C}' = \{c_3, \ldots, c_q\} = \mathcal{C} \setminus \{c_2\}$ to a subset of their endpoints $V' = \{v_3, \ldots, v_q\}$ such that $\tau(c_i) = v_i$ is an endpoint of c_i for each $i \in [3 .. q]$ where $v_i \neq v_j$ if $i \neq j$.

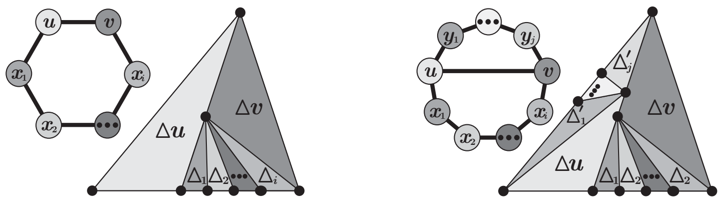
Claim 1. *If G is a strongly-connected outerplanar graph, then G has a strong reversed peeling order σ.*

Proof. Faces F_1 and F_2 of σ can be any pair of adjacent faces. For $i \in [2 .. q-1]$, assume that subgraph $G_i = F_1 \cup \cdots \cup F_i$, has the connected *chord subgraph* $H_i = c_2 \cup \cdots \cup c_i$, where c_i is the chord of face F_i that was added to G_{i-1}. At least one outeredge of G_i is a chord in G. Otherwise, the outerface of G_i, which is a separating cycle C in G, would have a cut-vertex in G—all the cut edges in G are chords, none of which can be in C—violating the biconnectivity of G. Since G is strongly-connected, every chord c in the outerface of G_i must be incident to some chord of H_i. Thus, the next face F_{i+1} in σ can be any remaining face whose chord c_{i+1} is an outeredge of G_i. □

Lemma 2. *If G is a strongly-connected outerplanar graph for which there exists a chord-to-endpoint assignment τ, then G has a proper touching triangle representation.*

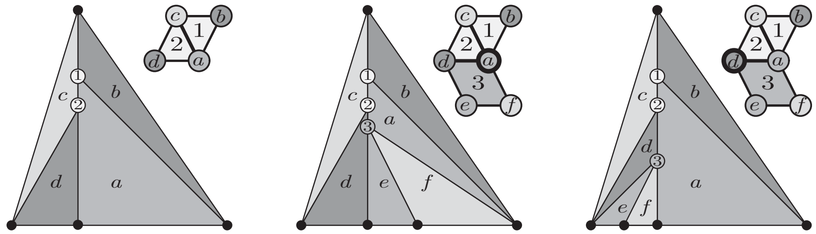
Proof. Let σ be the strong reversed peeling order of the faces $F_1, \ldots, F_q$ of G given by the order of chords as assigned by τ. We apply induction on k and assume that we have a proper TTG representation $\mathcal{R}_k$ for G_k, and show how to modify $\mathcal{R}_k$ to obtain $\mathcal{R}_{k+1}$ for $G_{k+1} = G_k \cup F_{k+1}$. When G_1 has a single face F_1 with the edge (u, v) connected by the chain of vertices $x_1, \ldots, x_i$, Fig. 2(a) gives a proper TTG representation $\mathcal{R}_1$ for G_1. Next, when G_2 has two faces with the common chord (u, v) (where $y_1, \ldots, y_j$ forms a chain in F_2), Fig. 2(b) gives a proper TTG representation $\mathcal{R}_2$ for G_2 where the triangle $\triangle u$ representing u in $\mathcal{R}_1$ was subdivided into a total of $j + 1$ triangles.

For $k \in [2 .. q-1]$, we also assume by induction on k that triangle $\triangle v_k$ representing $\tau(c_k) = v_k$ is a side triangle in $\mathcal{R}_k$. This holds in the base case of $k = 2$, since $\mathcal{R}_2$ in



(a) G_1 and $\mathcal{R}_1$ (b) G_2 and $\mathcal{R}_2$

Fig. 2. Proper TTG representations of strongly-connected outerplanar graphs with 1 or 2 faces



(a) $\mathcal{R}_2$ of G_2 (b) $\mathcal{R}_3$ of G_3 (c) $\mathcal{R}'_3$ of G_3

Fig. 3. For chord $c_3 = (a, d)$, either triangle $\triangle a$ or $\triangle d$ in $\mathcal{R}_2$ in (a) is divided into triangles $\triangle\tau(c_3), \triangle e, \triangle f$ to form $\mathcal{R}_3$ if $\tau(c_3) = a$ as in (b) or $\mathcal{R}'_3$ if $\tau(c_3) = d$ as in (c)

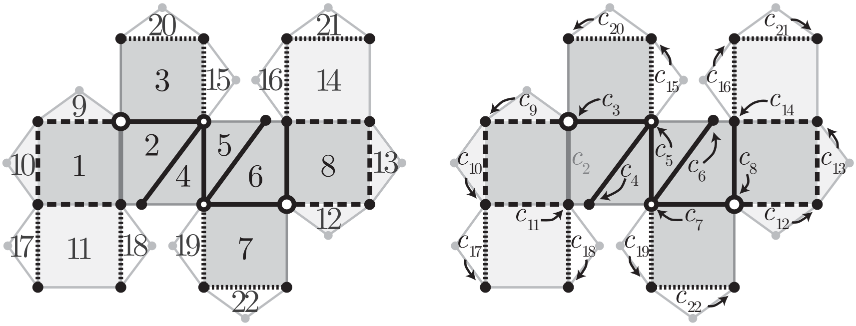
Fig. 2(b) only has side triangles. This allows us to subdivide $\triangle v_k$ in $\mathcal{R}_k$ a total of j_k times to obtain $\mathcal{R}_{k+1}$ in which an internal triangle now represents v_k and j_k new side triangles represent the j_k degree-2 vertices in F_k that were added to G_k to form G_{k+1}. Figure 3 illustrates this process. The chord $c_3 = (a, d)$ in G_3 either has $\tau(c_3) = a$ or $\tau(c_3) = d$. This results in $\mathcal{R}_3$ in Fig. 3(b) (or in $\mathcal{R}'_3$ in Fig. 3(c)) after dividing the side triangle $\triangle a$ (or $\triangle d$) in $\mathcal{R}_2$ of G_2 in Fig. 3(a) into the internal triangle $\triangle a$ (or $\triangle d$) and the two side triangles $\triangle e$ and $\triangle f$. As a consequence, each chord endpoint v_k has its representing triangle $\triangle v_k$ subdivided once, at which point $\triangle v_k$ is an internal triangle in $\mathcal{R}_{k'}$ for $k' > k$. However, this is not a problem in maintaining our inductive hypothesis for $k + 1$ since τ assigns each chord to a distinct vertex in G. □

Lemma 3. *If G is a strongly-connected outerplanar graph with at most two internal faces, then G is a proper touching triangle graph.*

Proof. We construct a chord-to-endpoint assignment τ, where Lemma 2 then implies that G is a proper TTG. If G has no internal faces, its connected chord subgraph H is acyclic, and hence, a tree. Let σ be a strong reversed peeling order of faces $F_1, \ldots, F_q$ for G given by Claim 1. Each face F_i was picked so that its chord c_i is a leaf edge in the subtree of chords $H_i = c_2 \cup \cdots \cup c_i$ of G_i. Thus, we can assign $\tau(c_i) = u_i$, where u_i is the endpoint of c_i that is a leaf node in H_i, for $i \in [3 .. q]$. Both endpoints of the first chord c_2 are left unassigned by τ.

If G has one internal face F, we apply Claim 1 and assign chords as before with the following exceptions: We set $F_1 = F$ and faces $F_2, \ldots, F_j$ as the adjacent faces of F_1, where face F_{i+1} is incident to face F_i for $i \in \{2 .. j - 1\}$. Since chord c_j forms the cycle C (edges of F_1) when added to H_j, we can set $\tau(c_j)$ to be the common endpoint of the chords c_j and c_2, which leaves one endpoint of c_2 unassigned.

Lastly, when G has two internal faces, we apply Claim 1 as follows: We set F_1 and F_k in σ to be the two internal faces in G such that k is minimal. The chords of G_k contain a path p connecting F_1 to F_k. For τ, we assign each chord of p to the endpoint first encountered along p so that c_2 and c_k each have exactly one assigned endpoint. To σ we add each remaining face F_i adjacent first to F_1 (and then to F_k) starting from the endpoints of p in cyclic order along each respective face. For τ, we assign chord c_i of each cycle to its newly added endpoint in H_i—except for the last chord, call it c', in F_1



(a) Strong reversed peeling order σ (b) Chord-to-endpoint assignment τ

Fig. 4. Example of determining σ and τ for strongly-connected outerplanar graph G with two internal faces F_1 and F_8. Subgraph G_8 (dark gray faces) is the minimal strongly-connected subgraph containing both F_1 and F_8, whose chord subgraph H_8 (solid chords) is a caterpillar. Subgraph G_{14} (light/dark gray faces) has faces $F_9, ..., F_{11}$ and $F_{12}, ..., F_{14}$ added in cyclic order along F_1 and F_8, resp., each starting from an endpoint of path p (white vertices). The chords of path p are assigned the endpoints first encountered along p from F_1 to F_8. The dashed chords of F_1 and F_8 are assigned endpoints next along each cycle (starting from endpoints of p). Chord c_i for $i \in \{4, 6, 15, ..., 22\}$ is assigned its endpoint that was not in the chord subgraph H_{i-1}.

(or in F_k). Given the greedy assignment of chords along p, c' has an available endpoint in common with c_2 in F_1 (or c_k in F_k). Each remaining face can be added to σ and have its chord assigned by τ as before. Figure 4 illustrates this procedure. □

For the next lemma, we consider the *representing dual graph* $\mathcal{G}_{\mathcal{R}}$ (the graph formed by the representation $\mathcal{R}$) of a proper TTG G. With respect to the triangular boundary T of a proper TTG representation $\mathcal{R}$, each vertex or edge of $\mathcal{G}_{\mathcal{R}}$ (common to one or more representing triangles) is either *external* if along the boundary T or *internal* if inside T. Likewise, we term the faces of $\mathcal{G}_{\mathcal{R}}$ as being either *external faces* if they correspond to corner or side triangles or *internal faces*, otherwise. We have the following relationships between G and $\mathcal{G}_{\mathcal{R}}$: (1) each vertex v in G corresponds to a bounded triangular face F_v in $\mathcal{G}_{\mathcal{R}}$ and (2) each bounded face f in G corresponds to an internal vertex v_F in $\mathcal{G}_{\mathcal{R}}$.

Clearly, the angle of a vertex v_F in $\mathcal{G}_{\mathcal{R}}$ along the face F_v in $\mathcal{G}_{\mathcal{R}}$ can be at most 180°. If the angle is less than 180°, then v_F is a corner of the triangular face F_v. Each internal vertex v_F in $\mathcal{G}_{\mathcal{R}}$ has at most one 180° angle since $deg(v_F) > 2$ in $\mathcal{G}_{\mathcal{R}}$. For example, the angle of vertex v_1 in $\mathcal{R}$ in Fig. 1(e) is 180° for face F_b, but not for faces F_a and F_m. Face F_v in $\mathcal{G}_{\mathcal{R}}$ representing v has exactly three vertices (each with an angle less than 180°) if either (i) F_v is an external face where $deg(v) = 2$ or (ii) F_v is an internal face where $deg(v) = 3$. Otherwise, F_v has at least $ch(v) = max\{deg(v) - 3, 0\}$ vertices in $\mathcal{G}_{\mathcal{R}}$ whose internal angles are 180° along the boundary of F_v. These correspond to $ch(v)$ faces in G that are incident to v.

We denote $ch(v)$ as the *charge* of v since each incident face F in G can *dissipate* at most one charge from v. This is done by having the internal angle for v_F be 180° along the face F_v representing v in $\mathcal{G}_{\mathcal{R}}$. However, face F can dissipate at most one charge from an incident vertex, since v_F has at most one 180° angle. For example, vertex d in

Fig. 1(d) has charge $ch(d) = 2$ dissipated by faces F_3 and F_{10}, where the corresponding internal vertices v_3 and v_{10} have 180° angles for face F_d in Fig. 1(e).

Thus, in order for G to be a TTG, there must exist a *discharge function*, $\pi : \mathcal{F}' \rightarrow V$ that assigns a subset faces $\mathcal{F}' \subseteq \mathcal{F}$ (the faces of G) to incident vertices to fully dissipate the *total charge* $ch(G) = \sum_{v \in V(G)} ch(v)$ of G. Hence, G cannot be a proper TTG graph if $av(G) = q - ch(G) < 0$, where $q = |\mathcal{F}|$ and $av(G)$ is the *availability* of G.

Lemma 4. *If G is a strongly-connected outerplanar graph with more than two internal faces, then G cannot be a proper touching triangle graph.*

Proof. We apply Claim 1 to get a strong reversed peeling order σ of the faces $F_1, \ldots, F_q$ of G, where the subgraph $G_i = F_1 \cup \cdots \cup F_i$ is strongly-connected. If G_i does not contain any internal faces, then H_i is a tree. When peeling face F_i from G_i to obtain G_{i-1}, chord c_i cannot have both of its endpoints u_i and v_i in H_{i-1}. Otherwise, c_i would form a cycle in H_i. Thus, one endpoint $v_i \notin H_{i-1}$ of c_i has $deg(v_i) = 2$ in G_{i-1}. While both degrees of u_i and v_i increased by 1 in going from G_{i-1} to G_i, only $deg(u_i) > 3$ in G_i, while $deg(v_i) = 3$ in G_i. Thus, the total charge $ch(G_i) = ch(G_{i-1}) + 1$ increases by one, so that the availability $av(G_i) = av(G_{i-1})$ remains constant.

If G_i contains a new internal face C that G_{i-1} does not, then both endpoints u_i and v_i of c_i must be in H_{i-1} in order for c_i to form the new cycle C in H_i. Hence, both $deg(u_i) > 3$ and $deg(v_i) > 3$ in G_i, so that $ch(G_i) = ch(G_{i-1}) + 2$, and as a result, $av(G_i) = av(G_{i-1}) - 1$ where the availability decreases by one.

Initially the availability is at most 2, where G_2 has maximum degree 3 with the two faces F_1 and F_2 so that $av(G_2) = 2$. Consequently, G can have at most two internal faces, before the availability drops below 0, preventing it from being a TTG. □

We conclude by combining Lemmas 3 and 4 to give the main theorem of the paper.

Theorem 5. *A strongly-connected outerplanar graph G has a proper touching triangle representation if and only if G has at most two internal faces.*

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