Chapter 1

Introduction

Nonlinearly interacting waves are often described by asymptotic equations. The derivation typically involves an ansatz for an approximate solution where higher order terms – the precise meaning of higher order term depends on the context and the relevant scales – are neglected. Often a Taylor expansion of a Fourier multiplier is part of that process.

There is an immediate consequence: This type of derivation leads to a huge set of asymptotic equations, and one should search for a general understanding of interacting nonlinear waves by asking for precise results for specific equations.

The most basic asymptotic equation is probably the nonlinear Schrödinger equation, which describes wave trains or frequency envelopes close to a given frequency, and their self-interactions. The Korteweg–de-Vries equation typically occurs as first nonlinear asymptotic equation when the prior linear asymptotic equation is the wave equation. It is one of the amazing facts that many generic asymptotic equations are integrable in the sense that there are many formulae for specific solutions, conserved quantities, Lax Pairs and Bi-Hamiltonian structures.

This text will focus on adapted function spaces and their recent application to a number of dispersive equations. They are build on functions of bounded p-variation, and their companion, the atomic space U^p. Combined with stationary phase resp. Strichartz estimates and bilinear refinements thereof, they provide an alternative to the Fourier restriction spaces $X^{s,b}$ which is better suited for scaling critical problems.

We discuss the method of stationary phase and dispersive estimates in Section 2, the application to the nonlinear Schrödinger equation in Section 3, the spaces U^p and V^p in Section 4, bilinear estimates in Section 5, and applications to nonlinear dispersive equations in Section 6.

In order to make these notes reasonably self-contained there are three appendices on Young's inequality, real and complex interpolation, on Bessel functions, and on the Fourier transform.

H. Koch et al., *Dispersive Equations and Nonlinear Waves: Generalized Korteweg–de Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps*, Oberwolfach Seminars 45,
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