Chapter 14
Karl Popper on Deduction

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Abstract We outline Karl Popper's theory of deduction, which he developed in the 1940s. In his theory it is assumed that a consequence relation is given or otherwise constructed by postulation. Logical operations, which may be available in this consequence relation, are then characterized by means of relational definitions, and logical operators are introduced as names for these operations by means of inferential definitions. Using logically structured sentences thus introduced, the inference laws for them are immediately obtained from the inferential definitions.

Keywords Karl Popper · Logic · Inferentialism · Logical constant · Deducibility · Quantification

14.1 Introduction

Karl Popper published his theory of deductive logic in a series of six articles between 1947 and 1949:

1. "Logic without Assumptions" (Popper 1947a),
2. "New Foundations for Logic" (Popper 1947b),
3. "Functional Logic without Axioms or Primitive Rules of Inference" (Popper 1947c,d),
4. "On the Theory of Deduction, Part I. Derivation and its Generalizations" (Popper 1948a,b),
5. "On the Theory of Deduction, Part II. The Definitions of Classical and Intuitionist Negation" (Popper 1948c,d),
6. "The Trivialization of Mathematical Logic" (Popper 1949).

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These articles are reprinted in *The Logical Writings of Karl Popper* (Binder, Piecha, and Schroeder-Heister 2022b), which also contains further published articles, unpublished manuscripts, contemporary reviews and Popper's correspondence on deductive logic. For an extensive presentation and discussion of his theory we refer to the first chapter of that book (Binder, Piecha, and Schroeder-Heister 2022a), on which this paper is mainly based. Further detailed investigations of Popper's theory are Lejewski (1974), Schroeder-Heister (1984), Schroeder-Heister (2006), Binder and Piecha (2017), Moriconi (2019), and Binder and Piecha (2021).

In Sect. 14.2 we describe Popper's structural framework for defining logical constants in a metalanguage restricted to positive logic with propositional quantification. In Sect. 14.3 we sketch Popper's general theory of deduction, which is the theory of "structural" rules (to use Gentzen's terminology) without taking into account the internal deductive form or power of sentences, and in Sect. 14.4 we discuss Popper's special theory of deduction, which deals with logical constants, their inferential definability and logical laws. In the final Sect. 14.5 we summarize the central aspects of Popper's theory and highlight its differences with respect to Tarski's approach to logical consequence.

Although we can cover only the main aspects of Popper's approach to logic here, it should be mentioned that he also obtained important concepts and results in logic first or, in some cases, independently of contemporary logicians. Examples are his consideration of the *tonk*-like connective "the opponent of a statement" (cf. Sect. 14.4.2), which trivializes logical systems containing it, the characterization of implication by means of Peirce's rule, the proposal of a dual-intuitionistic logic (formulated and investigated by his student Cohen in 1953), the discussion of conservative and non-conservative language extensions (cf. Sect. 14.4.2), the perhaps first formulation of a bi-intuitionistic logic after (and most probably without knowledge of) Moisil (1942), the idea of combining logics, his analysis of logicality (cf. Sect. 14.4.2), the axiomatic characterization of the substitution operation in his theory of quantification (cf. Sect. 14.4.5), and his results on several non-classical negations.

We note that Popper uses the terms "deduction", "derivation", and "inference" mostly synonymously, and likewise the terms "deducible", "derivable", "follows from" and "is a consequence of", and that we speak of theories of deduction where Popper would speak of theories of derivation or inference. Moreover, we use the terms "sentence" and "statement" mostly synonymously, while Popper has a preference for "statement". In order to avoid any misunderstanding from the beginning, we also note that rules of derivation are not rules in a calculus understood as a proof system. Popper does not develop a calculus in this sense, and what he calls rules of derivation are metalinguistically formulated statements about a deducibility relation.

14.2 The Structural Framework

For propositional logic, Popper considers pairs $(\mathcal{L}; a_1, \ldots, a_n/b)$ of an object language $\mathcal{L}$ containing statements $a_1, \ldots, a_n, b$ together with a deducibility relation / on $\mathcal{L}$, which today we would call a finite consequence relation in Tarski's sense.

In contradistinction to modern approaches to logic that start by defining a formal object language, Popper's approach does not presuppose any knowledge about the form or syntactic structure of the object language $\mathcal{L}$ under consideration and is in principle applicable both to formal and natural languages. For example, the conjunction of two statements a and b of the object language need not have any particular syntactic form like "$a \wedge b$" or "a and b". An object language may well be formally specified by an inductive definition, and Popper himself considers such languages. But it is not required, and Popper considers any sort of language, formal or non-formal, provided we know what it means that a sentence of the language follows from other sentences.

Popper's approach is based on the concept of deducibility (or "derivability"). It is the only undefined notion as far as propositional logic is concerned (this includes modal logic, which is also discussed by Popper). An operation of substitution is added for the treatment of first-order logic (Popper uses the terms "theory of quantification" or "quantification theory" instead of "first-order logic"; cf. Sect. 14.4.5). Deducibility is a relation, written with the slash /, that ranges over the object language and holds between finitely many premises $a_1, \ldots, a_n$ (for $n \geq 0$) and exactly one conclusion b. In /-notation, Popper writes

$$a_1, \ldots, a_n/b$$

to express that the statement b can be deduced from the statements $a_1, \ldots, a_n$. The case $n = 0$, in which no premises occur, was not yet considered in Popper (1947b). It was added later in Popper (1948a), where the so-called *D-notation* is introduced. In this notation, $D(a_1, a_2, \ldots, a_n)$ stands for $a_2, \ldots, a_n/a_1$, and the special case $D(a_1)$ corresponds to $/a_1$, meaning that a_1 is deducible without premises.

When Popper defines logical operations inferentially, this is carried out in a metalanguage, whose basic relation is deducibility /, and not in a syntactically specified object language. As a general example, let us consider a two-place logical operation $\circ$, which we call "connection" and which can stand for any operation such as conjunction or implication, or mutatis mutandis for operations of other arities such as negation or quantification. Then Popper relies on *relational definitions* of the form

$$c \text{ is a connection of } a \text{ and } b \text{ if and only if } \mathcal{R}(c, a, b)$$

and on *inferential definitions* of the form

$$c//a \circ b \text{ if and only if } \mathcal{R}(c, a, b),$$

where // stands for interdeducibility (or "mutual deducibility", to use Popper's term; cf. Sect. 14.3). In most cases the *defining condition* $\mathcal{R}(c, a, b)$ has the form of a rule, which is described in terms of deducibility /, metalinguistic conjunction, metalinguistic implication and metalinguistic universal quantification. Popper does not specify exactly the means of expression allowed in a defining condition, but from all contexts it is clear that positive logic is sufficient.

We use the following symbolic notation for metalinguistic expressions, which is similar to Popper's:

Symbol	$\rightarrow$	$\leftrightarrow$	&	(a)
Meaning	if-then	if and only if	and	for all a

The universal quantifier (a) ranges over statements a of the object language. Since equivalence is definable in terms of conjunction and implication, the fundamental logical operations used in the metalanguage to define arbitrary propositional logical operations of the object language are conjunction, implication and universal quantification (over statements). Metalinguistic disjunction could be added, but, as Popper notes, it is only needed to define modal operations (which we do not consider in this exposition). In Popper's theory of quantification (cf. Sect. 14.4.5) metalinguistic existential and universal quantification over statements and variables is used, for which we do not use symbols, however.

Due to the fact that only positive logic is permitted in the defining condition of an operation, the existence of this operation can never imply that the metalanguage is inconsistent. The trivial deducibility relation, which holds for all arguments, validates any defining condition $\mathcal{R}$ and therefore falsifies the negation of it, which means that not every metalinguistic statement is true. Trivialization of the object language does not imply trivialization of the metalanguage. That is, in any nonempty object language with a trivial deducibility relation (that is, $a_1, \ldots, a_n/b$ holds for any $a_1, \ldots, a_n, b$), every defining condition $\mathcal{R}$ for a logical operation is satisfied. This will become relevant in the discussion of the logical constant "the opponent of a statement" (*opp*) in Sect. 14.4.2.

For certain expressions of the metalanguage Popper uses a special vocabulary. Statements of the form

$$a_1, \ldots, a_n/b$$

are also called *absolute rules of derivation*, and statements of the form

$$a_1, \ldots, a_n/b \rightarrow c_1, \ldots, c_m/d$$

and iterated versions thereof are also called *conditional rules of derivation* or just *rules of derivation*. We emphasize again that rules of derivation are not rules in a calculus or proof system, but metalinguistically formulated statements about the deducibility relation. However, due to their specific form they can be read as descriptions of rules, as they tell us that from certain deducibility statements we may pass over to another deducibility statement. Thus we follow Popper in using the term "rule" to speak about such metalinguistic expressions. This means in particular that we will often speak of $\mathcal{R}(c, a, b)$ as "defining rules" instead of "defining conditions", even if they cannot be translated immediately into rules of some object language.

So far, the deducibility relation / has only been defined by saying that it ranges over an object language $\mathcal{L}$. The next step consists in providing what Popper calls a *basis* for this relation. A basis is a complete and independent set of rules, formulated in the metalanguage, that axiomatizes the deducibility relation /. Completeness is here defined with respect to Popper's notion of *absolute validity*, which is similar to the notion of validity obtained by allowing only structural rules of inference. Popper's idea seems to be that even if we abstract away from any concrete logical system (containing a specific set of logical constants) under consideration, we still have a rudimentary residuum of deduction consisting of structural inferences, like the inference from a statement a to the statement a.

Popper mainly uses two alternative bases, called Basis I and Basis II, which also occur in different versions. In another approach, Popper (1947c) uses an extended definition of conjunction, called basic definition (DB2), to ensure reflexivity and transitivity of / as well as exchangeability of premises. However, (DB2) corresponds to the defective Basis II of Popper (1947b), and is therefore just as problematic. We do not consider this approach here (it is discussed in Binder et al. 2022b, § 4.6).

Our presentation is based on the following version of Basis I that can be found in Popper (1947b) as well (cf. also Bernays 1965). It is given by a *generalized reflexivity principle* (Rg) together with a *generalized transitivity principle* (Tg):

$$a_1, \ldots, a_n / a_i \qquad (1 \leq i \leq n), \tag{Rg}$$

$$\left\{\begin{array}{l} \ a_1, \ldots, a_n / b_1 \\ \&\ a_1, \ldots, a_n / b_2 \\ \vdots \qquad \vdots \\ \&\ a_1, \ldots, a_n / b_m \end{array}\right\} \rightarrow (b_1, \ldots, b_m / c \rightarrow a_1, \ldots, a_n / c). \tag{Tg}$$

The principle (Tg) is a schematic rule, so its content cannot be expressed directly by means of a single formula of the metalanguage. Popper tried to improve on this by searching for a replacement of (Tg) that could be expressed directly in his metalanguage. However, this led to other problems due to the defective Basis II and the problematic basic definition (DB2) mentioned above (cf. Binder et al. 2022a, § 4.6).

The fact that (Tg) is parametrized by a natural number actually applies not only to the multiplicity of premises by m, but also to the multiplicity n of sentences on the left side of / within both (Rg) and (Tg). This could have been overcome easily by adopting a notation for finite sets of statements corresponding to contexts $\Gamma, \Delta, \ldots$ in Gentzen sequents, where the multiplicity m of premises can be modelled by conjunctively understood sets of sentences on the right hand side of /, as in the multiple premises the left hand side of / (that is, $a_1, \ldots, a_n$) is always the same. Obviously, Popper did not want to enlarge the formalized part of his metalanguage by such additional means of expression, insisting on the fact that deducibility / is the only primitive concept of the metalanguage, at least for propositional logic.

14.3 The General Theory of Deduction

Popper distinguishes between a general and a special theory of deduction. The general theory does not refer to any logical signs of the object language. It studies properties of statements and relations on statements that can be defined using only the deducibility relation. One such relation is *relative demonstrability*

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m,$$

which is defined by the following rule:

$$(c)(d_1)\ldots(d_k)((b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c) \rightarrow a_1, \ldots, a_n, d_1, \ldots, d_k/c).$$

It can be interpreted as derivability of the disjunction of $b_1, \ldots, b_m$ from the conjunction of $a_1, \ldots, a_n$. This interpretation is justified by the fact that for object languages containing conjunction $\wedge$ and disjunction $\vee$ one can show the following:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \;\leftrightarrow\; a_1 \wedge \ldots \wedge a_n \vdash b_1 \vee \ldots \vee b_m.$$

The concept of relative demonstrability gives thus an interpretation of Gentzen's (1935a, b) sequents. Object languages without conjunction or disjunction are not excluded, however.

Since for all $a_1, \ldots, a_n, b$ we have $a_1, \ldots, a_n/b \;\leftrightarrow\; a_1, \ldots, a_n \vdash b$, one can replace / by $\vdash$ in all formulas of the metalanguage. Note that, conversely, $\vdash$ can only be replaced by / if $\vdash$ has exactly one succedent b. We will make use of relative demonstrability in Sects. 14.4.3 and 14.4.4.

Relative demonstrability comprises further defined relations, also considered by Popper, as special cases. Namely *complementarity* (for $n = 0$), *demonstrability* (for $n = 0$ and $m = 1$), *contradictoriness* (for $m = 0$) and *refutability* (for $n = 1$ and $m = 0$).

Another useful relation is *mutual deducibility* (or *interdeducibility*) $a//b$, which can be defined directly with deducibility:

$$a//b \;\leftrightarrow\; (a/b \;\&\; b/a),$$

or by using relative demonstrability: $a//b \;\leftrightarrow\; (a \vdash b \;\&\; b \vdash a)$.

14.4 The Special Theory of Deduction

The general theory of deduction was purely structural (in Gentzen's terminology) and based solely on the deducibility of statements without regard for their individual form and their individual deductive power. The subject of Popper's special theory of derivation are relations between statements, which are logically complex or have a specific deductive power, and their components. The components are not necessarily subsentences, but sentences that are deductively related to the original sentence in a certain way, such as sentences a and b which are deductively related to a conjunction c of a and b, even if c does not syntactically contain a and b. Furthermore, the special theory of derivation deals with the logical laws emerging therefrom. It is based on the relational definitions of logical operations and the inferential definitions of logical operators.

14.4.1 Definitions of Logical Constants

Logical constants are characterized in terms of the role they play with respect to the deducibility relation $/$. Such characterizations proceed by what Popper calls *inferential definitions*. A sign of an object language $\mathcal{L}$ is a *logical constant* (Popper speaks of *formative signs*) if and only if it can be defined by an inferential definition. According to Popper (1947a, p. 286), "*inferential definitions* [. . .] are characterized by the fact that they define a formative sign by its logical force which is defined, in turn, by a *definition in terms of inference* (i.e., of '/')."

We use $\circ$ again to represent an arbitrary two-place logical operation called "connection" with $\mathcal{R}$ as its defining condition ("rule"), which is used in the *relational definition*

$$c \text{ is a connection of } a \text{ and } b \leftrightarrow \mathcal{R}(c, a, b)$$

and in the *inferential definition*

$$c//a \circ b \leftrightarrow \mathcal{R}(c, a, b). \qquad (\mathrm{D} \circ)$$

As already mentioned, $\mathcal{R}(c, a, b)$ is an expression of the metalanguage containing as relations the deducibility relation $/$, and maybe defined relations like relative demonstrability $\vdash$, which can always be eliminated in a given logical argument, however.

The relational definition makes no special assumption about the object language. It just singles out connections c of a and b if they are available in the object language (otherwise there simply is no connection of a and b). The inferential definition, however, requires existence and uniqueness, which means that it is based on the presupposition that there is *exactly one* connection c of a and b in the object language considered. Here "uniqueness requirement" and "exactly one" is understood *modulo interdeducibility*. That is, there is a connection c of a and b, and all other connections c' of a and b are interdeducible with c: $c /\!/ c'$. Note that in any case, every c' interdeducible with a connection c of a and b is itself a connection of a and b. This follows from the fact that according to the general theory of deduction, we have substitutivity of interdeducibles, which means that all our inferential concepts and results are invariant with respect to interdeducibility. Popper is fully aware of these existence and uniqueness requirements.

The inferential definition (D $\circ$) can be viewed as an *explicit definition* of a connective $\circ$. (D $\circ$) conservatively introduces into a language a sign for a new operator $\circ$, which is eliminable following the standard procedures used for the introduction and elimination of function symbols or definite descriptions. It should be noted, however, that (D $\circ$) is a definition of a *metalinguistic* function which associates with any a and b their connection $a \circ b$, which is unique up to interdeducibility, so $a \circ b$ essentially denotes an equivalence class of objectlinguistic sentences, none of which must have a special form. However, once we have reached that stage, we can, of course, introduce into our object language sentences of the form "$a \circ b$", where $\circ$ is now an objectlinguistic operator in the usual sense, and take it to be the objectlinguistic representative of $a \circ b$ (understood metalinguistically). Viewed that way, (D $\circ$) can be used to introduce a logical operator into a suitable object language, provided the connection operation as a relation between (not further specified) sentences is available.

In fact, we can even devise a formal object language in the usual way and lay down the rules $\mathcal{R}(a \circ b, a, b)$ for all a and b. In that case a connection $a \circ b$ of a and b with certain inferential properties always exists by stipulation. However, when proceeding in that manner, we must be aware (and Popper is) that such a stipulation may have undesired consequences up to the generation of inconsistencies. In any case, for (D $\circ$) to hold, we still must make sure that uniqueness is satisfied. If this is the case, we can proceed with the rule

$$\mathcal{R}(a \circ b, a, b) \qquad \text{(C} \circ\text{)}$$

as the *characterizing rule*, which corresponds to the definition (D $\circ$). If uniqueness is satisfied, (D $\circ$) and (C $\circ$) are obviously equivalent.

As an example, we consider the following relational definition of conjunction

$$c \text{ is a conjunction of } a \text{ and } b \;\leftrightarrow\; a, b/c \;\&\; c/a \;\&\; c/b,$$

where the metalinguistic condition "$a, b/c$ & c/a & c/b" on the right hand side formulates the standard introduction and elimination rules for conjunction in the form of consequence statements. Let us call this condition $\mathcal{R}_\wedge(c, a, b)$. Then the inferential definition of conjunction is

$$c /\!/ a \wedge b \;\leftrightarrow\; \mathcal{R}_\wedge(c, a, b),$$

and the characterizing rule is

$$\mathcal{R}_{\wedge}(a \wedge b, a, b).$$

Other definitions of conjunction can be given by choosing alternative metalinguistic conditions $\mathcal{R}$. We will see an example in Sect. 14.4.3.

14.4.2 Existence and Uniqueness in Inferential Definitions

What form might characterizing rules like $\mathcal{R}(a \circ b, a, b)$ be allowed to take? Should certain rules be disallowed because their use in a definition of form (D $\circ$) does not in fact define a logical constant, or does any rule $\mathcal{R}$ give rise to a definition of a logical constant? For Popper any characterizing rule which is equivalent to an inferential definition characterizes a logical constant. He calls such rules *fully characterizing* (cf., e.g., Popper 1948c, § VI):

(A) A rule $\mathcal{R}$ characterizing an operation $\circ$ is called *fully characterizing* if and only if it is equivalent to an inferential definition of $\circ$.

If $\mathcal{R}$ is given in the form $\mathcal{R}(a \circ b, a, b)$, this means uniqueness of $\mathcal{R}$ for its first argument. For Popper uniqueness is essential for any definition of a logical constant (cf. Popper 1948c, p. 324). That is, we have

(B) A rule of the form $\mathcal{R}(c, a_1, \ldots, a_n)$ is fully characterizing if and only if

$$(\mathcal{R}(a, a_1, \ldots, a_n) \;\&\; \mathcal{R}(b, a_1, \ldots, a_n)) \rightarrow a /\!/ b.$$

In other words, if and only if a rule $\mathcal{R}$ characterizes a statement c up to mutual deducibility, then $\mathcal{R}$ is fully characterizing c.

We distinguish between the definition (A) and the fact (B) because not every characterizing rule has the form $\mathcal{R}(c, a_1, \ldots, a_n)$ and can thus be used in a relational definition. Only logical constants, which are *fully* characterized, are always definable by rules of the form $\mathcal{R}(c, a_1, \ldots, a_n)$.

What about the existence requirement? The inferential definition (D $\circ$) is a proper definition only if, in addition to uniqueness, there exists a connection of a and b in the object language considered, for which we define a name. We may, of course, stipulate that there be a connection of a and b and even denote it by $a \circ b$, but this is nothing an inferential definition can do by itself without becoming creative. Unique connections must be there before we can single them out by means of an inferential definition. Popper is aware of this point. The existence of fully characterizing rules is Popper's *criterion of logicality*.

Forcing an operation to exist in an object language can make the object language inconsistent. However, if it is uniquely characterized, it is a logical constant. For example, Popper (1947a, p. 284) allows the following definition for "the opponent of a statement" (*opp*), with its characterizing rule:

$$a//opp(b) \leftrightarrow (c)(b/a \;\&\; a/c), \qquad \text{(D } opp\text{)}$$

$$(c)(b/opp(b) \;\&\; opp(b)/c). \qquad \text{(C } opp\text{)}$$

This obviously trivializes any system, since it implies $(c)(b/c)$ for any b. But this does not lead Popper to reject (D *opp*) as a definition. In a system, where *opp* exists (thus an inconsistent system), it is a logical constant because it is unique for trivial reasons. Historically, it is interesting to note that the connective *opp* is quite similar to the connective *tonk*, which was later introduced into the philosophical discussion by Prior (1960), with the intention to discredit inferentialism.

14.4.3 Inferential Definitions of Some Connectives

In order to illustrate Popper's special theory of deduction, we present inferential definitions and their characterizing rules for some standard connectives, now using the defined notion of relative demonstrability $\vdash$. We already discussed one possibility to define conjunction. Another possible definition of conjunction is the following, which can be dualized by swapping the left and right hand sides of $\vdash$ to obtain the shown definition of disjunction. Popper often uses duality in this way as it also allows him to dualize logical constants in the context of non-classical logics.

Conjunction $\wedge$:

$$a//b \wedge c \leftrightarrow (d)(a \vdash d \leftrightarrow b, c \vdash d), \qquad \text{(D} \wedge\text{)}$$

$$b \wedge c \vdash d \leftrightarrow b, c \vdash d. \qquad \text{(C} \wedge\text{)}$$

Disjunction $\vee$:

$$a//b \vee c \leftrightarrow (d)(d \vdash a \leftrightarrow d \vdash b, c), \qquad \text{(D} \vee\text{)}$$

$$d \vdash b \vee c \leftrightarrow d \vdash b, c. \qquad \text{(C} \vee\text{)}$$

Conditional $>$:

$$a//b > c \leftrightarrow (d)(d \vdash a \leftrightarrow d, b \vdash c), \qquad \text{(D} >\text{)}$$

$$d \vdash b > c \leftrightarrow d, b \vdash c. \qquad \text{(C} >\text{)}$$

Classical negation $\neg_k$:

$$a//\neg_k b \leftrightarrow (a, b \vdash \;\&\; \vdash a, b), \qquad \text{(D} \neg_k\text{)}$$

$$\neg_k b, b \vdash \;\&\; \vdash \neg_k b, b. \qquad \text{(C} \neg_k\text{)}$$

Unary tautology t:

$$a//t(b) \leftrightarrow (c)(b/a \leftrightarrow c/a), \qquad \text{(D } t\text{)}$$

$$(c)(b/t(b) \leftrightarrow c/t(b)). \qquad \text{(C } t\text{)}$$

Unary contradiction f:

$$a//f(b) \leftrightarrow (c)(a/b \leftrightarrow a/c), \qquad \text{(D } f\text{)}$$

$$(c)(f(b)/b \leftrightarrow f(b)/c). \qquad \text{(C } f\text{)}$$

For the last two unary connectives we have, for any a and b, $t(a)/\!/t(b)$ and $f(a)/\!/f(b)$. Thus one can simply write t and f. In other words, these connectives correspond to the nullary constants verum and falsum, and they will be used in this way in Sect. 14.4.5, (PF4).

In the context of non-classical logics, Popper inferentially defines, for example, several kinds of non-classical negations, such as:

Intuitionistic negation $\neg_i$:

$$a/\!/\neg_i b \leftrightarrow (c)(c \vdash a \leftrightarrow c, b \vdash), \tag{D $\neg_i$}$$

$$c \vdash \neg_i b \leftrightarrow c, b \vdash . \tag{C $\neg_i$}$$

Dual-intuitionistic negation $\neg_m$:

$$a/\!/\neg_m b \leftrightarrow (c)(a \vdash c \leftrightarrow \vdash c, b), \tag{D $\neg_m$}$$

$$\neg_m b \vdash c \leftrightarrow \vdash c, b. \tag{C $\neg_m$}$$

Both intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$ are logical constants, since their rules are fully characterizing. Popper observes that by adding classical negation $\neg_k$ to a language containing both intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$ one obtains

$$\neg_k a/\!/\neg_i a, \quad \neg_k a/\!/\neg_m a \quad \text{and} \quad \neg_i a/\!/\neg_m a.$$

That is, the addition of $\neg_k$ is a non-conservative language extension in this case.

14.4.4 From Inferential Definitions to Logical Laws

Given inferential definitions of certain logical constants, we immediately obtain from them basic laws for these constants, namely the right hand side $\mathcal{R}$ of the inferential definitions with the logically composed statement inserted for the leftmost variable.

For example, from the inferential definition of conjunction

$$c/\!/a \wedge b \leftrightarrow \mathcal{R}_\wedge(c, a, b)$$

with $\mathcal{R}_\wedge(c, a, b)$ being

$$a, b/c \;\&\; c/a \;\&\; c/b$$

we obtain $\mathcal{R}_\wedge(a \wedge b, a, b)$ by inserting $a \wedge b$ for c, that is, the introduction and elimination rules for conjunction:

$$a, b/a \wedge b \qquad\qquad a \wedge b/a \qquad a \wedge b/b.$$

More generally, for any inferential definition of an n-place propositional operator $*(a_1, \ldots, a_n)$ of the form

$$c /\!/ *(a_1, \ldots, a_n) \leftrightarrow \mathcal{R}_*(c, a_1, \ldots, a_n) \tag{D $*$}$$

we obtain the inference rules

$$\mathcal{R}_*(*(a_1, \ldots, a_n), a_1, \ldots, a_n)$$

that govern $*$ by substituting $*(a_1, \ldots, a_n)$ for c. As this instantiation is immediate and yields inference rules in a trivial way, Popper may speak of the "trivialization of logic", namely the metalinguistic deduction of valid inference rules immediately from a definition. *Given* the inferential definitions of logical constants in the form (D $*$), obtaining their governing rules is *trivial*.

This is, of course, also the case for inferential definitions which are formulated in terms of relative demonstrability $\vdash$, such as (D $\wedge$) and (D $\vee$), for example. For

$$a /\!/ b \wedge c \leftrightarrow (d)(a \vdash d \leftrightarrow b, c \vdash d) \tag{D $\wedge$}$$

we consider the substitution instance

$$a \wedge b \vdash a \leftrightarrow a, b \vdash a$$

of (C $\wedge$). The right hand side $a, b \vdash a$ holds by (Rg), hence $a \wedge b \vdash a$, from which we obtain the conjunction elimination rule $a \wedge b/a$. Similarly, one can show $a \wedge b/b$ and the introduction rule $a, b/a \wedge b$. In the case of

$$a /\!/ b \vee c \leftrightarrow (d)(d \vdash a \leftrightarrow d \vdash b, c) \tag{D $\vee$}$$

we consider the substitution instance

$$a \vee b \vdash a \vee b \leftrightarrow a \vee b \vdash a, b$$

of (C $\vee$). The left hand side $a \vee b \vdash a \vee b$ holds by (Rg), hence $a \vee b \vdash a, b$. The definition of $\vdash$ gives us the disjunction elimination rule

$$(c)((a/c \,\&\, b/c) \rightarrow a \vee b/c).$$

The disjunction introduction rules $a/a \vee b$ and $b/a \vee b$ follow in a similar way.

14.4.5 The Theory of Quantification

For propositional logic it was sufficient to consider pairs $(\mathcal{L}; a_1, \ldots, a_n/b)$ of an object language $\mathcal{L}$ and a deducibility relation / with a basis consisting of the rules (Rg) and (Tg), where each element of $\mathcal{L}$ is a statement, that is, something which has a truth value. Popper's first step in extending his framework of inferential definitions to a theory of quantification is to consider instead quadruples

$$(\mathcal{L}; \mathcal{P}; a_1, \ldots, a_n/b; a\tbinom{x}{y}),$$

which consist of a set $\mathcal{L}$ of *formulas*, a set $\mathcal{P}$ of *name-variables* (or pronouns), a *deducibility relation* / on $\mathcal{L}$ and a *substitution operation*

$$a\tbinom{x}{y}$$

which substitutes the name-variable y for the name-variable x in the formula a. Variables $a, b, \ldots$ now range over *formulas* in $\mathcal{L}$, and variables $x, y, \ldots$ range over name-variables in $\mathcal{P}$. Formulas can either be *open statements* (also called *statement-functions*) or *closed statements* (also called *statements*). Popper explicitly remarks that open statements do not have a truth value on their own. The deducibility relation / is axiomatized by the same rules (Rg) and (Tg) as in the case of propositional logic, but it now ranges over arbitrary formulas, not just closed statements.

The new substitution operation $a\tbinom{x}{y}$ is characterized by the following postulates (PF1) to (PF4) (which we present in a slightly simplified form) and by the six primitive rules of inference (S1) to (S6) given below.

$$\mathcal{L} \cap \mathcal{P} = \emptyset. \tag{PF1}$$

$$\text{If } a \in \mathcal{L} \text{ and } x, y \in \mathcal{P}, \text{ then } a\tbinom{x}{y} \in \mathcal{L}. \tag{PF2}$$

$$\text{For all } a \in \mathcal{L} \text{ there exists an } x \in \mathcal{P} \text{ such that for all } y \in \mathcal{P}\text{: } a\tbinom{x}{y}//a. \tag{PF3}$$

$$\text{There exist } a \in \mathcal{L} \text{ and } x, y \in \mathcal{P}\text{: } a/a\tbinom{x}{y} \to t/f. \tag{PF4}$$

Note that two kinds of metalinguistic quantifiers are used here: there is a universal and an existential quantifier ranging over statements $a \in \mathcal{L}$ and a universal and an existential quantifier ranging over name-variables $x \in \mathcal{P}$. For better readability, we do not use symbols for them, but write "for every a", "for every x" etc.

The postulates (PF1) and (PF2) are, in a way, only about the correct grammatical use of formulas and name-variables. The postulate (PF3) says that for every formula there is some name-variable not occurring in it. This is obvious if the set of name-variables is considered to be infinite, and if each formula is a finite object which can only mention a finite number of name-variables. The postulate (PF4), which

Popper considers to be optional, excludes degenerate systems in which only one object exists.

The six primitive rules of inference are as follows. They are valid for a concrete formalized object language and a substitution operation for that language.

$$\text{If, for every } z,\ a /\!/ a\binom{y}{z} \text{ and } b /\!/ b\binom{y}{z}, \text{ then } a /\!/ b \to a\binom{x}{y} /\!/ b\binom{x}{y}. \tag{S1}$$

$$a /\!/ a\binom{x}{x}. \tag{S2}$$

$$\text{If } x \neq y, \text{ then } (a\binom{x}{y})\binom{x}{z} /\!/ a\binom{x}{y}. \tag{S3}$$

$$(a\binom{x}{y})\binom{y}{z} /\!/ (a\binom{x}{z})\binom{y}{z}. \tag{S4}$$

$$(a\binom{x}{y})\binom{z}{y} /\!/ (a\binom{z}{y})\binom{x}{y}. \tag{S5}$$

$$\text{If } w \neq x,\ x \neq u \text{ and } u \neq y, \text{ then } (a\binom{x}{y})\binom{u}{w} /\!/ (a\binom{u}{w})\binom{x}{y}. \tag{S6}$$

The rules (S1) to (S6) characterize substitution as a structural operation. They cannot be brought into the form of an inferential definition of an operator of the object language. Hence, substitution cannot have the status of a logical constant according to Popper's criterion of logicality. Popper (1947c, p. 1216) writes: "The notation '$a\binom{x}{y}$' will be used as a (variable) metalinguistic name of the statement which is the result of substituting, in the statement a (open or closed), the variable y for the variable x, wherever it occurs. $a\binom{x}{y}$ is identical with a if x does not occur in a." Popper's rules for substitution may thus be viewed as an implicit characterization of a metalinguistic operation, and not as an inferential definition of a logical constant for object languages. It is worth noting that Popper gives an axiomatic characterization of the substitution operation. It can be compared to the theory of explicit substitution, which was developed much later (cf., e.g., Abadi et al. 1991).

In the next step two auxiliary concepts are defined with the help of both the deducibility relation and the substitution operation. If we work with some inductively defined formal object language, then we can easily specify the set of free variables of a formula by recursion on the structure of that formula. This possibility is excluded in Popper's approach, which is not restricted to formal languages. Popper therefore introduces the expression

$$a_{\hat{x}}$$

which can be read as "x does not occur among the free variables in a". Popper himself expresses this as "a does not depend on x", "*a-without-x*" and "x does not occur relevantly in a". The formula a does not depend on x if and only if substitution of some name-variable y for x does not change the logical strength of a. That is,

$$a /\!/ a_{\hat{x}} \;\leftrightarrow\; \text{for every } y\text{: } a /\!/ a\binom{x}{y}. \tag{D $a_{\hat{x}}$}$$

The second concept that Popper defines with the help of deducibility and substitution is *identity*. As Popper (1947b, p. 227f., fn 24) notes, one first has to extend the object language $\mathcal{L}$ to incorporate formulas of the form $Idt(x, y)$; this is achieved by the postulate

$$\text{If } x \text{ and } y \text{ are name-variables, then } Idt(x, y) \text{ is a formula.} \qquad \text{(P } Idt\text{)}$$

In addition, the characterizing rules for substitution have to be extended by rules of the following form:

$$(Idt(x, y))\binom{x}{z} /\!/ Idt(z, y).$$

$$(Idt(x, y))\binom{y}{z} /\!/ Idt(x, z).$$

$$\text{If } x \neq u \neq y, \text{ then } Idt(x, y)\binom{u}{z} /\!/ Idt(x, y).$$

With these preliminaries, Popper defines identity $Idt(x, y)$ in accordance to the idea that $Idt(x, y)$ should be the weakest statement strong enough to satisfy

$$Idt(x, y), a(x)/a(y)$$

as follows:

$$a /\!/ Idt(x, y) \leftrightarrow \text{(for every } b \text{ and } z\text{: } ((b /\!/ b_{\grave{x}} \;\&\; b /\!/ b_{\grave{y}}) \rightarrow a, b\tbinom{z}{x}/b\tbinom{z}{y}) \;\&$$
$$((\text{for every } c \text{ and } u\text{: } ((c /\!/ c_{\grave{x}} \;\&\; c /\!/ c_{\grave{y}}) \rightarrow b, c\tbinom{u}{x}/c\tbinom{u}{y}))) \rightarrow b/a)). \qquad \text{(D } Idt\text{)}$$

Finally, Popper (1947b) (with corrections and additions in Popper 1948e) gives inferential definitions for universal and existential quantification and derives some simple conclusions, without formally developing a meta-theory of quantification, since he only aims at showing that his approach is at least on a par with other proposed treatments of quantification.

Popper (1949, p. 725) expresses the intuition behind his inferential definition of universal quantification as follows: “The result of universal quantification of a statement a can be defined as the weakest statement strong enough to satisfy the law of specification, that is to say, the law ‘what is valid for all instances is valid for every single one’.”

Presupposing his rules of substitution, and writing Ax for the universal quantifier, the inferential definition and the characterizing rule for *universal quantification* are the following:

$$a_{\grave{y}} /\!/ Axb_{\grave{y}} \leftrightarrow \text{(for every } c_{\grave{y}}\text{: } c_{\grave{y}}/a_{\grave{y}} \leftrightarrow c_{\grave{y}}/b_{\grave{y}}\tbinom{x}{y}). \qquad \text{(D } Ax\text{)}$$

$$\text{For every } c_{\grave{y}}\text{: } c_{\grave{y}}/Axb_{\grave{y}} \leftrightarrow c_{\grave{y}}/b_{\grave{y}}\tbinom{x}{y}. \qquad \text{(C } Ax\text{)}$$

For clarification, we make a comparison with the more familiar rules for the universal quantifier ∀ of the (intuitionistic) sequent calculus, using the sequent sign ⊢ and writing $\varphi[x \mapsto y]$ for the result of substituting y for x in the formula φ, with the variable condition that y does not occur free in the conclusion of (⊢∀):

$$\frac{\Gamma, \varphi[x \mapsto t] \vdash \psi}{\Gamma, \forall x\varphi \vdash \psi}\ (\forall \vdash) \qquad\qquad \frac{\Gamma \vdash \varphi[x \mapsto y]}{\Gamma \vdash \forall x\varphi}\ (\vdash \forall).$$

By instantiating (C Ax) with $Axb\grave{y}$ and by using the rules (Tg) and (Rg) from the basis, we obtain the following rule

$$a, b\grave{y}\tbinom{x}{y}/c \to a, Axb\grave{y}/c,$$

which is a variant of the rule (∀⊢) where the name-variable y takes the role of the term t. Similarly, by instantiating (C Ax) with $c\grave{y}$ and reading the bi-implication from right to left we obtain the following rule, which corresponds to the rule (⊢∀) with the variable condition that y does not occur relevantly in c:

$$c\grave{y}/b\grave{y}\tbinom{x}{y} \to c\grave{y}/Axb\grave{y}.$$

The inferential definition of existential quantification corresponds to the following intuition (cf. Popper 1949, p. 725): "The result of existential quantification of the statement a can be defined as the strongest statement weak enough to follow from every instance of a." The inferential definition and the characterizing rule for the *existential quantifier* Ex are as follows:

$$a\grave{y}//Exb\grave{y} \ \leftrightarrow\ (\text{for every } c\grave{y} : a\grave{y}/c\grave{y} \ \leftrightarrow\ b\grave{y}\tbinom{x}{y}/c\grave{y}). \qquad (\text{D } Ex)$$

$$\text{For every } c\grave{y} : Exb\grave{y}/c\grave{y} \ \leftrightarrow\ b\grave{y}\tbinom{x}{y}/c\grave{y}. \qquad (\text{C } Ex)$$

In order to elucidate this definition, we make a comparison with the sequent calculus rules for the existential quantifier ∃, with the variable condition that y does not occur free in the conclusion of (∃⊢):

$$\frac{\Gamma, \varphi[x \mapsto y] \vdash \psi}{\Gamma, \exists x\varphi \vdash \psi}\ (\exists \vdash) \qquad\qquad \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x\varphi}\ (\vdash \exists).$$

Instantiating (C Ex) with $Exb\grave{y}$ and using the rules of the basis we can obtain the rule

$$a/b\grave{y}\tbinom{x}{y} \to a/Exb\grave{y},$$

which corresponds to (⊢∃); and by instantiating (C Ex) with $c_{\hat{y}}$ and reading the bi-implication from right to left, we obtain the following rule, which corresponds to (∃⊢):

$$b_{\hat{y}}\binom{x}{y}/c_{\hat{y}} \rightarrow Ex_{\hat{y}}/c_{\hat{y}}.$$

Popper does not consider the explicit definitions (D Ax) and (D Ex) to be improvements compared to the characterizing rules. They are given to show that universal and existential quantification can be defined using only his basis and the rules (S1) to (S6). He notices that these rules are not as simple as the rules of his basis. But he points out that the concept of "$a_{\hat{x}}$" can be avoided in these definitions (cf. Popper 1947b, p. 230, fn 26; added in the corrections and additions of Popper 1948e). Assuming $x \neq y$, one can use instead:

$$a\binom{y}{x}/Ax(b\binom{y}{x}) \leftrightarrow a\binom{y}{x}/b\binom{x}{y}, \qquad (Ax^*)$$

$$Ex(a\binom{y}{x})/b\binom{y}{x} \leftrightarrow a\binom{x}{y}/b\binom{y}{x}, \qquad (Ex^*)$$

$$a\binom{y}{x}/\!/Ax(b\binom{y}{x}) \leftrightarrow \text{(for every } c\text{: } c\binom{y}{x}/a\binom{y}{x} \leftrightarrow c\binom{y}{x}/b\binom{x}{y}\text{)}, \qquad (\text{D } Ax^*)$$

$$a\binom{y}{x}/\!/Ex(b\binom{y}{x}) \leftrightarrow \text{(for every } c\text{: } a\binom{y}{x}/c\binom{y}{x} \leftrightarrow b\binom{x}{y}/c\binom{y}{x}\text{)}. \qquad (\text{D } Ex^*)$$

Popper (1947b, § 7) considers his rules of quantification to be less complicated than those given by Hilbert and Ackermann (1928) or those given by Quine (1940, § 15), and he emphasizes that his rules in the end make use of only one logical concept, namely that of deducibility / as characterized by his basis.

14.5 Summary

Popper claims to lay "new foundations for logic", which also represent a "trivialization of mathematical logic" (cf. Popper 1947b, 1949). We saw that his approach to logic is based on the notion of deducibility $a_1, \ldots, a_n/b$, where by deducibility he does not mean the derivability in a formal system, but the semantical notion of logical consequence. Providing new foundations for logic thus means developing a theory of deducibility or logical consequence in a novel way. Tarski's notion of logical consequence is based on the idea of truth transmission: b follows logically from $a_1, \ldots, a_n$ if every interpretation which makes $a_1, \ldots, a_n$ true, makes b true as well. As Tarski had pointed out, this definition hinges on the definition of what a "logical constant" or "logical sign" is. An interpretation that makes $a_1, \ldots, a_n$ true and carries this truth over to b, can give all non-logical expressions in $a_1, \ldots, a_n, b$ an arbitrary meaning, while the meaning of the logical signs must be constant. Therefore, so one can argue, by providing a satisfying definition of what a logical constant is, we obtain a satisfying theory of logical consequence and thus of deducibility, given that the notion of truth itself is not problematic and is sufficiently clarified by Tarski's theory of truth. This situation is the systematic starting point of Popper's investigations, in particular of Popper (1947b).

There are nevertheless certain consequence laws which are independent of logical constants. These are the laws constituting a finite *consequence relation* in Tarski's sense. In a proof-theoretic setting, Gentzen (1935a,b) called them *structural rules* ("Struktur-Schlussfiguren"), where "structural" means independent of the "logical" form of the statements involved. Popper calls them "absolutely valid" since for their validity we need only be able to distinguish sentences from non-sentences, disregarding the internal structure of sentences. These absolutely valid rules are captured by what Popper calls a *basis*. As the validity of these rules is unproblematic, we can assume that, whenever we are dealing with deducibility, it is given as a finite consequence relation. Tarski's (1936) consideration of consequences from an infinite number of assumptions goes beyond Popper's finite proof-theoretic framework, which is in this respect more related to the work of Gentzen (1935a,b) and Hertz (1929), as was already pointed out by contemporary reviewers (cf. Beth 1948; Curry 1948a,b, 1949; Hasenjaeger 1949; Kleene 1949; McKinsey 1948). It is not clear to what extent, if at all, Popper was aware of these works when he conceived his approach.

In spite of this Tarskian motivation and starting point, the idea to define the logicality of operations and thus logical consequence for complex statements leads Popper to develop a conception entirely different from Tarski's. This conception has at least three central characteristics.

First, it is not assumed that a specified formal object language is given, in which logical operations are represented by functional expressions ("sentential functions"), which combine one or more sentences to a compound sentence or, in the quantifier case, operate on open statements. Whereas Tarski's approach of truth transmission under preservation of logical structure assumes that such a specification is given.

Second, logical operations are relationally and not functionally characterized. This relational view can leave open whether a logical operation always exists and is uniquely determined. Both features would have to be presupposed if one preferred a functional characterization of logical operations as implicit in standard logical notation. Tarski, in considering a structure based on logical constants, sticks to a functional view from the very beginning.

Third, the relational characterization of logical operations proceeds in terms of deducibility. For Popper, results of logical operations are characterized by their deductive or inferential behaviour. This turns the project of justifying deducibility upside down as compared to Tarski. Since we rely on a notion of consequence in the relational definition of logical operations, we can no longer use this notion of logical operation to define the validity of consequences along Tarskian lines. In fact, such a definition of validity in terms of truth transmission becomes obsolete, as deducibility must already be available at the level of logical operations. Thus the notions of truth and truth transmission do not play a role any more in the justification of logical inference. They are discarded in favour of deducibility as a primitive notion. Logical rules are set up and explained in an inferentialist framework without any recourse to truth.

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