The Logicality of Equality

Andrzej Indrzejczak

Abstract The status of the equality predicate as a logical constant is problematic. In the paper we look at the problem from the proof-theoretic standpoint and survey several ways of treating equality in formal systems of different sorts. In particular, we focus on the framework of sequent calculus and examine equality in the light of criteria of logicality proposed by Hacking and Došen. Both attempts were formulated in terms of sequent calculus rules, although in the case of Došen it has a nonstandard character. It will be shown that equality can be characterised in a way which satisfies Došen's criteria of logicality. In the case of Hacking's approach the fully satisfying result can be obtained only for languages with a nonempty, finite set of predicate constants other than equality. Otherwise, cut elimination theorem fails to hold.

Key words: equality, logical constants, sequent calculus

1 Introduction

It is difficult to find serious applications of logic that do not use equality. Not only it is necessary for development of mathematical theories but it plays also an important role in philosophical applications. Yet it is problematic to show that it is a logical constant having a similar behaviour to undisputable cases like extensional connectives or quantifiers. In this paper we try to look at the problem from the proof-theoretic perspective and ask if it is possible to characterize equality by means of rules satisfying some of the proposed criteria of logicality. For simplicity's sake we restrict our considerations to classical first-order logic (FOL) although obtained results may be easily transmitted to intuitionistic logic (see concluding remarks). The criteria which will be examined with respect to equality are those proposed by Hacking (1979) and Došen (1989), and the framework for our considerations is provided by Gentzen's

Andrzej Indrzejczak
Department of Logic, University of Łódź, Poland, e-mail: andrzej.indrzejczak@filhist.uni.lodz.pl

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sequent calculus (SC). This is partly determined by the fact that both approaches to criteria of logicality were proposed in this framework, although in the case of Došen it was not a standard variant of SC. Moreover, the framework of SC seems to be particularly well suited for investigations concerning the problems of criteria for logical constants and in general for investigations in proof-theoretic semantics (see, e.g., Schroeder-Heister 2016).

It will be convenient to start our considerations with some general remarks concerning equality since the problem of its logical status begins with the proper understanding of what the equality predicate stands for. It is tacitly and commonly assumed that a binary predicate, usually symbolised as =, is introduced to formal languages as a characterization of the identity relation. In fact, the words 'identity' and 'equality' are often treated as synonymous by a majority of mathematicians and logicians (not excluding the author). Usually, it does not lead to any problems, but in the case where equality is itself an object of study, we should be more careful. Therefore, we prefer to follow here such authors like Manzano and Moreno (2017) or Kahle (2016), in keeping a strict distinction between identity and equality. Identity is a relation between objects and equality between terms. The former is a semantic relation that holds trivially only between an object and itself, whereas the latter is a syntactical relation which may hold between any terms of the language. It is natural to postulate that the equality predicate expresses in the language the identity of objects denoted by its arguments, however, there are serious problems hidden in such identification. First of all it can be even doubtful if identity is a genuine relation, and if so, if it should be represented by some binary predicate. Wittgenstein (1922) presented such a view in his rejection of the very symbol of equality from his language. In fact, Wittgenstein's view can be formally developed in an interesting way, as was shown by Hintikka (1956) and Wehmeier (2014).

Even if we follow a standard practice of treating identity as a relation, one must be aware that the kind of the correspondence between the equality predicate and the identity relation is not very strict. In model-theoretic terms identity is just a diagonal relation on the product of the domain of a model. But the equality predicate, as characterised in axiomatic systems of first-order logic (see Section 3) cannot express identity only. Even in the case of a language with a finite number of predicates, one can find nonstandard models in which axioms of equality do not characterise identity. It seems that the second-order logic (SOL) is better in this respect. Well, if we admit that the second-order logic is a genuine logic we can define identity in terms of equivalence and the second-order universal quantification, by means of Leibniz' axiom (see Section 3). But the second-order logic is expressive enough to capture Peano's Arithmetic so only the logicist position makes this argument unproblematic. Moreover, this holds only in the standard semantics for which SOL is not complete. If we take Henkin's generalised models to regain completeness, we can again find models where Leibniz' axiom does not determine identity (see, e.g., Manzano, 2005).

Despite the deficiency of equality as a definition of identity (in FOL in particular) it does make sense to check if equality itself may be conceived as a logical constant and this is our aim. In Section 2 we establish the notation and recall the basic information on sequent calculi and properties of rules which are important for this task. Several ways of formalising equality in axiomatic and natural deduction systems are surveyed in Section 3. Ways of dealing with equality in sequent calculi are discussed in a separate section. In Section 5 we recall criteria of logicality formulated by Hacking and check if equality can be formalised in a way conforming to these desiderata. It appears that most of the criteria hold for our proposal but cut elimination is sensitive to the kind of language under consideration. In Section 6 we describe Došen's approach and a nonstandard structural variant of sequent calculus adapted to its realization. It seems that equality formalised in this kind of a system satisfies fully the conditions for logical constant but only if equality is not the only predicate of the language. We finish with some remarks concerning further applications and possible generalizations of presented approach.

2 Preliminaries

The notation applied in the paper is mostly standard. φ, ψ, χ will represent arbitrary formulae built by means of $\neg, \wedge, \vee, \rightarrow, \forall, \exists$ from atomic formulae, i.e., predicates followed by a list of terms. Following Gentzen's custom, we distinguish between bound and free occurrences, reserving $x, y, z, \ldots$ for representing the former and $a, b, c, \ldots$ for the latter, usually called parameters. Nothing essential depends on this distinction, although it simplifies a definition of substitution for terms. Other terms, if any, will be constructed from function symbols of any arity. We will use f, g, h and metalevel θ^n for their representation. Arbitrary terms will be represented as $\tau_1, \tau_2, \ldots$. Predicates will also be divided into parameters (schematic symbols) and predicate constants of specific languages determining their signature. Predicates of both categories will be represented either by A, R or, in the metalevel, by π^n. Incidentally, X will be used for bound (predicate) variable of the second-order and. $\varphi(\tau)$ denotes a fomula having at least one occurrence of τ and $\varphi[x/\tau]$ the result of the correct substitution of τ for all free occurrences of x. $\Gamma, \Delta, \Sigma, \ldots$ represent finite multisets of formulae. Eventually = will be used as a symbol of equality. In general, formulae of the form $\tau_1 = \tau_2$ are not counted as atomic, since = is considered as a logical constant.

Following Church's (1956) terminology we distinguish the following language variants of FOLI (FOL with identity):

1. pure FOLI with predicate and term variables/parameters and equality as the only predicate constant;
2. applied FOLI with additional other predicate constants;
3. simple applied FOLI with no predicate parameters (only constants);
4. simple FOLI with no predicate parameters and constants except equality.

Thus pure FOLI is just a schematic version of FOL with equality whereas several cases of simple applied FOLIs are specific languages characterised by their signatures. For example the language of simple applied FOLI of set theory has two binary predicate constants: = and $\in$. The cases of applied FOLI are of mixed character since in addition to constants they admit variables/parameters. This classification is essential for comparison of different possible characterisations of equality.

As our basic sequent calculus (SC) for classical logic FOL we will use a system which is essentially Gentzen's LK but with sequents built from multisets to avoid inessential complications. We also prefer to present all two-premiss rules in the multiplicative (or with independent contexts) version in contrast to Gentzen's original rules for $\wedge, \vee$. Again nothing essential hinges on this choice and other variants of SC can be also applied. The calculus consists of the following structural and logical rules:

$$(AX)\quad \varphi \Rightarrow \varphi \qquad\qquad (Cut)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(W\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(C\Rightarrow)\quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow C)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\neg\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\neg)\quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge\Rightarrow)\quad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Pi \Rightarrow \Sigma, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi\wedge\psi}$$

$$(\vee\Rightarrow)\quad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi\vee\psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad (\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi\vee\psi}$$

$$(\rightarrow\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi\rightarrow\psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad (\Rightarrow\rightarrow)\quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\rightarrow\psi}$$

$$(\forall\Rightarrow)\quad \frac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\forall)^1\quad \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

where a is not in Γ, Δ, φ

$$(\exists\Rightarrow)\quad \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\exists)\quad \frac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists x\varphi}$$

where a is not in Γ, Δ, φ

Formulae displayed in the schemata are active whereas those in (possibly empty) multisets Γ, Δ are parametric (or form the context). In particular, a unique formula in the antecedent or succedent of the conclusion is the principal formula of the respective rule application whereas active formulae in the premisses are called side-formulae. The notion of a proof is standard, i.e., a tree labelled with sequents where each leaf is an axiom and edges are regulated by the rules. The height of a proof is the number of nodes in maximal branches. For stating criteria of logicality it is important to focus on some of the characteristic features of the logical rules. First of all they are rules of introduction of a constant, either to the antecedent or to the succedent of a sequent. Moreover, using a terminology of Wansing (1999) (see also Poggiolesi, 2011), we can observe that well-behaved rules have the following properties:

1. Separation: a rule for a constant should not exhibit any other constants in its schema.
2. Weak symmetry: each rule should either introduce a constant to the antecedent or to the succedent; if both rules for a constant are of such a kind, the calculus is (simply) symmetric with respect to this constant.
3. Weak explicitness: a constant should be present only in the conclusion; if only one occurrence of it is present a rule is (simply) explicit.
4. Subformula property: only formulae present in the conclusion and their subformulae (modulo substitution of terms) are present in the premisses.

Rules satisfying these properties are also called canonical by Avron (2001). In what follows we tacitly assume that candidates on rules characterising a logical constants should have these features or some reasonable generalizations of them. But these are considered only as necessary conditions, for sufficiency we will examine additional requirements formulated by Hacking and Došen.

3 Approaches to equality

As we remarked our main tool will be SC but it is profitable to recall first how equality was (and is) usually dealt with in the framework of other proof systems, in particular, in axiomatic or natural deduction systems (ND). In the philosophical considerations we can often find a reference to the traditional characterisation of equality due to Leibniz. This approach may be formally presented as a formula of the second-order logic (SOL) which we call LA (Leibniz Axiom):

$$\tau_1 = \tau_2 \leftrightarrow \forall X(X\tau_1 \leftrightarrow X\tau_2).$$

Commonly, the left-right implication is called the principle of indiscernibility of identicals, whereas the converse is called the principle of identity of indiscernibles[1]. The latter principle implies immediately reflexivity of = whereas full LA is required to prove both symmetry and transitivity of = as implied by symmetry and transitivity of $\leftrightarrow$. Note also that LA may be weakened in two senses: (a) X may be restricted to atomic predicates (b) the rightmost equivalence may be replaced with implication. The restriction (b) is not independent from (a); LA is derivable from the weaker form (b) if we admit that not only predicates but complex formulae (of FOL) may be instantiated for X (see the proofs provided by Read, 2004 or Parlamento and Previale, 2019). This is perhaps not in conflict with the original intuitions of Leibniz since he seems to consider every context where respective terms may be exchanged salva veritate. On the other hand, in order to prove that equality is symmetric in restricted case (b) we must instantiate X with equalities. But this solution shows that LA in restricted form (b) cannot be treated as a definition of identity since it is either incomplete or circular. In the former case to obtain the full characteristics of identity

[1] Although some doubts may be raised against the correctness of the identification of these traditionally considered principles with this formula of SOL, see, e.g., Mates (1986).

we must add explicitly the condition of symmetry. Below we will show that using restricted form (b) of LA leads to other inadequacies as well.

On the other hand, restriction (a), i.e., restriction to atomic predicates as admissible instances of X, has some merits in the case of simple applied languages which was noted by Quine (1970). We can change the 'definiens' into conjunction of all possible cases. For example, if our language has one unary primitive predicate constant A and one binary R it takes the form:

$$\text{(LA}'\text{)} \qquad \tau_1 = \tau_2 \leftrightarrow ((A\tau_1 \leftrightarrow A\tau_2) \wedge \forall x((R\tau_1 x \leftrightarrow R\tau_2 x) \wedge (Rx\tau_1 \leftrightarrow Rx\tau_2)).$$

We already mentioned that this is not sufficient to obtain a real definition of identity in FOL but it can work as good stipulation of identity in the case of simple applied FOLI, i.e., languages with finite number of predicate constants.

If we restrict our considerations to FOL, a characterisation of equality in terms of LA is of no use in the case of pure or applied versions of language but still may have some heuristic value. In particular, on the ground of Hilbert systems one may distinguish two approaches which we call algebraic and Leibnizian. In the former, equality is characterised simply as a congruence on terms so we need to state first that it is an equivalence relation expressed by:

1. reflexivity axiom R: $\forall x(x = x)$;
2. symmetry axiom SYM: $\forall xy(x = y \rightarrow y = x)$
3. transitivity axiom TR: $\forall xyz(x = y \wedge y = z \rightarrow x = z)$.

This is enough for simple FOLI; in the case of simple applied versions of FOLI we must add two principles of congruence for every primitive atomic predicate and term:

1. Congruence of Predicates CP:

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n(x_1 = y_1 \wedge \ldots \wedge x_n = y_n \rightarrow$$
$$(\pi^n(x_1, \ldots, x_n) \rightarrow \pi^n(y_1, \ldots, y_n)),$$

where π^n is n-argument predicate symbol.

2. Congruence of Terms CT:

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n(x_1 = y_1 \wedge \ldots \wedge x_n = y_n \rightarrow$$
$$\theta^n(x_1, \ldots, x_n) = \theta^n(y_1, \ldots, y_n)),$$

where θ^n is n-argument function symbol.

This way of characterising equality is particularly elegant if we deal with simple applied first-order languages having only a small number of primitive predicate or function constants. It works better than characterization via LA′ since we obtain one axiom for every predicate instead of n equivalences for every n-ary predicate. Moreover, if we treat equality as a primitive atomic predicate it is not necessary for symmetry and transitivity of = to be explicitly added since they are provable by means of CP.

Authors dealing with pure or applied FOL, i.e., with predicate and function parameters usually prefer the latter approach which we called Leibnizian. It also requires reflexivity usually stated schematically as $\tau = \tau$ for every term, and the extensionality principle:

$$\text{(EP)} \qquad \forall xy(x = y \wedge \varphi[z/x] \rightarrow \varphi[z/y]),$$

where φ is arbitrary or atomic. The latter form is simpler to formulate since there is no problem with bound variables; moreover the general form is provable in extensional FOLI. In what follows we will keep the name EP for the version with arbitrary φ and call the version restricted to atoms Leibniz principle LP. One should note that it encodes one direction of LA, namely indiscernibility of identicals in weaker form (b), i.e., with $\leftrightarrow$ replaced by $\rightarrow$. It explains why we call this approach Leibnizian. One should also note that, contrary to ordinary custom, we defined EP (LP) by means of (correct) substitution of x, y for some free variable z. It is much more popular that it is characterised in terms of replacement:

$$\text{(EP}'\text{)} \qquad \forall xy(x = y \wedge \varphi \rightarrow \varphi[x//y]),$$

where $\varphi[x//y]$ denotes a replacement of some (not necessarily all) occurrences of x by y. It is perhaps intuitively more accessible but has some formal disadvantage since replacement is not an operation. To avoid this problem some authors define EP (LP) by means of a unique replacement:

$$\text{(EP}''\text{)} \qquad \forall xy(x = y \wedge \varphi(\ldots x \ldots) \rightarrow \varphi(\ldots y \ldots)),$$

where only one displayed occurrence of a variable (term) is taken into account. The last formulation is also simpler for arithmetization hence applied in the works dealing with Gödel's theorems. But it should be stressed that all these forms of characterization are equivalent. In particular, any possible application of EP$'$ is just a series of applications of EP$''$. Also SYM and TR are easily provable by any of these principles so it is not necessary to introduce them as primitive axioms.

Instead of REF we can find (for example in Tarski, 1941):

$$(\exists =) \qquad \exists x(x = \tau) \text{ where } x \text{ is not in } \tau.$$

This formula implies reflexivity by EP: $a = \tau \wedge a = \tau \rightarrow \tau = \tau$. In Mates (1965) the same characterization is applied but with universal closures of EP$'$ and $(\exists =)$.

There are also possible approaches which dispense with reflexivity, instead using just one formula which is equivalent to REF and EP. This approach is due to Wang (see Quine, 1966) and a dual axiom is due to Kalish and Montague (1957). On the ground of Hilbert system each one may be expressed by one axiom:

$$\exists x(x = \tau \wedge \varphi) \leftrightarrow \varphi[x/\tau], \text{ where } x \text{ is not free in } \tau$$

or

$$\forall x(x = \tau \rightarrow \varphi) \leftrightarrow \varphi[x/\tau], \text{ where } x \text{ is not free in } \tau.$$

In the framework of natural deduction (ND) the Leibnizian approach is prevalent although EP (LP) is usually presented as an inference rule of identity elimination:

(IE) $$\tau_1 = \tau_2, \varphi[x/\tau_1] \vdash \varphi[x/\tau_2],$$

whence reflexivity is treated as a zero-premiss rule of identity introduction:[2]

(II) $$\varnothing \vdash \tau = \tau.$$

Martin-Löf (1971) has shown that such a pair of rules is naturally induced for equality predicate in the general ND framework which satisfies normalization.

Kalish and Montague (1964) provide an ND system where the pair of rules is of the form:

(IE′) $\forall x(x = \tau \rightarrow \varphi) \vdash \varphi[x/\tau]$, where x is not free in τ,

(II′) $\varphi[x/\tau] \vdash \forall x(x = \tau \rightarrow \varphi)$, where x is not free in τ.

Both are clearly derived from the second axiom stated above.

There are also ND systems which operate not on formulae but on sequents of the form $\Gamma \vdash \varphi$, where Γ is a set (multiset, sequence) of active assumptions for φ[3]. In this setting rules for equality are formulated as follows:

(II″) $\Gamma \vdash \tau = \tau$,

(IE″) If $\Gamma \vdash \tau_1 = \tau_2$ and $\Delta \vdash \varphi[x/\tau_1]$, then $\Gamma, \Delta \vdash \varphi[x/\tau_2]$.

Finally let us point out the solution which is particularly important for our purposes. Read (2004) provides a rule of equality introduction as a proof construction rule of the form:

(RII) If $\Gamma, \varphi[x/\tau_1] \vdash \varphi[x/\tau_2]$, then $\Gamma \vdash \tau_1 = \tau_2$,
where φ is atomic and does not occur in Γ.

This rule is not sound in standard models for FOLI, however it is sound in so called Leibnizian models (see Read, 2004) and it may be shown that this class of models can equivalently characterise FOLI. Note that in the context of simple applied languages with a finite number of primitive predicates the corresponding result cannot be stated by means of such a rule; instead it may be stated by means of finite number of subproofs for each predicate constant. Something similar was proposed by Więckowski (2011) who provided a rule of the form:

(WRI) $$\varphi_1[x/\tau_1] \leftrightarrow \varphi_1[x/\tau_2], \ \ldots \ \varphi_n[x/\tau_1] \leftrightarrow \varphi_n[x/\tau_2] \ \vdash \ \tau_1 = \tau_2.$$

Note that here we have an inference rule since instead of subproofs we have a finite number of premisses. The drawback of this rule is that another constant, namely the

[2] Some authors add also inference rules corresponding to SYM and TR but of course they are derivable.

[3] The fact that we deal with sequents not with formulae is sometimes hidden since Γ consists not of formulae but of numbers referring to lines where assumptions were stated — see, e.g., Suppes (1957), Lemmon (1965).

equivalence connective, is present so the rule is not separate (see the previous section). It could be changed into a separate rule (i.e., with displayed equality only) by replacing each premiss $\varphi_i[x/\tau_1] \leftrightarrow \varphi_i[x/\tau_2]$ with a pair of subproofs $\varphi_i[x/\tau_1] \vdash \varphi_i[x/\tau_2]$ and $\varphi_i[x/\tau_2] \vdash \varphi_i[x/\tau_1]$. Note that it does not directly corresponds to Read's rule — the latter should have an additional subproof leading from $\varphi[x/\tau_2]$ (as an assumption) to $\varphi[x/\tau_1]$. The reasons why Read dispenses with the second subproof was explained above; WRI is based on LA (or rather LA′) whereas RII on LA with nonrestricted instantiation for X which enables replacement of equivalences by implication. Read's solution may be criticised not only from technical but also from philosophical point of view (see, e.g., Griffiths 2014, or Klev 2019) but deserves carefull examination. In what follows we will check how it works in the setting of SC.

4 Equality in sequent calculi

SC provides a framework which not only easily accomodates all approaches described so far but allows for several other solutions. An interesting thing is that in SC we can characterise equality not only by local rules but globally in the following way:

$$(\mathrm{SUB})\ \frac{\tau_1 = \tau_2, \Gamma[x/\tau_1] \Rightarrow \Delta[x/\tau_1]}{\tau_1 = \tau_2, \Gamma[x/\tau_2] \Rightarrow \Delta[x/\tau_2]} \qquad\qquad (\mathrm{REF})\ \Rightarrow \tau = \tau$$

where $\Gamma[x/\tau]$ denotes a uniform substitution of x by τ in all elements of Γ. Such solution was first introduced by Kanger (1957) but also proposed by Wang (1960) in the version where substitution is made only in Δ; this apparently weaker version is in fact sufficient. Essentially the same solution was used among others, by Mints (1968), Došen (1989), Seligman (2001), in several variants (for example with $\tau_1 = \tau_2$ only in the conclusion). Usually (SUB) is introduced in two variants where in the second we have $\tau_2 = \tau_1$, but it is redundant. Similar approach was also applied by Schroeder-Heister (1994) in the formalization of the free equality investigated in the setting of logic programming.

This form of introduction of equality is global, since we can treat a sequent as expressing a whole proof at some stage. Hence to imitate the application of this rules in ND we should rewrite the whole derivation. So in fact it is global in comparison with the solution proposed in ND setting. In SC this is reduced to the operation performed on the sequent not on the whole derivation. Such an approach has obvious virtues. One may easily prove everything which is needed to show that it is adequate; we obtain a proof of LP immediately. Moreover, cut elimination holds for it (see, e.g., Seligman, 2001 for a constructive proof; the version of Kanger is just a cut-free variant of G3). It is also worth noting that it was the most influential approach in automated theorem proving based on SC[4]. However, it seems that this approach is not fully convincing as a way for justifying equality as a logical constant. Even in the form proposed by Došen (see Section 6) equality is presented as something which

[4] Degtyarev and Voronkov (2001) present it as the only SC-based approach to equality formalization in automated deduction.

affects the whole sequent and looks like something of slightly different character from other logical constants which are characterised by local rules.

Other approaches are of local character and can be divided according to the possible ways of formalizing theories in the framework of SC. Negri and von Plato (2001) described four approaches to this question:

1. Addition of axiomatic sequents $\Rightarrow \varphi$ for each axiom φ.
2. Addition of “mathematical basic sequents” which consist of atomic formulae.
3. Addition of all axioms as a context in the antecedents of all provable sequents.
4. Addition of new rules corresponding to axioms.

In every class we can find SC formalizations of equality. The first approach may be called the "naive" since it treats SC in the same way as Hilbert system and does not refer to its specific features. Not surprisingly such a solution obstructs the application of specific virtues of SC; in particular cut elimination cannot be proved.

The second approach may be seen as a refinement of the first and was already applied by Gentzen in his formalization of Peano arithmetic. Restricted to equality it leads to addition of two axiomatic (atomic) sequents of the form:

$$\Rightarrow \tau = \tau$$

$$\tau_1 = \tau_2, \varphi[x/\tau_1] \Rightarrow \varphi[x/\tau_2] \text{ with } \varphi \text{ atomic}$$

In Takeuti (1987) one may find variants of this approach and the proofs of some of its features. It is interesting that although cut elimination cannot be proved in general for such a system it may be proved in restricted form. Let us call inessential any cut in which the cut formula is an equality, otherwise cut is essential. For Takeuti’s system it holds that all essential cuts are eliminable. Recently Parlamento and Previale (2019) proved an even stronger result showing that after an additional series of transformations cut can be eliminated from all proofs.

The third approach was also considered by Gentzen. Interestingly enough one can prove cut elimination for it but for any theorem φ of FOLI we do not obtain proofs of $\Rightarrow \varphi$ but of $\Gamma \Rightarrow \varphi$ where Γ is a collection of instances of REF and LP. Accordingly this approach does not provide an interesting tool for analysis of proofs.

There is a variant of this approach which in fact can be treated also as belonging to the last group. For each axiom φ an SC rule is postulated for elimination of this axiom, which is of the form:

$$\text{(AE)} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Such formalization of cut-free SC for FOLI is considered in Gallier (1986), where φ can be an instance of REF, CP or CT. Also Bell and Machover (1977) apply this approach in the tableau framework where it is represented just as a rule of introduction of suitable instances of REF, CT or CP on the branch. Note that in the context of SC this solution although applied in the cut-free version is in fact equivalent to addition of the special form of cut. This follows from the result that the cut elimination theorem is equivalent to the result showing eliminability of (AE) where φ is an arbitrary thesis (see Indrzejczak, 2017).

It seems that the most interesting approach, in particular for our purposes, is the last one. Gallier's solution is literaly speaking of this sort but not very satisfactory since (AE) is rather a trivial kind of rule mechanically applied to any formula. The subformula property does not hold and cut freeness is apparent, as we noted above. Generally speaking, it is not in any sense better than the first approach. What we need is the generation of genuine rules with active formulae in premisses and conclusions and satisfying possibly some welcome proof-theoretic properties like cut elimination or a reasonable form of the subformula property. Such solution which, in some specific form, was advocated by Negri and von Plato (2001) and applied to formalisations of several theories on the basis of SC, found many adherents. In particular, Troelstra and Schwichtenberg (1996) in the second edition of their well known textbook introduced this characterization of equality instead of the second one which was present in the first edition. Equality is characterised by means of two rules:

$$(RE)\ \frac{\tau_1 = \tau_2, \varphi[x/\tau_1], \varphi[x/\tau_2], \Gamma \Rightarrow \Delta}{\tau_1 = \tau_2, \varphi[x/\tau_1], \Gamma \Rightarrow \Delta} \qquad (REP)\ \frac{\tau = \tau, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

which are added to purely logical variant G3 . In general the specific features of Negri and von Plato's approach are connected with the fact that active formulae are atomic and occur only on one side of sequents. We will call this variant the one-sided approach; systems in Negri and Plato (2001) are in fact left- (or antecedent-)sided but in Negri and Plato (2011) right-(or succedent-)sided systems are also considered. Rules of this kind can safely be added without destroying all results concerning admissibility of structural rules, including cut. However, the rule-based approach to the characterisation of theories may be realised in many different ways, not necessarily as one-sided, despite its obvious virtues.

In order to put things in a systematic way we apply the following theorem (Indrzejczak, 2018b):

Theorem 4.1 (Rule-generation) *For any sequent* $\Gamma \Rightarrow \Delta$ *with* $\Gamma = \{\varphi_1, \ldots, \varphi_k\}$ *and* $\Delta = \{\psi_1, \ldots, \psi_n\}, k \geq 0, n \geq 0, k + n \geq 1$ *there are* $2^{k+n} - 1$ *equivalent rules captured by the general schema:*

$$\frac{\Pi_1, \Rightarrow \Sigma_1, \varphi_1,\ \ldots,\ \Pi_i \Rightarrow \Sigma_i, \varphi_i \qquad \psi_1, \Pi_{i+1} \Rightarrow \Sigma_{i+1},\ \ldots,\ \psi_j, \Pi_{i+j} \Rightarrow \Sigma_{i+j}}{\Gamma^{-i}, \Pi_1, \ldots, \Pi_i, \Pi_{i+1}, \ldots, \Pi_{i+j} \Rightarrow \Sigma_1, \ldots, \Sigma_i, \Sigma_{i+1}, \ldots, \Sigma_{i+j} \Delta^{-j}}$$

where $\Gamma^{-i} = \Gamma - \{\varphi_1, \ldots, \varphi_i\}$ *and* $\Delta^{-j} = \Delta - \{\psi_1, \ldots, \psi_j\}$ *for* $0 \leq i \leq k,\ 0 \leq j \leq n.$

It should be stressed that the proof of this theorem requires only applications of axioms and cut (see Indrzejczak, 2018b). Informally, it shows that for any sequent we can provide different rules which are interderivable with it. Premisses of these rules are obtained either by deleting some formula from the antecedent of the respective sequent and putting it in the succedent (of the respective premiss), or conversely, by deleting a formula in the succedent and putting it into the antecedent of a premiss. The conclusion of such a rule is provided by what remains intact in the input sequent. Let us see what kind of rules can be generated on the basis of LP (or EP if we wish), expressed as a sequent (1=) $\tau_1 = \tau_2, \varphi[x/\tau_1] \Rightarrow \varphi[x/\tau_2]$. We obtain the following equivalent rules:

$$(2=)\ \frac{\varphi[x/\tau_2], \Gamma \Rightarrow \Delta}{\tau_1 = \tau_2, \varphi[x/\tau_1], \Gamma \Rightarrow \Delta} \qquad (3=)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/\tau_1]}{\tau_1 = \tau_2, \Gamma \Rightarrow \Delta, \varphi[x/\tau_2]}$$

$$(4=)\ \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}{\varphi[x/\tau_1], \Gamma \Rightarrow \Delta, \varphi[x/\tau_2]} \qquad (5=)\ \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \quad \Pi \Rightarrow \Sigma, \varphi[x/\tau_1]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/\tau_2]}$$

$$(6=)\ \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \quad \varphi[x/\tau_2], \Pi \Rightarrow \Sigma}{\varphi[x/\tau_1], \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(7=)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/\tau_1] \quad \varphi[x/\tau_2], \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(8=)\ \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \quad \Pi \Rightarrow \Sigma, \varphi[x/\tau_1] \quad \varphi[x/\tau_2], \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta}$$

Each of them may be used to express LP as a rule of SC, and in fact all were used for that. For example, Negri and von Plato (2001) applied (2=) (but with the repetition of active formulae in the premiss to save admissibility of contraction — see the rule (RE) above) and Manzano (1999) prefers its dual form (3). Some authors, for example Parlamento and Previale (2019) used both (2=) and (3=) although it is redundant. Note also that both (2) and (3) may be seen as special forms of (SUB) (with singular antecedent and succedent containing only φ). (4) was applied by Reeves (1987), although in the framework of tableaux and in an apparently different way. He considers rules modelled not on LP but on CP, so the general schema would be rather:

$$(CP)\qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \tau_1 = \tau_1' \quad \dots \quad \Gamma_n \Rightarrow \Delta_n, \tau_n = \tau_n'}{\varphi(\tau_1, \dots \tau_n), \Gamma_1, \dots \Gamma_n \Rightarrow \Delta_1, \dots \Delta_n, \varphi(\tau_1', \dots \tau_n')}$$

and similarly for CT. Moreover, it deals with tableaux so the rules are branching downwards and there are no sequents but formulae as nodes.

Indrzejczak (2019) used (5=), whereas Baaz and Leitsch (2011) both (5=) and (6=) which is its dual. Nagashima (1966) used (7=) and Indrzejczak (2018a) (8=).

Let us discuss these rules in the light of properties required from well-behaved SC rules. Although all these rules are separate most of them are rather not satisfactory with respect to other features. Only (2=), (3=) and (7=) are weakly symmetric and explicit, in the sense that they may be treated as equality introduction rules. Together with REF treated as 0-premiss rule we may even say that such pair is symmetric. Only (2=) and (3=) satisfy the subformula property. In the remaining cases this property holds in the generalised sense: in any (cut-free) proof either subformulae of the proven sequent or atomic formulae occur.

What with cut elimination? Before we answer this question it should be established also in what form the reflexivity of equality is represented in the system. On the basis of the rule generation theorem, the only nontrivial rule (except axiomatic sequent REF) is the above mentioned:

$$\text{(REP)} \qquad \frac{\tau = \tau, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

which is used by Negri and von Plato (2001) (but also by Nagashima, 1966 and Gallier, 1986), whereas other authors prefer axiomatic sequents. However this is not an introduction but rather elimination rule. Of course we can think also of making use of Kalish and Montague (1964) solution from their ND system. Let us consider the possibility of application of the rule generation theorem to a sequent:

$$\forall x(x = \tau \rightarrow \varphi) \Rightarrow \varphi[x/\tau], \text{ where } x \text{ is not free in } \tau$$

corresponding to the reflexivity axiom. It may be expressed as a rule in the following ways:

$$(1)\ \frac{\varphi[x/\tau], \Gamma \Rightarrow \Delta}{\forall x(x = \tau \rightarrow \varphi), \Gamma \Rightarrow \Delta} \qquad (2)\ \frac{\Gamma \Rightarrow \Delta, \forall x(x = \tau \rightarrow \varphi)}{\Gamma \Rightarrow \Delta, \varphi[x/\tau]}$$

$$(3)\ \frac{\Gamma \Rightarrow \Delta, \forall x(x = \tau \rightarrow \varphi) \qquad \varphi[x/\tau], \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

Moreover, variants (2) and (3) may be improved in a way which dispenses with other constants:

$$(2')\ \frac{a = \tau, \Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \varphi[x/\tau]} \qquad (3')\ \frac{a = \tau, \Gamma \Rightarrow \Delta, \varphi[x/a] \qquad \varphi[x/\tau], \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

where a is not free in τ, Γ, Δ.

Such a rule is closer to the ordinary way of defining rules in SC setting since it is separated in a sense that no other constant is present in the schema. On the other hand, it is an elimination, not an introduction rule, similarly as (REP).

The last option for a nontrivial rule expressing reflexivity of equality is Read's rule from ND presented in the previous section. It may be used also in SC framework for formalization of reflexivity. It looks like this:

$$\frac{\varphi[x/\tau_1], \Gamma \Rightarrow \Delta, \varphi[x/\tau_2]}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2},$$

where φ is a predicate not present in Γ, Δ.

Such a rule is considered also by Parlamento and Previale (2019) whereas Restall (2020) considers a rule that is closer to Więckowski's solution (in fact he considers a stronger rule — see the next section):

$$(\Rightarrow=) \qquad \frac{\varphi[x/\tau_1], \Gamma \Rightarrow \Delta, \varphi[x/\tau_2] \qquad \varphi[x/\tau_2], \Gamma \Rightarrow \Delta, \varphi[x/\tau_1]}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}$$

where φ is a predicate not present in Γ, Δ.

These two rules seem to be the best choice from the syntactical point of view, since they are nontrivial equality introduction rules. Moreover they are separate, weakly symmetric (together with (2=), (3=) or (7=) even symmetric) and explicit. They also satisfy the subformula property in the generalised sense. Note also that for any simple applied language with n primitive predicates, these rules may be replaced with rules which do not refer to fresh predicate parameters. Consider a language having only n unary predicates, then a suitable rule will have just n (Read's variant) or $2n$ (Restall's variant) premisses. Of course in the case of languages having k-ary predicate constants (for $k > 1$) the situation is more complicated since for every such predicate we must have k- (Read variant) or $2k$-premisses which take into account all positions of τ_1, τ_2 as arguments of this predicate, exactly as in Quine's counterpart of LA.

Now let us consider which combinations of rules yield cut-free SC. At first we consider only how rules (2=)-(8=) behave in this respect together with (REF) or (REP). It is well known, as shown in Negri and Plato (2001), that SC with (REP) is cut free. In fact, also (2=) allows for cut elimination in LK again only with (REP). A similar situation holds for (6=). On the other hand (5=), (7=) and (8=) provide cut-free LK independently of the choice of (REP) or (REF). For systems with (3=) and (4=) it is not clear if cut elimination can be constructively proved. Consider the following:

$$\dfrac{(3=)\ \dfrac{\Gamma \Rightarrow \Delta, a = c}{a = b, \Gamma \Rightarrow \Delta, b = c} \qquad \dfrac{\Pi \Rightarrow \Sigma, Ab}{b = c, \Pi \Rightarrow \Sigma, Ac}\ (3=)}{a = b, \Gamma, \Pi \Rightarrow \Delta, \Sigma, Ac}\ (\mathrm{Cut})$$

It is neither possible to reduce the height of this cut or the complexity of cut-formula in the standard manner. A similar counterexample may be easily provided for (4=). Of course one can easily notice that in such cases the problem is connected with the fact that equalities are allowed as instances of φ in schemata of the respective rules. Of course if we think of rules for equality not only satisfying some desirable syntactic criteria for logicality, but also as being in a sense definitions of this constant, it would be desirable to restrict instances of φ to atomic formulae other than equalities. But then neither symmetry nor transitivity of equality can be proved. In fact, this holds for all seven rules considered in connection with (REF) or (REP).

Although most of the systems are cut-free, taking into account other properties, the only reasonable candidate for our aim is LK with (REF) and (7=). In the remaining combinations at least one rule is not a rule of equality introduction. Still (REF) is also rather a poor candidate for our aim. So eventually we should take (7) and ($\Rightarrow =$) (or its one-premiss version due to Read) as the pair of rules which, at least at the first sight, look better. In the next section we examine LK with such a pair of rules in the light of criteria of logicality proposed by Hacking.

5 Hacking's criterion

Hacking (1979) based his considerations on the criteria of logicality on the standard form of SC with canonical logical and structural rules[5]. He did not consider equality

[5] Although he is using sequents built from finite sets so contraction rules are dispensable.

but, as we will show, his analysis may be applied to this constant. In fact, Hacking follows closely Gentzen's suggestions concerning rules as possible definitions of constants. Hence rules of this kind have to satisfy all the properties we discussed in the preceding section with special attention paid to the subformula property. As we noticed, the two rules we have chosen satisfy this property only in the generalised sense but it seems that this generalisation is reasonable. However, this is not enough. In order to satisfy Hacking's requirements of logicality it should be possible to show three elimination theorems for a system consisting of such rules:

1. every axiom built from complex formulae must be eliminated in favor of atomic axioms;
2. every application of weakening rules introducing complex formulae must be eliminated in favor of weakening made on atomic formulae;
3. applications of cut must be eliminated.

The last one is Gentzen's famous Hauptsatz, whereas the remaining ones have a slightly weaker character since they do not postulate the complete elimination of purely logical axioms or weakening but only their reduction to the atomic level. In fact, if we consider purely logical versions of SC, i.e., without primitive structural rules, like G3 [6], then atomic axioms are present as primitive and all these requirements may be presented in a more uniform way as respective admissibility results. However, we opt for LK as our basis and check how it behaves when enriched with equality rules.

We start with SC for pure and applied FOLI. As we concluded the preceding section the only reasonable candidates are (7=) as the antecedent introduction rule, and Read's or Restall's rule as the succedent introduction rule. Accordingly we consider two variants, and in both (7=) will be taken (possibly in two symmetric versions) as the antecedent introduction rule. LKI1 is LK with:

$$(1{=}{\Rightarrow})\qquad \frac{\Gamma \Rightarrow \Delta, \varphi[a/\tau_1] \qquad \varphi[a/\tau_2], \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(1{\Rightarrow}{=})\qquad \frac{\varphi[a/\tau_1],\ \Gamma \Rightarrow \Delta,\ \varphi[a/\tau_2]}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}$$

where in the latter rule φ is an atomic predicate not in Γ, Δ (in antecedent introduction rules it is an arbitrary atomic formula).

Whereas in LKI2 we have instead:

$$(1{=}{\Rightarrow})\qquad \frac{\Gamma \Rightarrow \Delta, \varphi[a/\tau_1] \qquad \varphi[a/\tau_2], \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(2{=}{\Rightarrow})\qquad \frac{\Gamma \Rightarrow \Delta, \varphi[a/\tau_2] \qquad \varphi[a/\tau_1], \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

[6] See, e.g., Troelstra and Schwichtenberg (1996) or Negri and von Plato (2001).

$$(2{\Rightarrow}{=}) \qquad \frac{\varphi[a/\tau_1], \Gamma \Rightarrow \Delta, \varphi[a/\tau_2] \qquad \varphi[a/\tau_2], \Pi \Rightarrow \Sigma, \varphi[a/\tau_1]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \tau_1 = \tau_2}$$

where in the latter rule φ is an atomic predicate not in Γ, Δ.

The distinction between LKI1 and LKI2 follows from the way we treat equalities. If they are treated as atomic formulae, LKI1 is sufficient; otherwise we need LKI2. The key point is how to prove symmetry in both systems; in the former we have the following proof:

$$(1{=}{\Rightarrow})\ \dfrac{(1{\Rightarrow}{=})\ \dfrac{A\tau_1 \Rightarrow A\tau_1}{\Rightarrow \tau_1 = \tau_1} \qquad \tau_2 = \tau_1 \Rightarrow \tau_2 = \tau_1}{\tau_1 = \tau_2 \Rightarrow \tau_2 = \tau_1}$$

where $\varphi[a/\tau_i]$ is $a = \tau_1[a/\tau_i]$.

In LKI2 it looks like that:

$$(2{\Rightarrow}{=})\ \dfrac{(2{=}{\Rightarrow})\ \dfrac{A\tau_2 \Rightarrow A\tau_2 \qquad A\tau_1 \Rightarrow A\tau_1}{\tau_1 = \tau_2, A\tau_2 \Rightarrow A\tau_1} \qquad \dfrac{A\tau_1 \Rightarrow A\tau_1 \qquad A\tau_2 \Rightarrow A\tau_2}{\tau_1 = \tau_2, A\tau_1 \Rightarrow A\tau_2}\ (1{=}{\Rightarrow})}{\tau_1 = \tau_2 \Rightarrow \tau_2 = \tau_1}$$

Both systems are complete and it is not difficult to extend them to cover complex terms. However suitable rules of the antecedent introduction for equality must be modified in the way which ensures derivability of CT. It is only necessary to require that in two atomic formulae which are side formulae in both premisses the respective occurrences of terms being arguments of principal formula, may appear not only as arguments of this predicate but also as arguments of complex terms being its arguments. Under this generalised understanding a proof of the simplest case of CT in LKI2 for some unary operation f looks like this:

$$(2{\Rightarrow}{=})\ \dfrac{(1{=}{\Rightarrow})\ \dfrac{Af\tau_1 \Rightarrow Af\tau_1 \qquad Af\tau_2 \Rightarrow Af\tau_2}{\tau_1 = \tau_2, Af\tau_1 \Rightarrow Af\tau_2} \qquad \dfrac{Af\tau_2 \Rightarrow Af\tau_2 \qquad Af\tau_1 \Rightarrow Af\tau_1}{\tau_1 = \tau_2, Af\tau_2 \Rightarrow Af\tau_1}\ (2{=}{\Rightarrow})}{\tau_1 = \tau_2 \Rightarrow f\tau_1 = f\tau_2}$$

There are other solutions which work as well as LKI2 but are simpler. A close inspection of proofs needed to prove completeness shows that it is also possible to obtain two variants of LKI1 keeping the proviso concerning equalities in LKI2 (i.e., that they are not treated as atoms being possible instances of the antecedent introduction rule). Instead of one 2-premiss rule $(2{\Rightarrow}{=})$ we can add two one-premiss rules: It is enough either to use both rules of the antecedent introduction, as in LKI2, or to add a symmetric version of $(1{\Rightarrow}{=})$ (corresponding to the right premiss of $(2{\Rightarrow}{=})$):

$$(1{\Rightarrow}{=}') \qquad \frac{\varphi[a/\tau_2],\ \Gamma \Rightarrow \Delta,\ \varphi[a/\tau_1]}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}$$

(with the same proviso concerning fresh φ). It may be easily checked that suitable proofs of symmetry and transitivity may be obtained from the proofs in LKI2 stated above by deleting derivations of one branch. This provides an adequate SC for FOL with equality and simplifies many proofs since the branching factor is lower.

The last thing concerns the question how these variants of SC fare with respect to Hacking's criteria of logicality. We have noticed above that the subformula property is generalised but in the sense which is acceptable. As for eliminability conditions one may easily check that the first two hold for all stated versions of LKI. What with cut elimination? First of all note that in this respect LKI1 is not a suitable system. Consider the case where the cut formula is an equality which is a principal formula of the last applied rule, and moreover side formulae of $(1{=}{\Longrightarrow})$ application were equalities:

$$\dfrac{\dfrac{\varphi(\tau_1), \Gamma \Rightarrow \Delta, \varphi(\tau_2)}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2} \qquad \dfrac{\Pi \Rightarrow \Sigma, \tau_1 = \tau_3 \qquad \tau_2 = \tau_3, \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

In this case, there is no possibility of making a reduction either on the height or on the degree (complexity) of cut formula.

Perhaps LKI2, or modified versions of LKI1 with additional rule but excluding equality as atomic, are better. Again consider the case where the cut formula is an equality which is a principal formula of the last applied rule:

$$\dfrac{\dfrac{\varphi(\tau_1), \Gamma \Rightarrow \Delta, \varphi(\tau_2) \qquad \varphi(\tau_2), \Gamma \Rightarrow \Delta, \varphi(\tau_1)}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2} \qquad \dfrac{\Pi \Rightarrow \Sigma, \psi(\tau_1) \qquad \psi(\tau_2), \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

(The argument for the right premiss deduced by $(2{=}{\Longrightarrow})$ is symmetric). We assume that all atomic formulae have complexity 0 but equalities have complexity 1 since = is treated on a par with other logical constants. In order to apply induction on the complexity of the cut formula we should however first unify side formulae in the premisses of both applications of equality rules. The situation is analogous to the case of reduction made when the cut formula is introduced by quantifier rules. In this case we first apply substitution to fresh parameter of the premiss of $(\Rightarrow \forall)$ (or $(\exists\Rightarrow)$) and then we can make a reduction by making cut on the premisses. In the present case, if a similar kind of substitution is possible, the following reduction of cut degree would do:

$$\dfrac{\dfrac{\dfrac{\Pi \Rightarrow \Sigma, \psi(\tau_1) \qquad \psi(\tau_1), \Gamma \Rightarrow \Delta, \psi(\tau_2)}{\Pi, \Gamma \Rightarrow \Sigma, \Delta, \psi(\tau_2)} \qquad \psi(\tau_2), \Pi \Rightarrow \Sigma}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

where in the middle sequent $\psi(\tau_1), \psi(\tau_2)$ were substituted for unique occurrences of $\varphi(\tau_1), \varphi(\tau_2)$ in the leftmost premiss of the previuos figure. The problem is that no corresponding result for substitution of fresh atomic formulae can be proved in the presence of rules for antecedent introduction. Consider the following example which shows the source of the problem:

$$\dfrac{\tau_3 = \tau_2, \varphi(\tau_1), \Gamma_1 \Rightarrow \Delta_1, \varphi(\tau_2) \qquad \dfrac{\varphi(\tau_2) \Rightarrow \varphi(\tau_2) \qquad \varphi(\tau_3), \Gamma_2 \Rightarrow \Delta_2, \varphi(\tau_1)}{\tau_3 = \tau_2, \varphi(\tau_2), \Gamma_2 \Rightarrow \Delta_2, \varphi(\tau_1)}}{\tau_3 = \tau_2, \Gamma \Rightarrow \Delta, \tau_1 = \tau_2}$$

Assuming that it is a proof of the left premiss of the considered cut application we cannot change φ for ψ since the middle top sequent will not be an axiom.

This shows that in case of pure and applied FOLI our solution does not satisfy the most important condition in the list of logicality criteria provided by Hacking. Since it is not applicable to simple FOLI there is only one possibility — that it works for simple applied versions of FOLI. Certainly it works for all languages having only one unary predicate constant. In this case all the rules of LKI2 (or variants of LKI1) have the same shape but with $\varphi(\tau)$ being always $A\tau$ and with no side condition for the succedent introduction rules. All proofs are intact, moreover, the problem connected with the proof of cut elimination does not hold since there is only one atomic formula in both premisses of cut, i.e., both $\varphi(\tau)$ and $\psi(\tau)$ are $A\tau$ and the reduction of cut degree holds. What with languages having richer signature? Following Quine's recipe mentioned in Section 3 we must suitably modify both rules of succedent introduction: In the case of ($1\Rightarrow=$) for each n-ary predicate constant we must use n premisses with respective term as an argument in all positions, for ($2\Rightarrow=$) we must introduce $2n$ of such premisses. For example suitable form of ($1\Rightarrow=$) for the language with unary A and binary R which was considered in Section 3 for the illustration of LA′ looks like that:

$$\frac{A\tau_1, \Gamma_1 \Rightarrow \Delta_1, A\tau_2 \qquad R\tau_1 a, \Gamma_2 \Rightarrow \Delta_2, R\tau_2 a \qquad Ra\tau_1, \Gamma_3 \Rightarrow \Delta_3, Ra\tau_2}{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}$$

where a is not in $\Gamma, \Delta, \tau_1, \tau_2$.

Rules for the antecedent introduction remain intact but φ is now an instance of an arbitrary predicate constant with at least one occurrence of suitable term. Such versions of SC with succedent introduction rules having no fixed number of premisses but relative to the signature are not very satisfactory. In particular, from the proof-search point of view in the case of richer signature the branching factor is too big to use them in practice. However, we are not concerned here with practical application but with satisfiability of theoretical desiderata and from this point of view this solution works. In particular, cut elimination can be proved since every application of ($2\Rightarrow=$) (and ($1\Rightarrow=$) in variants of LKI1) has always among its premisses the one which has identical side formula as the ones occuring in actual premisses of ($1=\Rightarrow$) or ($2=\Rightarrow$).

6 Došen's criterion

In this section we focus on the criterion of logicality proposed by Došen. It can be seen as a successful refinement of a proposal due to Popper (1947a; 1947b) concerned with some conception of proof-theoretical semantics which was however not articulated in a satisfactory way. Popper tried to characterize constants by means of inferential definitions which yield double-valid rules characterising constants of the form:

$$(\rightarrow)\ \frac{\varphi,\ \chi\ \Rightarrow\ \psi}{\overline{\chi \Rightarrow \varphi \rightarrow \psi}} \qquad\qquad (\wedge)\ \frac{\varphi,\ \psi\ \Rightarrow\ \chi}{\overline{\varphi \wedge \psi \Rightarrow \chi}}$$

$$(\vee)\ \frac{\varphi \Rightarrow \chi \quad \psi \Rightarrow \chi}{\varphi \vee \psi \Rightarrow \chi} \qquad\qquad (\neg)\ \frac{\varphi, \psi \Rightarrow \chi}{\neg\chi, \psi \Rightarrow \neg\varphi}$$

Popper's project was criticised by Kleene, Curry, and many others, however it was convincingly shown by Schroeder-Heister (1984) (see also Schroeder-Heister, 2006 and Binder and Piecha, 2021) that his works contain an interesting proposal for a slightly weaker plan of providing criteria for being a logical constant. Such an enterprise was undertaken much later by Došen, first in his doctoral thesis and then in Došen (1989). It seems that if we do not use them as a way for establishing a variant of proof theoretic semantics but only as an independent criterion of logicality it works well. In fact, the opinions on the very nature of the relationships between criteria of logicality and proof-theoretic semantics differ strongly. Despite of Došen's own remarks that in his proposal rules are not intended as definitions of constants[7] it may be treated as a good framework even for developing proof-theoretic semantics. The idea of the application of double-line rules as definitional rules reappeared in such significantly different frameworks like Koslow's structural approach to logics (see Koslow, 1992), Sambin's Basic Logic (see Sambin, Battilotti, and Faggian, 2000), or categorical logic (see Maruyama, 2016). Moreover, this strong interpretation of Došen's proposal is provided independently by Gratzl and Orlandelli (2017) and by Restall (2019). In particular, the former work proposes an interesting explanation of the reasons for such a choice in terms of harmony. Since the 1960s, harmony was treated as crucial for the explanation of proper rules for defining logical constants, and a lot of work was offered in which a clarification of this notion was provided[8]. Harmony is in general understood as a kind of balance between two kinds of rules, introduction and elimination in ND, or antecedent and succedent introduction in SC. However, this notion is explicated in many different ways. Gratzl and Orlandelli (2017) proposed an explanation of harmony as a kind of deductive equilibrium which was first proposed by Tennant (2010). Their approach may be seen as an improvement of Tennant's solution in the sense that it is purely local, i.e., an analysis of a constant in terms of rules is independent of what other constants are already present in the language. Moreover, in contrast to other approaches to harmony it allows for the unique determination of one kind of rules from the other kind and vice versa.

Došen's system described below serves as an exemplification of his theory of criteria of logicality. The starting point of analysis of logical constants is the conviction that logic is the science of formal proofs. Hence basic formal proofs are of purely structural character, i.e., where only structural rules were applied[9]. It follows that an expression is logical iff it is analysable in purely structural terms. As he emphasized in the title of his paper — logical constants are punctuation marks. An analysis should satisfy three conditions:

[7] In particular, the proposed criteria do not necessarily satisfy the criteria of eliminability and non-creativity required from the well-stated definitions.

[8] See, e.g., Schroeder-Heister (2012), Poggiolesi (2011) or Kürbis (2019).

[9] It is in a sense a development of Hertz's programme (see Hertz, 1929).

1. any sentence (sequent) with constant to be analysed should be equivalent to a sentence (sequent) without it;
2. the analysis must be adequate (i.e., sound and complete);
3. the constant must be uniquely characterised.

His rules provide such an analysis in the sense that on the one side we have only structural sequents, i.e., with no constant displayed. According to Došen, in order to claim that an expression is a logical constant it is necessary to find such a double-valid rule which after addition to structural rules allows for obtaining a full characterisation of this constant. Došen (1989) proposed a structural version of LK, which we will call SLK, in the language without negation but with $\bot$ and $\top$ but we introduce a rule for negation instead. In this system the set of structural rules is primitive and not eliminable. Every constant is characterised by means of only one, but double-line (i.e., invertible) rule:

$$(\rightarrow)\ \frac{\varphi,\ \Gamma \Rightarrow\ \Delta,\ \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \qquad (\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad (\neg)\ \frac{\varphi,\ \Gamma \Rightarrow\ \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\vee)\ \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\forall)^1\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi} \qquad (\exists)^1\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta}$$

where in the latter two rules a is not in Γ, Δ, φ. In each case in addition to the rule of introduction we have also a rule of elimination if we read the rule upside down. Every rule is then a counterpart of a suitable equivalence characterising the respective constant within Scott's theory of consequence relations (see Scott, 1974). In what follows we will use notation ($\downarrow\rightarrow$) and ($\uparrow\rightarrow$) for suitable halves of the rule for implication and similarly for other constants.

The first condition is obviously satisfied for all rules. The second one can be shown by providing a proof of equivalence with some standard version of SC which is known to be adequate; our version of LK is sufficient. Since one half of each of Došen's rules corresponds to a suitable introduction rule such a proof amounts in principle to demonstration of derivability of the remaining rules by means of structural rules only. For example if we take his rule for implication, then ($\downarrow\rightarrow$) is exactly our ($\Rightarrow\rightarrow$) and the following shows derivability of ($\rightarrow\Rightarrow$) by means of ($\uparrow\rightarrow$):

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \dfrac{(\uparrow\rightarrow)\ \dfrac{\varphi \rightarrow \psi \Rightarrow \varphi \rightarrow \psi}{\varphi, \varphi \rightarrow \psi \Rightarrow \psi} \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi, \varphi \rightarrow \psi, \Pi \Rightarrow \Sigma}\ (\text{Cut})}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\text{Cut})$$

whereas the converse derivability goes like that:

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \qquad \dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi}\ (\rightarrow\Rightarrow)}{\varphi, \Gamma \Rightarrow \Delta, \psi}\ (\text{Cut})$$

Note that both demonstrations of derivability are structural and moreover in addition to the respective logical rules they use only axioms and cut. Such demonstration of interderivability is sufficient to show that rules of LK are harmonious in the sense explained by Gratzl and Orlandelli (2017). One may easily check that the remaining halves of Došen's rules are also interderivable with $(\wedge\Rightarrow)$, $(\neg\Rightarrow)$, $(\Rightarrow\vee)$, $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$. However, in the case of interderivability of $(\wedge\Rightarrow)$ and $(\Rightarrow\vee)$ with $(\uparrow\wedge)$ and $(\downarrow\vee)$ we must additionally use contraction and weakening. Derivations are still structural but some authors tend to be careful with that and admit as fully satisfactory analyses only those where axioms and cut are the only rules (see, e.g., Gratzl and Orlandelli, 2017). However, it is easy to provide a remedy. In fact, Došen's rules were based on the original Gentzen's rules for LK which are additive and for these rules interderivability requires only axioms and cut. On the other hand we have chosen all rules to be multiplicative. If we change Došen's rules for conjunction and disjunction into:

$$(\wedge)\ \frac{\varphi, \psi,\ \Gamma \Rightarrow\ \Delta}{\overline{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}} \qquad\qquad (\vee)\ \frac{\Gamma \Rightarrow\ \Delta, \varphi, \psi}{\overline{\Gamma \Rightarrow \Delta, \varphi \vee \psi}}$$

we can provide interderivability proofs for our version of LK by means of axioms and cut only.

On the other hand note that showing derivability of $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$ in structural variant requires additional structural rule of substitution:

$$\text{(SUB)}\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma[a/\tau] \Rightarrow \Delta[a/\tau]}$$

which is necessary to enable unrestricted instantiation (modulo correct substitution) of terms in these two rules.

The last condition, i.e., uniqueness may be demonstrated as the provability for each constant $*$ of two sequents: $*\varphi \Rightarrow \star\varphi$ and $\varphi \Rightarrow *\varphi$, where $\star$ is a notational variant having the same rule. Suitable proofs are trivial in this setting (although not in general — see, e.g., Došen, 1985).

How can equality be characterised in this framework? In fact, Došen proposed a rule which is of global character:

$$(=)\qquad \frac{\Gamma[a/\tau_1] \Rightarrow \Delta[a/\tau_1]}{\overline{\tau_1 = \tau_2, \Gamma \Rightarrow \Delta}}$$

In Section 2 we explained why such kind of rules cannot be treated as providing a criterion of logicality. What kind of local rules can be used instead? The obvious candidates are double versions of the rules for succedent introduction which we examined in the last section:

$$(1{=})\qquad \frac{\varphi[a/\tau_1],\ \Gamma \Rightarrow\ \Delta,\ \varphi[a/\tau_2]}{\overline{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}}$$

$$(2{=})\qquad \frac{\varphi[a/\tau_1], \Gamma \Rightarrow \Delta, \varphi[a/\tau_2] \qquad \varphi[a/\tau_2], \Gamma \Rightarrow \Delta, \varphi[a/\tau_1]}{\overline{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}}$$

where in both rules φ is atomic predicate not in Γ, Δ. The first one is an obvious SC version of Read's rule and the second of Restall's one. Let us consider a pure or applied FOLI as formalised by Došen's SLK in two variants: SLK1 (with (1=)) and SLK2 (with (2=)). We start with the latter, moreover we add, similarly as in LKI2, the proviso that equalities are not atomic predicates, so only predicate parameters (and other predicate constants in applied version) are admitted as instances of φ. This system satisfies the first condition so we must check the second, i.e., adequacy. Both directions of (2=) are cases of (2⇒=) and (4=) respectively so SLK2 is sound. Of course we can also easily provide demonstrations of the interderivability of (1=↑) and (2=↑) with (1=⇒) and (2=⇒) which is enough to show that the equality rules of LKI1 and LK2 are harmonious in the sense of Gratzl and Orlandelli (2017). To show that SLK (in both versions) is complete we must be able to prove the reflexivity axiom and LP which are immediate:

$$(\downarrow 2=)\ \frac{A\tau \Rightarrow A\tau \qquad A\tau \Rightarrow A\tau}{\Rightarrow \tau = \tau}$$

$$(\uparrow 2=)\ \frac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_2, \varphi[a/\tau_1] \Rightarrow \varphi[a/\tau_2]}$$

where φ is any atomic predicate (parameter or constant).

However, it is not sufficient. Since equalities cannot be instances of φ we must provide also proofs of symmetry and transitivity for =:

$$\begin{array}{l} (2{=}\uparrow) \\ (2{=}\downarrow) \end{array} \frac{\dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_2, A\tau_2 \Rightarrow A\tau_1} \qquad \dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_2, A\tau_1 \Rightarrow A\tau_2}\ (2{=}\uparrow)}{\tau_1 = \tau_2 \Rightarrow \tau_2 = \tau_1}$$

for transitivity we derive:

$$\begin{array}{l} (2{=}\uparrow) \\ (\text{Cut}) \end{array} \frac{\dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_2, A\tau_1 \Rightarrow A\tau_2} \qquad \dfrac{\tau_2 = \tau_3 \Rightarrow \tau_2 = \tau_3}{\tau_2 = \tau_3, A\tau_2 \Rightarrow A\tau_3}\ (2{=}\uparrow)}{\tau_1 = \tau_2, \tau_2 = \tau_3, A\tau_1 \Rightarrow A\tau_3}$$

and

$$\begin{array}{l} (2{=}\uparrow) \\ (\text{Cut}) \end{array} \frac{\dfrac{\tau_2 = \tau_3 \Rightarrow \tau_2 = \tau_3}{\tau_2 = \tau_3, A\tau_3 \Rightarrow A\tau_2} \qquad \dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_2, A\tau_2 \Rightarrow A\tau_1}\ (2{=}\uparrow)}{\tau_1 = \tau_2, \tau_2 = \tau_3, A\tau_3 \Rightarrow A\tau_1}$$

which together by (2=↓) yield $\tau_1 = \tau_2, \tau_2 = \tau_3 \Rightarrow \tau_1 = \tau_3$.

It is important to note that in both proofs the use of both premisses of (2=) is essential. It is not possible to provide a proof of symmetry and transitivity of = in SLK1 with the same proviso. On the other hand, if we admit equalities as possible instances of φ, SLK1 appears also to be an adequate formalization of pure or applied FOLI, similarly as LKI1. Proofs of symmetry and transitivity look like that:

$$\begin{array}{l} (1{=}\downarrow) \\ (\text{Cut}) \end{array} \frac{\dfrac{A\tau_1 \Rightarrow A\tau_1}{\Rightarrow \tau_1 = \tau_1} \qquad \dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2}{\tau_1 = \tau_1, \tau_1 = \tau_2 \Rightarrow \tau_2 = \tau_1}\ (1{=}\uparrow)}{\tau_1 = \tau_2 \Rightarrow \tau_2 = \tau_1}$$

where in the application of (1=↑) on the right $\varphi[a/\tau_i]$ is $a = \tau_1[a/\tau_i]$.

$$\dfrac{\tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2 \qquad \dfrac{\tau_2 = \tau_3 \Rightarrow \tau_2 = \tau_3}{\tau_1 = \tau_2, \tau_2 = \tau_3 \Rightarrow \tau_1 = \tau_3}\,(1{=}\uparrow)}{\tau_1 = \tau_2, \tau_2 = \tau_3 \Rightarrow \tau_1 = \tau_3}\,(\mathrm{Cut})$$

where in the application of (1=↑) on the right $\varphi[a/\tau_i]$ is $\tau_1 = a[a/\tau_i]$. Since the proof of LP is correct also with (1=) and the proof of reflexivity by (1=) is involved in the proof of symmetry above we are done. Also the proofs of uniqueness are directly obtainable in both versions of SLK. The above proofs show also that simple FOLI may be also formalised by means of SLK1; it is enough to change in the proof of reflexivity $A\tau \Rightarrow A\tau$ for $\tau = \tau \Rightarrow \tau = \tau$.

We leave the problem of formalization of simple applied versions of FOLI in SLK — it may follow the way described in the preceding section. The only difference is that now such many-premiss rules are treated as double valid. In particular, in the case of the language with just one unary predicate we can use just (2=) without proviso that φ is fresh.

There is no problem with the treatment of complex terms in the structural variant provided we will treat substitution of terms in the same way as in the systems of the preceding section. For illustration we can show how to provide a proof of CT for a binary operation f of the form: $\forall xyzv(x = y \wedge z = v \rightarrow fxz = fyv)$

We can prove:

$$(2{=}\downarrow)\,\dfrac{(2{=}\uparrow)\,\dfrac{a = b \Rightarrow a = b}{a = b, Afac \Rightarrow Afbc} \qquad \dfrac{a = b \Rightarrow a = b}{a = b, Afbc \Rightarrow Afac}\,(2{=}\uparrow)}{a = b \Rightarrow fac = fbc}$$

In a similar way we prove $c = d \Rightarrow fbc = fbd$. Since $fac = fbc, fbc = fbd \Rightarrow fac = fbd$ is provable as an instance of transitivity we obtain $a = b, c = d \Rightarrow fac = fbd$ by two applications of cut, and then the result by $(\wedge), (\rightarrow), (\forall)$. This proof generalizes for any n-ary operation.

We finish this section with the remark that similarly as in the case of other constants, there is another possibility of characterizing equality by means of invertible rule. We can use the antecedent-based rule:

$$(=3) \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi[a/\tau_1], \varphi[a/\tau_2] \qquad \varphi[a/\tau_1], \varphi[a/\tau_2], \Gamma \Rightarrow \Delta}{\tau_1 = \tau_2, \Gamma \Rightarrow \Delta}$$

This rule has an advantage of having no side condition on φ except that it is atomic. However, showing that it satisfies Došen's criteria is slightly more involved and we do not pursue this task here.

7 Final comments

The results of our analysis do not provide a decisive answer to the problem that was posed concerning logicality of equality. They show that the status of equality as a logical constant is dependent not only on the criteria which are taken into account but also sensitive to the version of the language to which equality is appended. In this respect the structural variant of Došen works better. All criteria of logicality hold for pure, applied and simple applied variants of FOLI introduced in Section 6. Moreover, invertible rules of this calculus allow us to show that standard introduction rules from Section 5 are harmonious in the sense advocated by Gratzl and Orlandelli (2017). However, in the latter case, i.e., SC realizing Hacking's approach, it is striking that the most important condition, namely cut eliminability, fails for the pure and applied version of FOLI. On the other hand it can be proved for many other sequent formalizations of FOLI but with non-canonical equality rules. Such rules however cannot be considered as proof-theoretic characterisations of a logical constant. Note also that simple FOLI does not satisfy the criteria of logicality in either formalization. Neither cut elimination holds for LKI variants, nor the first condition of Došen's analysis holds if equality is a sole predicate.

Our analysis was provided in the setting of standard SC and its slightly nonstandard variant but still based on the standard notion of a sequent. One may ask if the application of some other generalised setting may show in a more decisive way that equality satisfies the criteria of logicality. For example, in Hacking (1979) modal operators are taken as a negative example since in the setting of standard SC they cannot be characterised by means of canonical rules. But one can find more satisfying solutions on the ground of generalised formalisations. For example, in the setting of display logic one can provide rules for modal operators satisfying Hacking's demands (see, e.g., Wansing, 1999).

One possible generalization which works for equality can be based on using sequents with terms occuring on a par with formulae. The idea of formal systems with terms treated as fully fledged elements of deductions is not new. For the first time it was introduced by Jaśkowski (1934) in his first system of ND. Quite recently the idea was independently undertaken and developed in the setting of ND by Textor (2017) and Gazzari (2019). In the framework of SC it was developed by Restall (2019) and in Indrzejczak (2021). It seems that in this slightly generalised setting not only equality but also several kinds of term-forming operators may be formalised in a way important for further development of proof-theoretic semantics.

In this study we have restricted considerations to classical FOLI but the results we obtained may be easily extended to Intuitionistic version by restricting sequents to single-succedent ones. It is an open problem how to adapt this kind of analysis to other predicates of similar character applied in nonclassical logics.

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References

Avron, A. and I. Lev (2001). Canonical propositional gentzen-type systems. In: *Proceedings of IJCAR'01*. Vol. 2083. LNCS, 529–543.

Baaz, M. and A. Leitsch (2011). *Methods of Cut-Elimination*. Springer.

Bell, J. L. and M. Machover (1977). *A Course in Mathematical Lgic*. Amsterdam: North-Holland.

Binder, D. and T. Piecha (2021). Popper on quantification and identity. In: *Karl Popper's Science and Philosophy*. Ed. by Z. Parusnikova and D. Merrit. To appear. Springer.

Church, A. (1956). *Introduction to Mathematical Logic*. Vol. I. Princeton: Princeton University Press.

Degtyarev, A. and A. Voronkov (2001). Equality reasoning in sequent-based calculi. In: *Handbook of Automated Reasoning vol I*. Ed. by A. Robinson and A. Voronkov. Elsevier, 611–706.

Došen, K. (1985). Sequent-systems for modal logic. *Journal of Symbolic Logic* 50, 149–159.

– (1989). Logical constants as punctuation marks. *Notre Dame Journal of Formal Logic* 30, 362–381.

Gallier J., H. (1986). *Logic for Computer Science*. New York: Harper and Row.

Gazzari, R. (2019). The calculus of natural calculation. In: *Proof-Theoretic Semantics: Assesment and Future Perspectives. Proceedings of the Third Tübingen Conference of Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Tübingen, 123–139.

Gratzl, N. and E. Orlandelli (2017). Double-line harmony in sequent calculi. In: *The Logica Yearbook 2016*. Ed. by P. Arazim and T. Lavicka. College Publications, 157–171.

Griffiths, O. (2014). Harmonious rules for identity. *The Review of Symbolic Logic* 7, 499–510.

Hacking, I. (1979). What is logic? *Journal of Philosophy* 76, 285–319.

Hertz, P. (1929). Über axiomensysteme für beliebige satzsysteme. *Mathematische Annalen* 101, 457–514.

Hintikka, J. (1956). Identity, variables and impredicative definitions. *Journal of Symbolic Logic* 21, 225–245.

Indrzejczak, A. (2017). Tautology elimination, cut elimination and s5. *Logic and Logical Philosophy* 26, 461–471.

– (2018a). Cut-free modal theory of definite descriptions. In: *Advances in Modal Logic 12*. Ed. by G. B. et al. College Publications, 387–406.

– (2018b). Rule-generation theorem and its applications. *The Bulletin of the Section of Logic* 47, 265–281.

– (2019). Fregean description theory in proof-theoretical setting. *Logic and Logical Philosophy* 28, 137–155.

– (2021). A novel approach to equality. *Synthese* 199, 4749–4774.

Jaśkowski, S. (1934). On the rules of suppositions in formal logic. *Studia Logica* 1, 5–32.

Kahle, R. (2016). Towards a proof-theoretic semantics of equalities. In: *Advances in Proof-theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 153–160.

Kalish, D. and R. Montague (1957). Remarks on descriptions and natural deduction. *Archiv. für Mathematische Logik und Grundlagen Forschung* 3, 65–73.

– (1964). *Logicical Techniques of Formal Reasoning*. New York.

Kanger, S. (1957). *Provability in Logic*. Stockholm: Almqvist & Wiksell.

Klev, A. (2019). The harmony of identity. *Journal of Philosophical Logic* 48, 867–884.

Koslow, A. (1992). *A Structuralist Theory of Logic*. Cambridge: Cambridge University Press.

Kürbis, N. (2019). *Proof and Falsity. A Logical Investigation*. Cambridge: Cambridge University Press.

Lemmon, E. J. (1965). *Beginning Logic*. London: Nelson.

Manzano, M. (1999). *Model Theory*. Oxford: Oxford University Press.

– (2005). *Extensions of First-Order Logic*. Cambridge: Cambridge University Press.

Manzano, M. and M. C. Moreno (2017). Identity, equality, nameability and completeness. *The Bulletin of the Section of Logic* 46, 169–196.

Martin-Löf, P. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: *Proceedings of the Second Scandinavian Logic Symposium*. Ed. by J. E. Fenstad. North-Holland.

Maruyama, Y. (2016). Categorical harmony and paradoxes in proof-theoretic semantics. In: *Advances in Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 95–114.

Mates, B. (1965). *Elementary Logic*. Oxford: Oxford University Press.

– (1986). *The Philosophy of Leibniz. Metaphysics and Language*. Oxford: Oxford University Press.

Mints, G. (1968). Some calculi of modal logic. *Trudy Mat. Inst. Steklov* 98. in Russian, 88–111.

Nagashima, T. (1966). An extension of the craig-schütte interpolation theorem. *Annals of the Japan Association for the Philosophy of Science* 3, 12–18.

Negri, S. and J. von Plato (2001). *Structural Proof Theory*. Cambridge: Cambridge University Press.

– (2011). *Proof Analysis. A Contribution to Hilbert's Last Problem*. Cambridge: Cambridge University Press.

Parlamento, F. and F. Previale (2019). *The elimination of atomic cuts and the semishortening property for Gentzen's sequent calculus with equality*. on-line first. The Review of Symbolic Logic.

Poggiolesi, F. (2011). *Gentzen Calculi for Modal Propositional Logic*. Springer.

Popper, K. (1947a). Logic without assumptions. *Proceedings of the Aristotelian Society* 47, 251–292.

– (1947b). New foundations for logic. *Mind* 56.

Quine, W. V. (1966). *Set Theory and its Logic*. Harvard University Press.

– (1970). *Philosophy of Logic*. Prentice Hall.

Read, S. (2004). Identity and harmony. *Analysis* 64, 113–119.

Reeves, S. V. (1987). Adding equality to semantic tableau. *Journal of Automated Reasoning* 3, 225–246.

Restall, G. (2019). Generality and existence 1: quantification and free logic. *The Review of Symbolic Logic* 12, 354–378.

– (2020). Assertions, denials, questions, answers and the common ground. URL: http://consequently.org/presentation.

Sambin, G., G. Battilotti, and C. Faggian (2000). Basic logic: reflection, symmetry, visibility. *Journal of Symbolic Logic* 65, 979–1013.

Schroeder-Heister, P. (1984). Popper's theory of deductive inference and the concept of a logical constant. *History and Philosophy of Logic* 5, 79–110.

– (1994). Definitional reflection and the completion. In: *Extensions of Logic Programming. Fourth International Workshop, St. Andrews, Scotland, April 1993, Proceedings*. Ed. by R. Dyckhoff. Vol. 798. Berlin/Heidelberg/New York: Springer Lecture Notes in Artificial Intelligence, 333–347.

– (2006). Popper's structuralist theory of logi. In: *Karl Popper: A Centenary Assesment, vol III: Science*. Ed. by I. Jarvie, K. Milford, and D. Miller. Ashgate Publishing: Aldershot, 17–36.

– (2012). Proof-theoretic semantics. In: *Stanford Encyclopedia of Philosophy*. URL: https://plato.stanford.edu/entries/proof-theoretic-semantics/.

– (2016). Open problems in proof-theoretic semantics. In: *Advances in Proof-theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 253–283.

Scott, D. (1974). Rules and derived rules. In: *Logical Theory and Semantical Analysis*. Ed. by S. Stenlund, 147–161.

Seligman, J. (2001). Internalization: the case of hybrid logics. *Journal of Logic and Computation* 11, 671–689.

Suppes, P. (1957). *Introduction to Logic*. Princeton: Van Nostrand.

Takeuti, G. (1987). *Proof Theory*. Amsterdam: North-Holland.

Tarski, A. (1941). *Introduction to Logic*. Oxford University Press.

Tennant, N. (2010). Harmony in a sequent setting. *Analysis* 70, 462–468.

Textor, M. (2017). Towards a neo-brentanian theory of existence. *Philosophers' Imprint* 17, 1–20.

Troelstra, A. S. and H. Schwichtenberg (1996). *Basic Proof Theory*. Oxford: Oxford University Press.

Wang, H. (1960). Toward mechanical mathematics. *IBM Journal of Research and Development* 4, 2–22.

Wansing, H. (1999). *Displaying Modal Logics*. Dordrecht: Kluwer Academic Publishers.

Wehmeier, K. (2014). How to live without identity – and why. *Australasian Journal of Philosophy* 90, 761–777.

Więckowski, B. (2011). Rules for subatomic derivations. *Review of Symbolic Logic* 4, 219–236.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*. Brace and Co., New York: Harcourt.